VILNIUS UNIVERSITY

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ON TWO MODELS OF RANDOM GRAPHS

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VILNIAUS UNIVERSITETAS

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Foreword

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Introduction

Formal introduction

This dissertation consists of two parts. The object of the first part of the dissertation is random intersection graphs and random intersection digraphs. The goal of the work was to determine certain asymptotic properties of such graphs (or digraphs). They include (a) the birth threshold for fixed-size complete subgraphs in the random intersection digraph; (b) the clique number of sparse random intersection graphs; (c) the chromatic number of random uniform intersection graphs. An additional goal was to better understand the connection of random intersection graphs and large real-world networks.

Random intersection graphs have been actively studied in the last decade. It has been shown that this model can produce instances with positive clustering coefficient and other commonly observed properties of real-world networks (such as the Internet, social and biological networks). The applications of such models include wireless networking, classification and epidemiology, see [21,22,46,53]. The work is also relevant from the computer science point of view: we consider classical NP-hard problems, but we restrict attention to a particular, rather general and practically important family of distributions of graphs.

The object of the second part of the dissertation is minor-closed classes of graphs without $k+1$ disjoint minors in $\mathcal{B}$, where a set $\mathcal{B}$ consists of 2-connected graphs. The problem of this part is to enumerate such classes asymptotically and prove properties of typical graphs in them. We study two general types of $\mathcal{B}$. As part of the work, we aim to answer a question of Bernardi, Noy and Welsh in this case.

The results in Part II build on the work of McDiarmid on addable minor-closed classes. The theory of graph minors has many applications in theoretical computer science and "has made a fundamental impact both outside the graph theory and within" [38]. Asymptotic enumeration of minor-closed classes was originally motivated by a particular case relevant both theoretically and practically, the planar graphs. Results of this kind can usually be directly applied to average-case complexity analysis of graph algorithms where the input is a uniformly random graph with some natural restrictions [12, 40, 48]. Another algorithmic application highlighted in the literature is as follows. The ability to count often gives knowledge how to construct large instances [52,85]. This can be used for system testing. The proofs in the second part are mostly based on combinatorial and probabilistic arguments (as opposed to the approach that uses mainly analysis of generating functions), and the results often hold with rather general conditions.

In the next two sections we specify the models and review the propositions that we prove in the thesis, this is done for each part separately.

Random intersection graphs

Let $S_1, S_2, \ldots, S_n$ be finite sets. The pairs uv where $u \neq v$ and $S_u \cap S_v \neq \emptyset$ define edges of a graph on the vertex set $[n] = \{1, \ldots, n\}$. This graph is called the *intersection graph* of $S_1, \ldots, S_n$, see Figure 1.

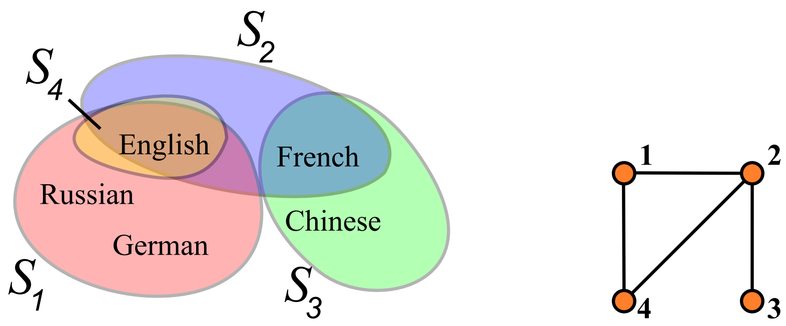

Figure 1: An intersection graph representing all communicating pairs, when, for example, S_v is the set of languages spoken by person v.

If the sets $S_1, \ldots, S_n$ are random subsets of some (finite) ground set W of m *attributes* (or keys), we obtain a random intersection graph. The first authors to consider such random graphs were Karoński, Scheinerman and Singer-Cohen (1999, [58]). They studied the *binomial random intersection graph* model $G(n, m, p)$, where an attribute w is added to the set S_v independently at random with probability p, for each pair (w, v), $w \in W$ and $v \in [n]$.

Godehardt and Jaworski [53] introduced a more general "active" random intersection graph $G(n, m, P)$, where each set S_v is generated independently at random in two stages: first the size X_v is drawn according to the probability distribution P, then a uniformly random subset S_v of size X_v is drawn (without replacement) from W. We give a more detailed description of this and related models in Section 1.1.

Studying random intersection graphs is motivated by a belief that they share some properties with large empirical networks. Large empirical networks are often observed to be sparse (the average number of neighbours of a vertex is bounded) and have a non-negligible clustering coefficient (which is the conditional probability that three randomly chosen vertices make up a triangle, given that the first two are neighbours of the third one). Unlike in many other models, the parameters of random intersection graphs can be chosen in a way that the resulting random instances have these two properties simultaneously.

For any collection $\{S_1, S_2, \ldots, S_n\}$ of subsets of W there is a unique *dual* collection $\{T_w\}_{w \in W}$ of subsets of $[n]$, where $T_w = \{v : w \in S_v\}$. In terms of intersection graphs, it can be understood as follows: each attribute $w \in W$ corresponds to a clique on the vertex set T_w; edges of all the cliques T_w define the set of edges of the intersection graph. We call cliques T_w *monochromatic*.

In their paper Karoński, Scheinerman and Singer-Cohen determined for which choices of p, the binomial random intersection graph $G(n, m, p)$ contains a clique on h vertices with high probability (when n and m are large). They solved the problem for any fixed h by showing that it is enough to consider a finite number of configurations of pairwise intersecting sets. Figure 2 shows two such configurations in the case $h = 4$.

Formalisms. Statements such as "$G(n, m, p)$ has a clique of size h with high probability" should be rigorously interpreted as follows. We consider a sequence of random graphs $\{G(n), n = 1, 2, \ldots\}$, where $G(n) = G(n, m, p)$ and $m = m(n)$, $p = p(n)$. For $n = 1, 2, \ldots$ we let $A = A(n)$ be the event that $G(n)$ has a clique of size h. Then a statement like "A holds with high probability" means that $\mathbb{P}(A(n)) \to 1$ as $n \to \infty$. Informal statements about $D(n, m, p_-, p_+)$ and $G(n, m, P)$ should be interpreted similarly. For example, when we talk about the parameter P, we actually have in mind a sequence of probability measures $\{P(n), n = 1, 2, ..\}$.

For a sequence $\{X_n, n = 1, 2, \ldots\}$ of random variables (for example X_n may be the size of a maximum clique in $G(n, m, p)$), we informally write that X_n is "asymptotically" $f(n)$ if for any $\epsilon > 0$ $\mathbb{P}(|X_n - f(n)| > \epsilon f(n)) \to 0$ as $n \to \infty$. A standard notation $X_n = f(n)(1 + o_P(1))$ will be used in the subsequent chapters.

Small subgraphs in random intersection digraphs

In Chapter 2 we ask a similar question for a related *binomial random intersection digraph* model $D(n, m, p_-, p_+)$. In this model, proposed by Bloznelis [16], each vertex $v \in [n]$ is assigned not one, but two random subsets, S_v^- and S_v^+. Each attribute $w \in W$ is included into S_v^- with probability p_- and into S_v^+ with probability p_+ independently. Then the random binomial intersection digraph is a directed graph on the vertex set $[n]$ with arcs $\{uv : S_u^- \cap S_v^+ \neq \emptyset\}$. Such a digraph makes sense if we interpret S_v^- and S_v^+ as sets of attributes (qualities) that v "likes" and "possesses" respectively.

We determine ranges of parameters for which $D(n, m, p_-, p_+)$ contains a copy of the complete directed graph on h vertices with a high probability. Depend-

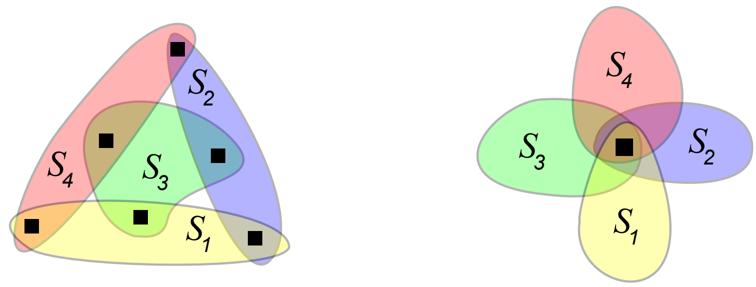

Figure 2: Two different configurations of intersecting sets that yield a clique on four vertices.

ing on the relationship between parameters p_-, p_+ and m (all of them can vary with n), four different patterns of intersecting sets can be most likely to realise the complete digraph on h vertices, and two of these patterns do not have an undirected counterpart.

Largest clique

But what can we say about the size of a maximum clique (*the clique number*) of a random intersection graph? In general, the problem is difficult due to two reasons. Firstly, the local clustering property causes a lot of dependence between the edges of the random graph. Secondly, the clique number can grow with n, and so it no longer suffices to consider only a finite number of patterns of sets. In Chapter 3 a solution for *sparse* uniform random intersection graphs $G(n, m, P)$ is presented. Here "sparse" means that m and P are such that the expected number of edges is linear in n.

Let $D(n)$ be the degree of a random vertex (or, equivalently, of vertex 1) in $G(n, m, P)$. We find that the clique number of $G(n, m, P)$ depends on the tail of the distribution of $D(n)$. If $D(n)$ is "asymptotically" power-law distributed with index $\alpha \in (1, 2)$ (for example, a Pareto distributed random variable X with $\mathbb{P}(X > t) = t^{-\alpha}$ is power-law with index α) then the largest clique is "asymptotically" of polynomial size. The order of the clique number in this case is the same as in a much simpler model without clustering studied by Janson, Łuczak and Norros (2010, [55]).

Meanwhile, if the degree variance is bounded ($\sup_n VarD(n) < \infty$), then the largest clique is with high probability "almost" monochromatic (generated by a single attribute, as in Figure 2, right) and its size is "asymptotically" logarithmic. This phenomenon is specific to random intersection graphs, and the clique number here is closely related to the maximum load problem: if N balls are thrown randomly to m bins, what is the maximum number of balls a bin receives? Both in the "power-law" and the "bounded variance" regimes our results are optimal up to the first-order asymptotic term. These two regimes cover most of the interesting choices of the parameters for sparse $G(n, m, P)$.

Furthermore, for each of the two main regimes there is a simple algorithm for finding large cliques. We prove that with high probability $G(n, m, P)$ is such, that the corresponding algorithm outputs a clique of asymptotically optimal size and terminates in polynomial time. These algorithms have a potential to be used and studied with large scale real-world graphs. A reader interested to see the simple pseudocode is welcome to jump directly to Section 3.4.

Chromatic index of random uniform hypergraphs

A *hypergraph* H is a pair (V, E), where V is a set and E is a collection of subsets of V called hyperedges, or simply *edges*. Intersection graphs are hypergraphs, where we put emphasis on pairwise intersections of edges. The *chromatic number* of a graph is the least number of colours needed to colour its vertices, so that no two neighbours receive the same colour. The *chromatic index* of a hypergraph is the least number of colours needed to colour its edges so that no pair of intersecting edges receives the same colour.

In Chapter 4 we study the chromatic index of $\mathbb{H}^{(k)}(m, n)$, the random hypergraph on the vertex set $[m]$ and with n edges drawn independently with replacement from all subsets of $[m]$ of size k. The problem is equivalent to the problem of determining the chromatic number of $G(n, m, k)$, the uniform random intersection graph with n vertices, m attributes and all subsets of size k. In the case when k is constant and n is much larger than m, a result by Pippenger and Spencer (1989, [82]) implies the answer. That result holds for arbitrary 'almost regular' hypergraphs, not just the random ones. For random hypergraphs we extend their result slightly and allow k to grow slowly with n. To do this, we exhibit a simple greedy algorithm (different from Pippenger and Spencer's one) and prove that it colours the edges with an (asymptotically) optimal number of colours.

Empirical aspects

In Section 1.2 we plot certain statistics for large real-world networks (such as, for example, the actor affiliation networks, where two actors are declared adjacent if they had a role in the same movie) and random intersection graphs with corresponding parameters. The statistics include assortativity and counts of pairs with given number of common neighbours.

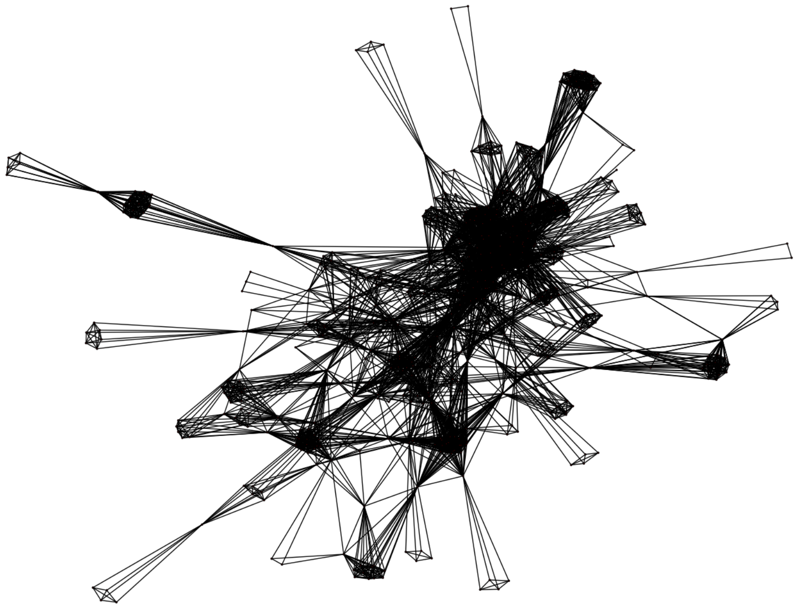

Figure 3: The Lithuanian actor affiliation network (data from IMDB: `http://www.imdb.com`). The 'union of cliques' structure, where each clique consists of actors participating in the same film, can clearly be seen here.

Minor-closed classes of graphs

The second part of this thesis is concerned with graphs that do not contain certain subgraphs.

Connected graphs that do not have any cycle are *trees*. Acyclic, but not necessarily connected graphs are called *forests*. Given a class of labelled graphs $\mathcal{A}$ (for example, the class of trees), we denote by $\mathcal{A}_n$ the restriction of $\mathcal{A}$ to graphs on the vertex set $[n] = \{1, .., n\}$. We study

(*) the asymptotic number of graphs in $\mathcal{A}_n$;

(**) the structure of a typical graphs in $\mathcal{A}$; more precisely, properties of a uniformly random graph from $\mathcal{A}_n$.

For example, the classic result of Cayley (1868) states that there are n^{n-2} trees on the vertex set $[n]$. Rényi (1959) proved that the number of forests on the same vertex set is $\sqrt{e}n^{n-2}(1 + o(1))$ as n tends to infinity.

Part II starts with investigation of the class of graphs that do not have $k + 1$ vertex-disjoint cycles. Erdős and Pósa (1965, [45]) showed that there is a constant c_k such that each graph that does not contain $k + 1$ disjoint cycles has a set of at most c_k vertices, whose removal results in an acyclic graph (a forest). It is known that the smallest possible c_k is of order $k \ln k$ [38]. (The important thing here is that no matter how large a graph is, if it has at most k disjoint cycles then we can "destroy" all of its cycles by removing just a few vertices.)

In Chapter 6 we present a proof that only k vertices are enough for typical graphs without $k+1$ disjoint cycles. A uniformly random such graph on the vertex set $[n]$ for large n is shown to be very close in distribution to the following simple construction a) pick a uniformly random set $S \subset [n]$ of size k; b) put a uniformly random forest on the remaining vertices $[n] \setminus S$; c) for each pair $\{x, y\} \subset [n]$ with at least one element in S, add the edge xy independently at random with probability 1/2.

Given a graph G, the *contraction* of an edge $e = xy \in E(G)$ is the following operation: merge the endpoints x and y of e into a new vertex v_{xy}, so that v_{xy} becomes adjacent to all of the former neighbours of x and y. A graph H is called a *minor* of G if it can be obtained from G by applying a series of edge deletions, vertex deletions and edge contractions, see Figure 4.

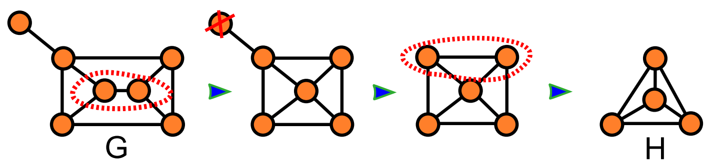

Figure 4: A sequence of vertex deletions and edge contractions showing that H is a minor of G.

A class of graphs $\mathcal{A}$ is *minor-closed* if for any $G \in \mathcal{A}$ every minor of G also belongs to $\mathcal{A}$. Minor-closed classes of graphs is the subject of the theory of graph minors developed by Robertson and Seymour in a series of more than twenty papers (1983-2004). One of the results is the following: each minor-closed class $\mathcal{A}$ can be characterised by a finite list $\mathcal{B}$ of minimal *excluded minors*. That is, to test whether a graph G is *not* in the class $\mathcal{A}$, it suffices to check whether any of the finitely many graphs in $\mathcal{B}$ is a minor of G. We denote this by $\mathcal{A} = \text{Ex}\,\mathcal{B}$. For example, by an earlier work of Kuratowski (1930) and Wagner (1937) the class of planar graphs (graphs drawable on the plane so that edges can intersect only at their endpoints) can be characterised by two minimal non-planar excluded minors: $K_{3,3}$ and K_5 (here $K_{t,t}$ is the complete bipartite graph with both parts of size t and K_t is the complete graph on t vertices). Counting and studying properties of random planar graphs and other minor-closed classes of graphs has received a lot of attention in the last decade, we review the work most relevant to this thesis in Chapter 5. Some other examples of minor-closed classes are forests, series-parallel graphs, outerplanar graphs, graphs embeddable in a fixed surface (for example, the torus), graphs with a bounded treewidth, graphs knotlessly embeddable in

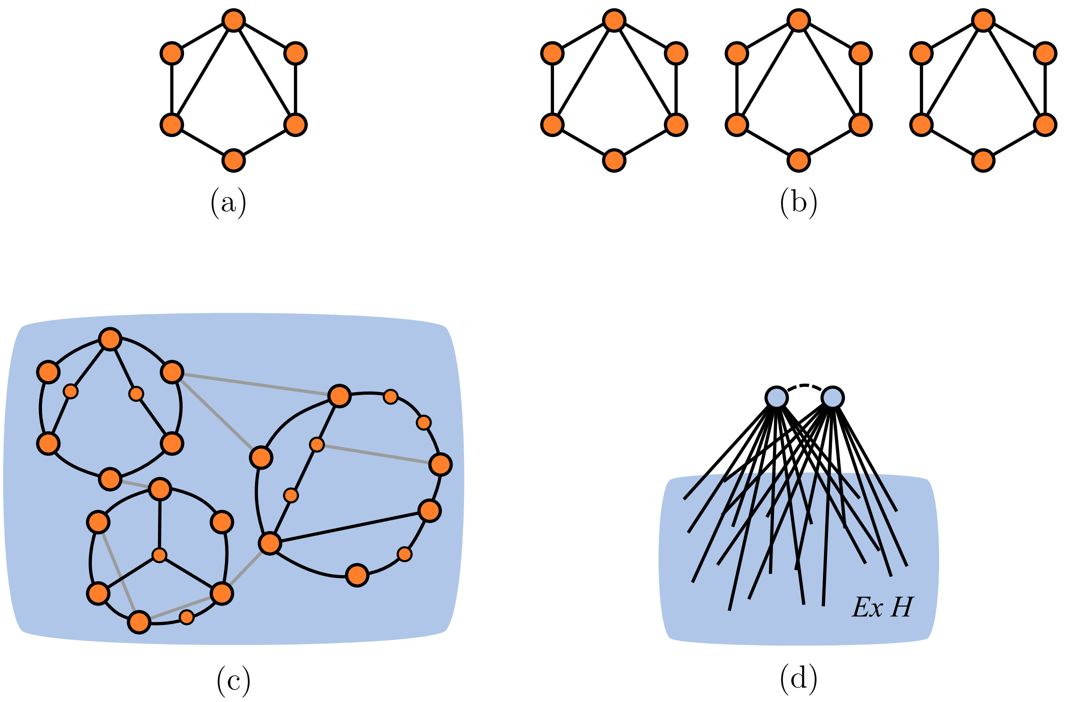

Figure 5: Illustration of a class of graphs handled in Chapter 7: (a) the graph H; (b) the forbidden minor: 3 disjoint copies of H; (c) a graph with three disjoint forbidden minors; (d) a typical graph without three disjoint minors H.

Euclidean 3-space, etc.

The class of graphs containing at most $k+1$ disjoint cycles is also minor-closed; the forbidden minor is $k+1$ disjoint copies of K_3. Chapter 7 generalizes results of Chapter 6 to classes with at most k disjoint excluded minors from a given fixed set $\mathcal{B}$ (with repetitions allowed). For the generalisation to work, the excluded minors in $\mathcal{B}$ have to necessarily satisfy a certain restriction: the class of graphs $\operatorname{Ex}\mathcal{B}$ must not contain arbitrarily large fans (a fan is a graph consisting of a path together with a vertex joined to each vertex on the path).

We postpone the formal statements of our theorems until Chapter 7; now we will just discuss one example, a straightforward application of our results with $k = 2$ and a set $\mathcal{B}_0 = \{H\}$, consisting of a particular graph H on six vertices shown in Figure 5 (a). Our result concerns the class $\mathcal{A}$ of graphs with the excluded minor shown in Figure 5 (b). Graphs that violate the requirement are, for instance, as in Figure 5 (c) (two of the subgraphs are *subdivisions* of H, the third can be seen to have H as a minor by contracting two edges incident to vertices marked with the smaller circles).

Our result implies that a uniformly random graph in $\mathcal{A}_n$ essentially consists of a random graph in $\operatorname{Ex}\mathcal{B}$ on $n-2$ vertices, two "apex vertices", and edges incident to each of the apex vertices which appear independently with probability 1/2.

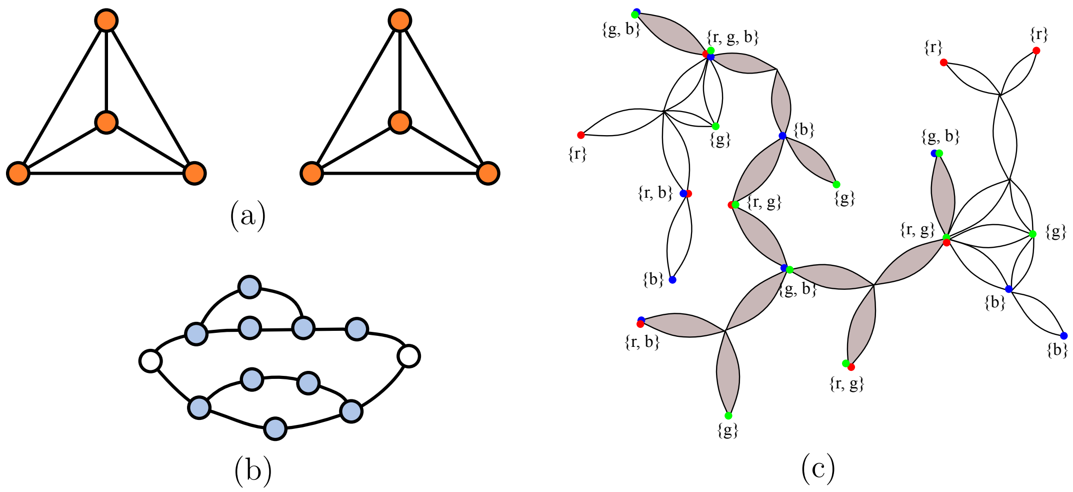

Figure 6: (a) A forbidden minor: two disjoint K_4; (b) a series-parallel network; (c) the "core" of a typical graph without two disjoint minors K_4. To complete the graph, add three new "apex" vertices x, y, z and join them to each node coloured r[ed], g[reen] and b[lue] respectively; replace leaf-like shapes by (non-series) series-parallel networks, and attach more series-parallel graphs at each vertex arbitrarily. Neither of x, y, z is allowed to create a K_4 minor alone.

With very high probability these two apex vertices are the only vertices that have linearly many neighbours, see Figure 5 (d).

The last two chapters are devoted to the next layer of disjoint forbidden minors. In Chapter 8 we prove results for general classes $\mathcal{B}$, such that $\mathrm{Ex}\,\mathcal{B}$ contains all fans, but $\mathcal{B}$ is "good enough". We show that a different general construction dictates the asymptotic number of graphs without $k+1$ disjoint excluded minors in $\mathcal{B}$. The motivating case behind quite general results of Chapter 8 was a particular set $\mathcal{B} = \{K_4\}$. The graphs without a minor K_4 are known as *series-parallel* graphs. This class is important in computer science, and has been used to study algorithms for problems that are hard for general graphs.

In Chapter 9 we obtain precise first-order asymptotics for the number of graphs without $k+1$ disjoint minors K_4. We prove that for large n, a typical such graph G on $\{1, \ldots, n\}$ has a unique set S with the following properties:

(i) S is of size $2k+1$;

(ii) for any $x \in S$, the removal of $S \setminus \{x\}$ from G results in a series-parallel graph;

(iii) each vertex in S has linearly many neighbours.

A rather complete picture of typical graphs from this class is developed. Figure 6 illustrates some of the basic features.

Overview of the methods

Many different methods are used throughout the dissertation. The most important of them are discussed in more detail in the background chapters, Chapter 1 and Chapter 5.

Inequalities from probability theory and probabilistic method. We use various classic probabilistic methods such as the first and second moment method, Chebyshev's inequality quite intensively in Part I of the thesis. In addition, we use some less standard results from Ramsey theory and the theory of random graphs.

Concentration inequalities. In both parts an indispensable tool is Chernoff bounds for sums of independent random variables. Most of the applications require only that the bounds are exponential, the constant in the exponent is not important. In Chapter 4 we use more sophisticated concentration inequalities for martingales, from McDiarmid (1998) [72].

Differential equations method. This method is based on concentration inequalities for martingales. It was developed by Wormald [101] in the context of random graph processes, though Karp and Sipser had applied similar techniques in their earlier work [59]. The main idea is to show that a trajectory of a parameter of a random process is highly concentrated around its mean at all time steps. The curve for the mean is a solution of a system of differential equations. We apply this method for the random hypergraph edge colouring problem.

Theory of graph minors. Several major results in the theory of graph minors by Robertson and Seymour, see [89], are the starting point in the proofs of Part II. One of the key facts is that graphs with a planar excluded minor have a bounded tree-width.

Singularity analysis. Methods based on generating functions play an important role in Part II. While in Chapters 6, 7 and 8 we only make use of very simple results, such as the "exponential formula", Chapter 9 contains a full application of the singularity analysis method: we obtain decompositions of relevant classes, convert them to exponential generating functions and use either general theorems or our own observations in complex analysis to extract the asymptotic coefficients. Most of the tools of this kind can be found in the book "Analytic Combinatorics" by Flajolet and Sedgewick [48]. For counting tree-like structures we also find the work of Meir and Moon (1989, [79]) very general and useful.

Computational tools. We used computer to aid some of our work. Numerical estimates presented in Part II were carried out with the symbolic computation system `Maple`. Empirical analysis of real networks in Section 1.2 required larger scale computation, this was implemented using `Python` with its packages `numpy` and `matplotlib` and executed in the cluster of the Digital Science and Computing Centre of the Faculty of Mathematics and Informatics, Vilnius University. Some programming with `C++` and `Python` was used to explicitly construct all possible graphs related to classes with few disjoint minors K_4. Most of the illustrations in this thesis were created using `Xfig` and `Inkscape`. The dissertation itself was prepared and compiled with `XeLaTeX`.

The methods and proofs presented in this thesis are mathematically rigorous. The empirical parameters of large real networks were only evaluated for particular graphs and served mainly for illustratory purposes. Statistical inference or hypothesis testing using random intersection graph models can be seen as a potential future work in the area.

Content, originality and novelty

The content presented in this thesis has been created and prepared by the author of the thesis together with his co-authors.

The results obtained in the dissertation are original and all of them can be considered as new. Most of the problems of Part I had been considered by other authors with related but different models. In our work we propose several methods not used in this context before (applications of the balls and bins problem, extremal combinatorics and differential equations). The important phenomenon of largest clique being generated by a single attribute has been earlier also discovered by two other groups of researchers in related but more restricted models. Part II explores an entirely new type of minor-closed classes and similar results were unknown before. Some of the intermediate lemmas extend previously known ones.

Each of the results chapters is based on an article that has either been published or submitted for publication.

Section 1.2 – on paper [20] with M. Bloznelis and J. Jaworski and paper [24] with M. Bloznelis[1].

Chapter 2 – on paper [62];

[1] In these two works I essentially carried out only the empirical analysis.

Chapter 3 – on paper [25] with M. Bloznelis;

Chapter 4 – on paper [66] with K. Rybarczyk.

Chapter 6 – on paper [64] with C. McDiarmid;

Chapter 7 – on paper [65] with C. McDiarmid,

Chapter 8 and Chapter 9 – on paper [63].

The papers [24,62,64,65] have been published, the remaining papers have been submitted for publication.

Part I

Random intersection graphs

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Conclusion

Results of Part I

1. In Chapter 2 we considered sequences of random intersection graphs $\{D(n)\}$ where $D(n) = D(n, m, p_-, p_+)$ and $m = m(n)$, $p_- = p_-(n)$ and $p_+ = p_+(n)$. We defined the birth threshold function τ such that $\tau(n, m, p_-, p_+) \to \infty$ (respectively, 0) implies that $D(n)$ contains (respectively, does not contain) a copy of the complete directed graph $\overrightarrow{K}_h$ whp. Next, we introduced the notion of a diclique cover of a digraph. We showed that there are several possible cases of the relationship of the parameters m, p_- and p_+, and to each case corresponds one or more simple diclique covers. The "in-star" and the "out-star" covers that realise the birth threshold when p_- is much larger than p_+ (respectively, p_+ much larger than p_-), were not possible in the undirected case.

2. In Chapter 3 we considered sequences $\{G(n)\}$ of sparse 'active' random intersection graphs $G(n) = G(n, m, P)$, where $m = m(n)$, $P = P(n)$. We introduced a power-law tail condition (**??**) for the normalized random subset size $Y(n) = \sqrt{\frac{n}{m}}X(n)$, where $X(n)$ is distributed according to $P(n)$. We determined the asymptotic clique number in $G(n)$ (it is polynomial in n) when $Y(n)$ satisfies this condition with index $\alpha \in (1, 2)$ for a wide range of sequences $m = m(n)$, including the case $m = \Theta(n)$ that yields a non-vanishing clustering coefficient. Secondly, we considered the case where $G(n)$ is sparse and $\mathbb{E}Y = \Theta(1)$ and $VarY = \Theta(1)$. In this case we showed that the largest clique in $G(n)$ is monochromatic (plus possibly a stochastically bounded number extra vertices) and we proved that the total variation distance of $\omega'(G(n))$ and the size of the maximum bin when $(mn)^{1/2}\mathbb{E}Y(n)$ balls are thrown into m bins tends to zero. Thirdly, we described algorithms to find a clique of asymptotically optimal size in each of the above cases, and showed their correctness and efficiency. Finally, we proved a technical result on the relation of $Y(n)$ and the degree of a random vertex of $G(n)$.

3. In Chapter 4 we introduced a randomized greedy algorithm for colouring edges of the random uniform hypergraph $\mathbb{H}^{(k)}(n, m)$ and proved that there is a constant c_ϵ such that if $k \geq 2$, $k \leq c_\epsilon \ln\left(\frac{n}{\ln \bar{d}}\right)$ and $k \leq c_\epsilon \ln\left(\frac{\bar{d}}{\ln n}\right)$ then the algorithm properly colours the edges of $\mathbb{H}^{(k)}(n, m)$ with $\lceil \bar{d}(1+\epsilon) \rceil$ colours and probability at least $1 - \frac{2}{n} - \frac{2}{\bar{d}}$. For a sequence $\{H(n), n = n_0, n_0+1, \dots\}$ where $k = k(n)$, $m = m(n)$ and $H(n) = \mathbb{H}^{(k)}(n, m)$ satisfies the above condition this yields $\chi'(H(n)) = \bar{d}(1 + o_P(1))$.

4. In Section 1.2 we presented plots illustrating that parameters in real-world networks can, with interesting exceptions, be matched closely with those in random intersection graphs. This direction requires further research.

To sum up, we determined the behaviour of several important parameters in random intersection graphs. Notably, we made progress in the most practically relevant regime of sparse graphs with positive clustering.

Results of Part II

1. In Chapter 6 we proved that $|(\mathrm{Ex}\,(k+1)K_3)_n| = (1 - e^{-\Omega(n)})|(\mathrm{apex}^k\,\mathcal{F})_n|$. Using this, we obtained precise asymptotic counting formula for graphs without $k+1$ disjoint cycles. We showed that with probability $1 - e^{-\Omega(n)}$ a uniformly random graph R_n from $(\mathrm{Ex}\,(k+1)K_3)_n$ contains a unique vertex feedback set (blocker) of size k, we determined the asymptotic probability that R_n is connected and investigated the asymptotic distribution of the number of components, chromatic and clique numbers of R_n.

2. In Chapter 7 we generalized results of Chapter 6 and proved that $|(\mathrm{Ex}\,(k+1)\mathcal{B})_n| = (1 - e^{-\Theta(n)})|(\mathrm{apex}^k\,\mathcal{A})_n|$, as long as the class $\mathcal{A} = \mathrm{Ex}\,\mathcal{B}$ is addable and does not contain all fans. We showed that this implies that such a class $\mathrm{Ex}\,(k+1)\mathcal{B}$ has a growth constant $2^k\gamma(\mathcal{A})$, i.e., the answer to the question of Bernardi, Noy and Welsh in this case is positive. We expressed asymptotics of $|(\mathrm{Ex}\,(k+1)\mathcal{B})_n|$ in terms of $|\mathcal{A}_n|$. Next, we showed that with probability $1 - e^{-\Omega(n)}$ a random graph $R_n \in_u \mathrm{Ex}\,(k+1)\mathcal{B}$ contains a unique $\mathcal{B}$-blocker S of size k, such that each vertex in the blocker has a linear degree. We also generalized proofs of other asymptotic properties (connectivity, components, clique and chromatic number, etc.).

3. In Chapter 8 we considered addable classes $\mathcal{A} = \mathrm{Ex}\,\mathcal{B}$ such that $\mathcal{A}$ contains all fans, but not all 2-fans, nor all bipartite graphs $K_{3,t}$. We showed that there is a constant k_0, such that for $k \ge k_0$, $\overline{\gamma}(\mathrm{Ex}\,(k+1)\mathcal{B}) = \overline{\gamma}(\mathrm{rd}_{2k+1}\,\mathcal{B})$ and for a subsequence n_l realising this upper limit, a random graph $R_{n_l} \in_u \mathrm{Ex}\,(k+1)\mathcal{B}$ has no $\mathcal{B}$-blocker of size smaller than $2k$ with probability $1 - e^{-\Omega(n_l)}$ as $l \to \infty$. We also proved that if we add a further condition that the minimal excluded minors are 3-connected and $\mathcal{A}$ does not contain all wheels, then $\mathrm{Ex}\,(k+1)\mathcal{B}$ has a growth constant. To obtain these results we proved two non-trivial graph-theoretical lemmas.

4. In Chapter 9 we proved that for $k = 1, 2, \ldots$ there are constants c_k, γ_k such that $|(\mathrm{Ex}\,(k+1)K_4)_n| = (1 - e^{-\Omega(n)})|(\mathrm{rd}_{\,2k+1}\,K_4)_n| \sim c_k n^{-3/2}\gamma_k^n n!$. We proved that $R_n \in_u \mathrm{Ex}\,(k+1)K_4$ whp has a unique redundant blocker of size $2k+1$, and each vertex in this blocker has a linear degree. Along the way we obtained decompositions for classes related to $\mathrm{rd}_{\,2k+1}\,K_4$ and proved a lemma for enumerating trees where leaves, internal vertices and edges are replaced with objects of different type. Lastly, we considered class $\mathrm{Ex}\,(k+1)\{K_{2,3}, K_4\}$ and showed that it behaves very differently.

The work explores a new subarea of asymptotic enumeration, i.e., counting graphs with few disjoint excluded minors. We saw that such classes are combinatorially tractable and have an interesting structure. We made progress on two rather general families of disjoint excluded minors, though an infinite number of unresolved important cases (i.e., no two disjoint minors K_5) remain.

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