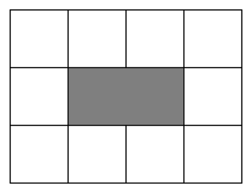
Mathematical Competition for Students of the
Department of Mathematics and Informatics of Vilnius University
Problems and Solutions

Paulius Drungilas[1], **Artūras Dubickas**[2]

2018-02-17

PROBLEMS

Problem 1. Is it possible to place the integers from 1 to 10 in the unshaded boxes of



the table in such a way that the four sums of numbers in two rows of four boxes and in two columns of three boxes

a) are all equal to 20?
b) are all equal to 16?

Problem 2. Is it true that for each positive integer n there exists a positive integer m such that $n|m$ and the sum of the digits of m equals n?

Problem 3. Find all pairs of positive integers (m, n) for which there exists a polynomial with real coefficients $P(x, y)$ satisfying the following four conditions

(1) $\deg_x P = m$,
(2) $\deg_y P = n$,
(3) $P(x, y) > 0$ for all $(x, y) \in \mathbb{R}^2$,
(4) $\inf_{(x,y)\in\mathbb{R}^2} P(x, y) = 0$

or prove that there are no such pairs (m, n).

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function satisfying $f(1) > 2$ and $f(2) < 3$. Prove that $f(x) = x + 1$ for some $x \in (1, 2)$.

Each problem is worth 10 points.

[1]Institute of Mathematics, Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, Vilnius LT-03225, Lithuania, http://www.mif.vu.lt/∼drungilas/

[2]Institute of Mathematics, Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, Vilnius LT-03225, Lithuania, http://www.mif.vu.lt/∼dubickas/

PROBLEMS WITH SOLUTIONS

Problem 1. Is it possible to place the integers from 1 to 10 in the unshaded boxes of the table in such a way that the four sums of numbers in two rows of four boxes and in two columns of three boxes

a) are all equal to 20?
b) are all equal to 16?

Answer: a) Yes; b) No.

Solution. Denote by x the sum of the four corner numbers. The sum of the integers from 1 to 10 is 55. Therefore, the sum of the numbers in the two rows and the two columns is $55 + x$.

Now, in case a) we get $4 \cdot 20 = 55 + x$, which implies $x = 25$. For example, we can fill the table as shown below.

10	5	1	4
8			7
2	3	6	9

In case b) one gets $4 \cdot 16 = 55 + x$, which implies $x = 9$. However, the sum of four corner numbers is at least $1 + 2 + 3 + 4 = 10$, a contradiction. □

Problem 2. Is it true that for each positive integer n there exists a positive integer m such that $n|m$ and the sum of the digits of m equals n?

Answer: Yes.

Solution 1. Write $n = 2^a 5^b k$, where a, b are nonnegative integers and $k \in \mathbb{N}$ is coprime to 10. Set $\ell = \max(a, b)$. By Euler's theorem, there is a positive integer s for which we have $10^s \equiv 1 \pmod{k}$. Consider the integer

$$m = 10^{s\ell} + 10^{s(\ell+1)} + \cdots + 10^{s(\ell+n-1)}.$$

We will show that m has the required property. Indeed, m has n (decimal) digits equal to 1 and other digits equal to 0. Hence, the sum of the digits of m equals n. Furthermore, m is divisible by $10^{s\ell}$, and so by $2^a 5^b$. Also, by the choice of s and m, the number m modulo k is zero, since $k|n$. Now, as $\gcd(2^a 5^b, k) = 1$, we conclude that m is divisible by $2^a 5^b k = n$. □

Solution 2. Consider n^2 positive integers 10^j, where $j = 0, 1, \dots, n^2 - 1$. Since there are n possible remainders modulo n, i.e., $0, 1, \dots, n-1$, by Dirichlet's box principle, at least n of those n^2 integers, say, $10^{k_1}, \dots, 10^{k_n}$, where $0 \leqslant k_1 < \cdots < k_n \leqslant n^2 - 1$, modulo n give the same remainder r. (Here, $r \in \{0, 1, \dots, n-1\}$.) Thus, the sum of those integers $m = 10^{k_1} + \cdots + 10^{k_n}$ is divisible by n. It is also clear that the sum of digits of m equals n. □

Problem 3. Find all pairs of positive integers (m, n) for which there exists a polynomial with real coefficients $P(x, y)$ satisfying the following four conditions

(1) $\deg_x P = m$,
(2) $\deg_y P = n$,
(3) $P(x, y) > 0$ for all $(x, y) \in \mathbb{R}^2$,
(4) $\inf_{(x,y)\in\mathbb{R}^2} P(x, y) = 0$

or prove that there are no such pairs (m, n).

Answer: All pairs $(m, n) \in \mathbb{N}^2$, where both m and n are even.

Solution. Suppose first that one of the numbers, say, m is odd, and assume that such a polynomial $P(x, y)$ exists. Let us write this polynomial in the form

$$P(x, y) = x^m Q_m(y) + \cdots + x Q_1(y) + Q_0(y),$$

where $Q_m(y), \dots, Q_1(y), Q_0(y)$ are some polynomials in y and $Q_m(y)$ is not identically zero. Select any $y_0 \in \mathbb{R}$ for which $q_m = Q_m(y_0) \neq 0$ and set $q_j = Q_j(y_0)$ for $j = 0, \dots, m-1$. Then, $P(x, y_0) = q_m x^m + \cdots + q_1 x + q_0$ is polynomial in x of odd degree. Therefore, there is $x_0 \in \mathbb{R}$ for which $P(x_0, y_0) < 0$, contrary to the condition (3). This proves that there is no such polynomial P in case m is odd. The proof for n odd is exactly the same.

Now, assume that both m and n are even. For $m \geqslant n$ let us consider the polynomial

$$P(x, y) = (xy - 1)^n + x^m.$$

It is clear that $\deg_x P = m$ and $\deg_y P = n$, so the conditions (1) and (2) are satisfied. Also, since m, n are even, we have $P(x, y) \geqslant 0$ with equality only for $(x, y) \in \mathbb{R}^2$ satisfying $xy - 1 = 0$ and $x = 0$. This is clearly impossible. Hence, $P(x, y) > 0$ for any pair $(x, y) \in \mathbb{R}^2$, which means that the condition (3) is also satisfied. Finally, for each $N \in \mathbb{N}$ selecting $(x_N, y_N) = (N^{-1}, N)$ we find that $P(x_N, y_N) = P(N^{-1}, N) = N^{-m}$, which tends to 0 as $N \to \infty$. This proves that the condition (4) is also satisfied. For $m < n$, by the same argument, the polynomial $P(x, y) = (xy - 1)^m + y^n$ satisfies all four conditions. □

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function satisfying $f(1) > 2$ and $f(2) < 3$. Prove that $f(x) = x + 1$ for some $x \in (1, 2)$.

Proof. Let $A \subset [1, 2]$ be a set of points a such that $f(a) \geqslant a+1$. Then, $1 \in A$ and $2 \notin A$. Set $\xi := \sup A$. Then, for any $\varepsilon > 0$ there is $a \in A$ such that $\xi - \varepsilon < a \leqslant \xi$. Hence,

$$\xi + 1 - f(\xi) \leqslant \xi + 1 - f(a) \leqslant a + \varepsilon + 1 - f(a) \leqslant \varepsilon.$$

Since ε is arbitrary, this yields $f(\xi) \geqslant \xi + 1$. Therefore, $\xi \in A$ and so $\xi \in [1, 2)$.

We claim that $f(\xi) = \xi + 1$ and $\xi \in (1, 2)$, so that ξ is one of the required x. Suppose $\delta := f(\xi) - \xi - 1 > 0$. Take any $y > \xi$ such that $y < \min\{\xi + \delta, 2\}$. Then, $y + 1 > f(y)$, since otherwise $y \in A$. Now, as f is non-decreasing, we find that

$$y + 1 > f(y) \geqslant f(\xi) = \xi + 1 + \delta > y + 1,$$

which is impossible. Hence, $\delta = 0$ and so $f(\xi) = \xi + 1$. Finally, from $f(1) > 2$ it follows that $\xi \neq 1$, so ξ belongs to the interval $(1, 2)$. $\square$