Invariance principle for multiparameter summation processes and its applications

Vaidotas Zemlys

Vilnius University
Universite des Sciences et Technologies de Lille

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Outline

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   1. Invariance principle for i.i.d. variables in space $\mathrm{H}^o_\alpha([0,1]^d, \boldsymbol{H})$.
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3. Applications
   1. CUSUM type statistics (FCLT for panel regresion residuals).
   2. Generalisation of epidemic alternative for two-dimensional case.

What is the invariance principle?

- Convergence of partial sums process in some space of functions. Generalization of the Central Limit Theorem.
- Example of partial sums process:

$$\xi_n(t) = S_{[nt]} + (nt - [nt])X_{[nt]+1}, \quad S_k = X_1 + \cdots + X_k$$

- Invariance principle:

$$n^{-1/2}\xi_n \xrightarrow{C[0,1]} W \Leftrightarrow EX_1^2 < \infty$$

- W - Wiener process, $EW(t)W(s) = \min(t, s)$.

The change point problem

- Having a sample we want to test whether there is a point where the mean changes.
- Null and alternative hypotheses:

$$H_0 : EX_i = \mu_0$$

against

$$H_A : \exists k^* \text{ such that } EX_i = \mu_0 + (\mu_1 - \mu_0)\mathbf{1}\left(k^* < i \leq n\right)$$

Change point is known

- Simple idea: compare the means.

$$\frac{1}{k^*}S_{k^*}-\frac{1}{n-k^*}(S_n-S_{k^*})=\frac{n}{k^*(n-k^*)}\left(S_{k^*}-\frac{k^*}{n}S_n\right)$$

- Denote

$$R=(S_{k^*}-\frac{k^*}{n}S_n)$$

- Assume $k^*/n\to c$. Then under null due to CLT we have

$$n^{-1/2}R\to N(0,\sigma^2c(1-c))$$

- Under alternative we have

$$n^{-1/2}R=n^{1/2}\frac{k^*}{n}\left(1-\frac{k^*}{n}\right)(\mu_1-\mu_0)+O_P(1)\to\infty$$

Change point is unknown

- If the change point is unknown, "look" for it:

$$Q = \max_{1<k<n} |S_k - \frac{k}{n} S_n|$$

- Remember $\xi_n = S_{[nt]} + (nt - [nt])X_{[nt]+1}$:

$$Q = \max_{1<k<n} |\xi_n(k/n) - k/n\xi_n(1)|$$

- We have that Q is actually a functional of ξ_n.

$$Q = f(\xi_n), \text{ where } f(x) = \sup_{0<t<1} |x(t) - x(1)|$$

- Functional $f : C[0,1] \to R$ is continuous in $C([0,1])$.
- Invariance principle and continuous mapping gives us:

$$n^{-1/2} Q \xrightarrow{D} \sup_{0<t<1} |W(t) - tW(1)|$$

Epidemic alternatives

- We want to test epidemic: the mean changes then reverts to previous value.
- Alternative hypothesis: $EX_i = \mu_0 + (\mu_1 - \mu_0)\mathbf{1}\left(k^* < i \leq m^*\right)$
- The same argumentation gives test statistic:

$$UI(n,\alpha) = \max_{1\leq i<j\leq n} \frac{|S_j - S_i - (j-i)/nS_n|}{[(j-i)/n]^\alpha}$$

- Division by $[(j-i)/n]^\alpha$ let us detect shorter epidemics.

Epidemic alternatives II

- Shortness of detected epidemic depends on α.
- Under alternative

$$n^{-1/2} UI(n,\alpha) \geq \frac{l^{*(1-\alpha)}}{n^{1/2-\alpha}} \left(1 - \frac{l^*}{n}\right) |\mu_1 - \mu_0| + O_P(1)$$

- If $l^* = n^\gamma$ then

$$n^{-1/2} UI(n,\alpha) \to \infty \text{ for } \gamma > \frac{1/2-\alpha}{1-\alpha}$$

Hölder space

- Problem:

$$f(x) = \sup_{0<s<t<1} \frac{|x(t) - x(s) - (t-s)x(1)|}{|t-s|^{\alpha}}$$

f is not continous in $C[0,1]$!

- Functional $f(x)$ is continuous in Hölder space:

$$\mathrm{H}^{o}_{\alpha}([0,1]) = \{x \in C[0,1] : |x(t+h) - x(t)| = o(h^{\alpha})\}.$$

- If $n^{-1/2}\xi_n \to W$ in Hölder space then

$$n^{-1/2} UI(n,\alpha) \to \sup_{0<s<t<1} \frac{|W(t) - W(s) - (t-s)W(1)|}{[t-s]^{\alpha}}$$

- Note $0 < \alpha < 1/2$, since W “lives only” in $\mathrm{H}^{o}_{\alpha}([0,1])$ for $\alpha < 1/2$.

Short conclusion

- Invariance principle gives us easy way to get limiting distributions for tests for the change point problems.
- For statistical applications it makes sense to investigate invariance principle in different spaces of functions.

Panel data

- Usual situation: we observe some random variable at given time points (price of stock), or we observe group of variables at certain given time (Inflation rate of EU countries in 2007). In both cases our observation have one index.
- Panel data - observations of group of random variables at certain time points. Yearly inflation rate history of EU countries. Observed variables have two indexes: X_{it}, i - individuals, t - time.
- We have double index, hence the need to use invariance principle for multi-indexed processes.

Thesis goals

- Investigate invariance principle for multiparameter processes
- Investigate change point problems in panel data.

General summation process

- Double indexed sum:

$$S_{nm} = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{ij}$$

- In one-dimensional setting we have

$$\xi_n(i/n) = S_i$$

- In two-dimensional setting then put

$$\xi_{nm}(i/n, j/m) = S_{ij}$$

General summation process II

- For $\boldsymbol{j}, \boldsymbol{n} \in \mathbb{N}^d$ write $\boldsymbol{j} \le \boldsymbol{n}$ iff $j_i \le n_i$, $i = 1, .., d$.
- Continuous version, $\boldsymbol{t} \in [0,1]^d$

$$\xi_{\boldsymbol{n}}(\boldsymbol{t}) = \sum_{\boldsymbol{j} \le \boldsymbol{n}} |R_{\boldsymbol{n},\boldsymbol{j}}|^{-1} |R_{\boldsymbol{n},\boldsymbol{j}} \cap [0, \boldsymbol{t}]| X_{\boldsymbol{j}},$$

where

$$R_{\boldsymbol{n},\boldsymbol{j}} := [(j_1 - 1)/n_1, j_1/n_1] \times \cdots \times [(j_d - 1)/n_d, j_d/n_d].$$

and

$$[0, \boldsymbol{t}] = [0, t_1] \times \cdots \times [0, t_d]$$

- Note $\xi_{\boldsymbol{n}}(\boldsymbol{j}/\boldsymbol{n}) = S_{\boldsymbol{j}}$

Invariance principle

- Limiting process Wiener sheet:
  $EW(\boldsymbol{t})W(\boldsymbol{s}) = \min(t_1, s_1) \dots \min(t_d, s_d)$
- When

$$(n_1 \dots n_d)^{-1/2} \xi_{\boldsymbol{n}} \to W?$$

Known results

- Bass (1985), Alexander, Pyke (1986), space of continuous functions. Necessary and sufficient condition $EX_1^2 < \infty$. Not multiindex.
- Erickson (1981), Hölder function space. The FCLT holds in $H_\alpha([0,1]^d)$, if $E|X_{\mathbf{1}}|^q < \infty$, for $q > d/(1/2 - \alpha)$.
- Lots of results for $\boldsymbol{n} = (n, \dots, n)$ and different dependence assumptions

I. i. d. random variables

- Računauskas, Suquet, Zemlys (2007)

$$(n_1 \dots n_d)^{-1/2} \xi_{\boldsymbol{n}} \xrightarrow[m(\boldsymbol{n}) \to \infty]{\mathcal{D}} W \text{ in the space } H_\alpha^o([0,1]^d),$$

if and only if

$$n_1 \dots n_d \mathbf{P}\left(|X_{\mathbf{1}}| > n_1^{1/p} n_2^{1/2} \dots n_d^{1/2}\right) \to 0, \text{ when } m(\boldsymbol{n}) \to \infty,$$

where $p = 1/(1/2 - \alpha)$.

Improvements

- Also proved for case when $X_{\mathbf{j}}$ are random elements from Hilbert space $\mathbb{H}$.
- Improves on Erickson's result, weakens moment assumption.
- For $d = 1$ moment condition is $\lim_{t\to 0} t^p P(|X_1| > t) = 0$. Proven by Račkauskas and Suquet.
- For $d > 1$ in general case $\sup_{t>0} t^p P(|X_1| > t) < \infty$, but if we assume $\boldsymbol{n} = (n, \dots, n)$:

$$\lim_{t\to\infty} t^{\frac{2d}{d-2\alpha}} P(|X_{\mathbf{1}}| > t) = 0$$

Series schema: definitions

- $(X_{\boldsymbol{n},\boldsymbol{k}},\ \mathbf{1} \le \boldsymbol{j} \le \boldsymbol{k_n}),\ \boldsymbol{n} \in \mathbb{N}^d$
-
$$S_{\boldsymbol{n}}(\boldsymbol{k}) = \sum_{\boldsymbol{j} \le \boldsymbol{k}} X_{\boldsymbol{n},\boldsymbol{j}}, \quad b_{\boldsymbol{n}}(\boldsymbol{k}) = \sum_{\boldsymbol{j} \le \boldsymbol{k}} \sigma^2_{\boldsymbol{n},\boldsymbol{k}}$$
- $b_i(k) = b_{\boldsymbol{n}}(k^1_{\boldsymbol{n}}, \dots, k^{i-1}_{\boldsymbol{n}}, k, k^{i+1}_{\boldsymbol{n}}, \dots, k^d_{\boldsymbol{n}})$
- "Adaptive" grid:
$$Q_{\boldsymbol{n},\boldsymbol{j}} := \Big[b_1(j_1 - 1),\ b_1(j_1)\Big) \times \cdots \times \Big[b_d(j_d - 1),\ b_d(j_d)\Big)$$
- Summation process
$$\xi_{\boldsymbol{n}}(\boldsymbol{t}) = \sum_{\mathbf{1} \le \boldsymbol{j} \le \boldsymbol{n}} |Q_{\boldsymbol{n},\boldsymbol{j}}|^{-1} |Q_{\boldsymbol{n},\boldsymbol{j}} \cap [0, \boldsymbol{t}]| X_{\boldsymbol{j}}$$

Series schema results:

- If

$$\mu_{\boldsymbol{n}}(\boldsymbol{t}) = \sum_{\boldsymbol{1} \leq \boldsymbol{k} \leq \boldsymbol{k_n}} \mathbf{1}\left(\boldsymbol{B}(\boldsymbol{k}) \in [0, \boldsymbol{t}]\right) \sigma^2_{\boldsymbol{n},\boldsymbol{k}} \to \mu(\boldsymbol{t})$$

- Then (with moment conditions), Zemlys (2007):

$$\xi_{\boldsymbol{n}} \xrightarrow{D} G, \text{ in } \mathrm{H}^o_\alpha([0,1]^d)$$

where $\{G(\boldsymbol{t}), \boldsymbol{t} \in [0,1]^d\}$ - Gaussian with

$$EG(\boldsymbol{t})G(\boldsymbol{s}) = \mu(\min(t_1, s_1), \ldots, \min(t_d, s_d))$$

Time series regression

- Model:

$$y_t = \mathbf{x}_t \boldsymbol{\beta} + u_t, \quad t = 1, \ldots, T \tag{1}$$

- Regression residuals

$$\widehat{u}_t = y_t - \mathbf{x}_t \widehat{\boldsymbol{\beta}} = u_t - \mathbf{x}_t(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})$$

  where $\widehat{\boldsymbol{\beta}}$ is least squares solution of (1).
- Summation process $\xi_T(z) = T^{-1/2} \sum_{t=1}^{[Tz]} \widehat{u}_t$
- Ploberger, Kramer (1992):

$$T^{-1/2} \xi_T \to W(z) - zW(1)$$

CUSUM test

- Null hypothesis: $\boldsymbol{\beta}_t = \boldsymbol{\beta}_0$.
- Alternative hypothesis: $\boldsymbol{\beta}_t$ depends on t.
- Under null

$$\sup_{0\leq z\leq 1} |\xi_T(z)| \to \sup_{0\leq z\leq 1} |W(z) - zW(1)|.$$

Panel regression

- Ordinary least squares

$$y_{it} = \mathbf{x}_{it}\boldsymbol{\beta} + u_{it}, \quad i = 1, \dots, n; t = 1, \dots, T$$

- Fixed effects

$$y_{it} = \mu_i + \mathbf{x}_{it}\boldsymbol{\beta} + u_{it}, \quad i = 1, \dots, n; t = 1, \dots, T$$

Invariance principle

If invariance principle holds for u_{it}:

$$(nT)^{-1/2}\xi_{nT} \xrightarrow{D} W$$

then

- Ordinary least squares

$$(nT)^{-1/2}\xi_{nT}^{(OLS)}(u,v) \to W(u,v) - uvW(1,1)$$

- Fixed effects

$$(nT)^{-1/2}\xi_{nT}^{(FE)}(u,v) \to W(u,v) - vW(u,1)$$

Local alternatives

Suppose

$$\boldsymbol{\beta}_{ij} = \boldsymbol{\beta} + \frac{1}{\sqrt{nm}} \boldsymbol{g}\left(\frac{i}{n}, \frac{j}{m}\right)$$

- Ordinary least squares

$$(nT)^{-1/2} \xi_{nT}^{(OLS)}(t,s) \xrightarrow{D} W(t,s) - tsW(1,1) + \int_0^t \int_0^s \boldsymbol{c}' \boldsymbol{g}(u,v) du dv - ts\boldsymbol{c}' \int_0^1 \int_0^1 \boldsymbol{g}(u,v) du dv$$

- Fixed effects

$$(nT)^{-1/2} \xi_{nT}^{(FE)}(t,s) \xrightarrow{D} W(t,s) - sW(t,1) + \int_0^t \int_0^s \boldsymbol{c}' \boldsymbol{g}(u,v) du dv - s \int_0^t \int_0^1 \boldsymbol{c}' \boldsymbol{g}(u,v) du dv,$$

Definitions

- (H_0) : X_{ij} have all the same mean μ_0.
- (H_A) : There are integers $1 < a^* \le b^* < n$, $1 < c^* \le d^* < m$ and a constand $\mu_1 \ne \mu_0$ such that

$$EX_{ij} = \mu_0 + \mu_1 \mathbf{1}\left((i,j) \in [a^*, b^*] \times [c^*, d^*] \cap \mathbb{N}^2 \right)$$

- Test statistic:

$$DUI(nm, \alpha) = \max_{\substack{1 \le a < b \le n \\ 1 \le c < d \le m}} \frac{|\Delta^1_{b-a}\Delta^2_{d-c} S_{bd} - (s_b - s_a)(t_d - t_c) S_{nm}|}{\max\{s_b - s_a, t_d - t_c\}^\alpha}$$

where $s_i = i/n$, $t_j = j/m$, for $i = 1, \dots, n$, $j = 1, \dots, m$ and

$$\Delta^1_{b-a}\Delta^2_{d-c} S_{bd} = S_{bd} - S_{ad} - S_{bc} + S_{ac}$$

Null hypothesis

$$T_\alpha(x) = \sup_{\mathbf{0}\le \boldsymbol{s}<\boldsymbol{t}\le \mathbf{1}} \frac{|x(\boldsymbol{t})-x(s_1,t_2)-x(t_1,s_2)+x(\boldsymbol{s})-(t_1-s_1)(t_2-s_2)x(1,1)|}{|\boldsymbol{t}-\boldsymbol{s}|^\alpha}$$

is continuous in $\mathrm{H}_\alpha^o([0,1]^2)$ hence

$$(nm)^{-1/2} DUI(nm,\alpha) \to T_\alpha(W)$$

Alternative hypothesis

Assume under (H_A) that the X_{ij} are independent and $\sigma_0^2 = \sup_{\boldsymbol{n}} \operatorname{var}(X_{\boldsymbol{n}})$ is finite. If

$$\lim_{\boldsymbol{n}\to\infty} (nm)^{1/2} \frac{h_{nm}}{d_{nm}^{\alpha}} |\mu_1 - \mu_0| \to \infty,$$

where

$$h_{nm} = \frac{k^* l^*}{nm}\left(1 - \frac{k^* l^*}{nm}\right) \text{ and } d_{nm} = \max\left\{\frac{k^*}{n}, \frac{l^*}{m}\right\},$$

then

$$(nm)^{-1/2} DUI(nm, \alpha) \to \infty$$

Open problems

- Extend invariance principle for dependent random variables
- More general alternative hypothesis for epidemic alternative
- Improve rates for alternative hypothesis