Hölderian functional limit theorem for multi-indexed summation processes

A. Račkauskas[1,3], Ch. Suquet[2], V. Zemlys[1,2]

[1]Vilniaus University

[2]Universite des Sciences et Technologies de Lille

[3]Institute of Mathematics and Informatics

27th of June, 2006.

Polygonal line

- Usual polygonal line process

$$\xi_n(t) = S_{[nt]} + (nt - [nt])X_{[nt]+1}, \quad t \in [0,1],$$

where

$$S_k = X_1 + X_2 + \ldots + X_k, \quad k = 1, 2, \ldots$$

- Donsker-Prokhorov theorem

$$n^{-1/2}\xi_n \xrightarrow{\mathcal{D}} W \text{ in the space } C[0,1] \Leftrightarrow EX_1^2 < \infty,$$

Hölder spaces

- $H_\alpha([0,1]^d,\mathbb{R})$ – space of continuous functions $x : [0,1]^d \to \mathbb{R}$, satisfying $w_\alpha(x,1) < \infty$, where

$$w_\alpha(x,\delta) = \sup_{\boldsymbol{t},\boldsymbol{s}\in[0,1]^d, |\boldsymbol{t}-\boldsymbol{s}|<\delta} \frac{|x(\boldsymbol{t}) - x(\boldsymbol{s})|}{(|\boldsymbol{t}-\boldsymbol{s}|)^\alpha}.$$

- $H_\alpha^o([0,1]^d,\mathbb{R})$ subset of $H_\alpha([0,1]^d,\mathbb{R})$ of functions satisfying

$$\lim_{\delta\to 0} w_\alpha(x,\delta) = 0$$

- Norm

$$\|x\|_\alpha := \|x\|_\infty + w_\alpha(x,1)$$

Generalisation of D-P

$$n^{-1/2}\xi_n \xrightarrow[n\to\infty]{\mathcal{D}} W, \text{ in the space } H_\alpha^o([0,1]).$$

- Lamperti (1962) proved sufficient condition:

$$E|X_1|^p < \infty,$$

where $p > 1/(1/2-\alpha)$ and $0 < \alpha < 1/2$.

Generalisation of D-P

$$n^{-1/2}\xi_n \xrightarrow[n\to\infty]{\mathcal{D}} W, \text{ in the space } H_\alpha^o([0,1]).$$

- Lamperti (1962) proved sufficient condition:

$$E|X_1|^p < \infty,$$

  wherre $p > 1/(1/2 - \alpha)$ and $0 < \alpha < 1/2$.

- Račkauskas and Suquet (2001) proved necessary and sufficient condition:

$$\lim_{t\to\infty} t^p \mathbf{P}(|X_1| \geq t) = 0 \text{ with } p = \frac{1}{1/2 - \alpha}.$$

Multi-dimensional case

- $(X_{\boldsymbol{j}}, \boldsymbol{j} \in \mathbb{N}^d)$ – zero mean independent real random variables.
- $\boldsymbol{n} = (n_1, \dots, n_d)$,

$$\xi_{\boldsymbol{n}}(\boldsymbol{t}) = \sum_{\boldsymbol{j} \leq \boldsymbol{n}} |R_{\boldsymbol{n},\boldsymbol{j}}|^{-1} |R_{\boldsymbol{n},\boldsymbol{j}} \cap [\boldsymbol{0}, \boldsymbol{t}]| X_{\boldsymbol{j}},$$

where

$$R_{\boldsymbol{n},\boldsymbol{j}} := [(j_1 - 1)/n_1, j_1/n_1] \times \cdots \times [(j_d - 1)/n_d, j_d/n_d].$$

Alternative definitions

- Alexander and Pyke (1986)

$$Z_n(A) = \sum_{\boldsymbol{j}} |(nA) \cap (\boldsymbol{j} - 1, \boldsymbol{j}]|$$

- Erickson (1981)

$$\zeta_{\boldsymbol{n}}(A) = \sum_{\boldsymbol{j} \le \boldsymbol{n}} |R_{\boldsymbol{n},\boldsymbol{j}}|^{-1} |R_{\boldsymbol{n},\boldsymbol{j}} \cap A| X_{\boldsymbol{j}},$$

- Khoshnevisan (2002)

$$\boldsymbol{S}_n(\boldsymbol{t}) = \int_{[0,n\boldsymbol{t}]} \Xi(ds)$$

where

$$\Xi(s) = |[[\boldsymbol{s}], \boldsymbol{s}]| X_{[\boldsymbol{s}]+\mathbf{1}}$$

Functional central limit theorem

- Weak convergence of $\{\pi(\boldsymbol{n})^{-1/2}\xi_{\boldsymbol{n}}(\boldsymbol{t}), \boldsymbol{n} \in \mathbb{N}^d\}$, when $m(\boldsymbol{n}) \to \infty$, where $m(\boldsymbol{n}) := n_1 \wedge \ldots \wedge n_d$, $\pi(\boldsymbol{n}) = \prod n_i$.
- Weak convergence of $\{n^{-d/2} Z_n(A), n \in \mathbb{N}\}$, when $n \to \infty$.
- Weak convergence of $\{\pi(\boldsymbol{n})^{-1/2}\zeta_{\boldsymbol{n}}(A), \boldsymbol{n} \in \mathbb{N}^d\}$, when $m(\boldsymbol{n}) \to \infty$

Limiting process – Wiener sheet

- For $\boldsymbol{t} \in [0,1]^d$:

Definition

$W(\boldsymbol{t})$ – Gaussian with $EW(\boldsymbol{t}) = 0$ and
$EW(\boldsymbol{t})W(\boldsymbol{s}) = (t_1 \wedge s_1) \dots (t_d \wedge s_d)$.

Limiting process – Wiener sheet

- For $\boldsymbol{t} \in [0,1]^d$:

Definition

$W(\boldsymbol{t})$ – Gaussian with $EW(\boldsymbol{t}) = 0$ and
$EW(\boldsymbol{t})W(\boldsymbol{s}) = (t_1 \wedge s_1) \dots (t_d \wedge s_d)$.

- For A - Borel subset of $[0,1]^d$:

Definition

$W(A)$ – Gaussian with $EW(A) = 0$ and $EW(A)W(B) = |A \cap B|$,
for Borels $A, B \subset [0,1]^d$.

Previous results

- Bass (1985), Alexander, Pyke (1986), continuous function space. Sufficient conditions: $EX_1^2 < \infty$ and metric entropy condition on collection of sets.
- Erickson (1981), Hölder function space. Relationship of moment, metric entropy and Hölder smoothness conditions. For collection $\mathcal{Q}_d = \{[\mathbf{0}, \boldsymbol{t}], \boldsymbol{t} \in [0,1]^d\}$, the FCLT holds in $H_\alpha([0,1]^d)$, if $E|X_{\mathbf{1}}|^q < \infty$, for $q > d/(1/2 - \alpha)$.

Main result

$$\pi(\boldsymbol{n})^{-1/2}\xi_{\boldsymbol{n}} \xrightarrow[m(\boldsymbol{n})\to\infty]{\mathcal{D}} W \text{ in the space } H_\alpha^o\big([0,1]^d\big),$$

if and only if

$$n_1 \dots n_d \mathbf{P}\big(|X_{\mathbf{1}}| > n_1^{1/p} n_2^{1/2} \dots n_d^{1/2}\big) \to 0, \text{ when } m(\boldsymbol{n}) \to \infty,$$

where $p = 1/(1/2 - \alpha)$.

Equivalent condition

Conditions

$$n_1 \dots n_d \mathbf{P}\left(|X_{\mathbf{1}}| > n_1^{1/p} n_2^{1/2} \dots n_d^{1/2}\right) \to 0, \text{ when } m(\boldsymbol{n}) \to \infty.$$

and

$$\sup_{t>0} t^p \mathbf{P}(|X_{\mathbf{1}}| > t) < \infty$$

when $d > 1$ are equivalent.

Remarks

Erickson's condition: $E|X_1|^q < \infty$, when $q > d/(1/2 - \alpha)$.
New condition:
$n_1 \dots n_d \mathbf{P}\left(|X_\mathbf{1}| > n_1^{1/p} n_2^{1/2} \dots n_2^{1/2}\right) \to 0$, when $m(\boldsymbol{n}) \to \infty$.

- The result of Erickson is improved for collection $\mathcal{Q}_d = \{[\mathbf{0}, \boldsymbol{t}], \boldsymbol{t} \in [0,1]^2\}$. You "need" d moments less.

$$\sup_{t>0} t^p \mathbf{P}(|X_\mathbf{1}| \geq t) < \infty.$$

- If path is fixed: $\boldsymbol{n} = (n, \dots, n)$ you need less moments than in one dimensional case

$$\lim_{t\to\infty} t^{\frac{d}{d/2-\alpha}} \mathbf{P}(|X_\mathbf{1}| \geq t) = 0.$$