Functional limit theorem for double-indexed summation processes

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Polygonal line

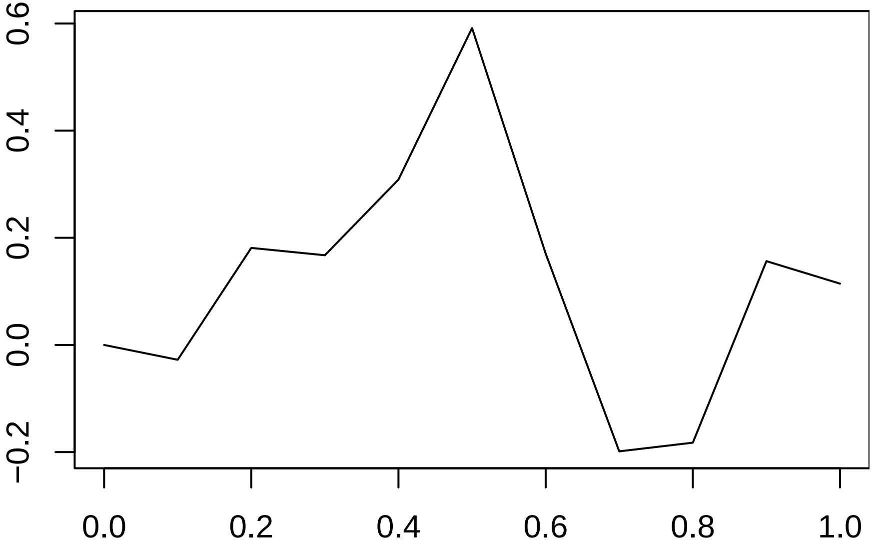
Usual polygonal line process

$$\xi_n(t) = S_{[nt]} + (nt - [nt])X_{[nt]+1}, \quad t \in [0, 1],$$

where

$$S_k = X_1 + X_2 + \ldots + X_k, \quad k = 1, 2, \ldots$$

Polygonal line



Donsker-Prokhorov theorem

$$n^{-1/2}\xi_n \xrightarrow{\mathcal{D}} W \text{ in the space } C[0,1] \Leftrightarrow EX_1^2 < \infty,$$

where $W(t)$ - Wiener process or Brownian motion.

- $W_0 = 0$.
- $W_t \sim N(0,t)$.
- $EW_tW_s = t \wedge s$.

Hölder spaces

- $H_\rho([0,1]^d,\mathbb{R})$ – space of continuous functions $x:[0,1]^d\to\mathbb{R}$, satisfying $w_\rho(x,1)<\infty$, where

$$w_\rho(x,\delta)=\sup_{\boldsymbol{t},\boldsymbol{s}\in[0,1]^d,|\boldsymbol{t}-\boldsymbol{s}|<\delta}\frac{|x(\boldsymbol{t})-x(\boldsymbol{s})|}{\rho(|\boldsymbol{t}-\boldsymbol{s}|)}.$$

- $H_\rho^o([0,1]^d,\mathbb{R})$ subset of $H_\rho([0,1]^d,\mathbb{R})$ of functions satisfying

$$\lim_{\delta\to 0} w_\rho(x,\delta)=0$$

- Norm

$$\|x\|_\rho := \|x\|_\infty + w_\rho(x,1)$$

Generalisation of D-P

$$n^{-1/2}\xi_n \xrightarrow[n\to\infty]{\mathcal{D}} W, \text{ in the space } H_\rho^o([0,1]),\ \rho(h) = h^\alpha$$

- Lamperti (1962) proved sufficient condition:

$$E|X_1|^p < \infty,$$

where $p > 1/(1/2 - \alpha)$ and $0 < \alpha < 1/2$.

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  wherre $p > 1/(1/2 - \alpha)$ and $0 < \alpha < 1/2$.

- Račkauskas and Suquet (2001) proved necessary and sufficient condition:

$$\lim_{t\to\infty} t^p \mathbf{P}(|X_1| \geq t) = 0 \text{ with } p = \frac{1}{1/2 - \alpha}.$$

Multi-dimensional case

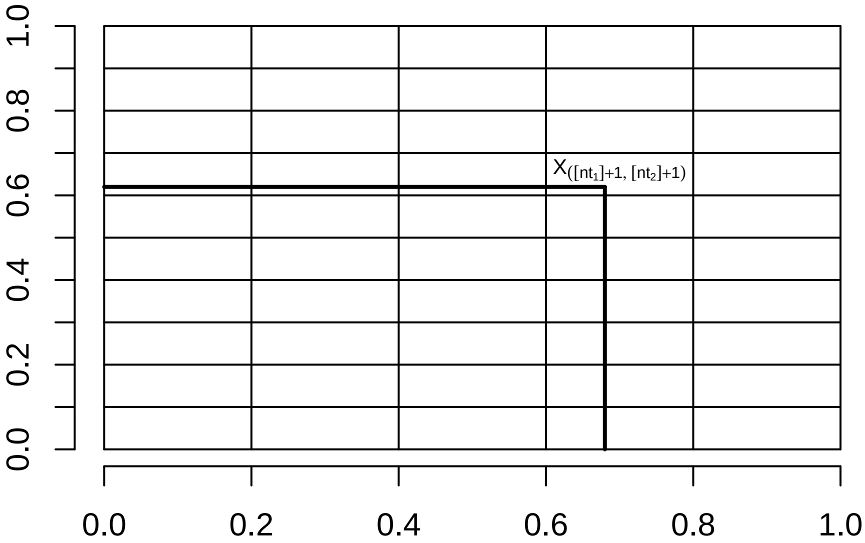
- $(X_{\boldsymbol{j}}, \boldsymbol{j} \in \mathbb{N}^d)$ – zero mean independent real random variables.
- $\boldsymbol{n} = (n_1, \ldots, n_d)$, A - Borel subset of $[0,1]^d$

$$\xi_{\boldsymbol{n}}(A) = \sum_{\boldsymbol{j} \le \boldsymbol{n}} |R_{\boldsymbol{n},\boldsymbol{j}}|^{-1} |R_{\boldsymbol{n},\boldsymbol{j}} \cap A| X_{\boldsymbol{j}},$$

where

$$R_{\boldsymbol{n},\boldsymbol{j}} := [(j_1 - 1)/n_1, j_1/ni_1] \times \cdots \times [(j_d - 1)/n_d, j_d/n_d].$$

Two dimensional case $A = [0, t_1] \times [0, t_2]$



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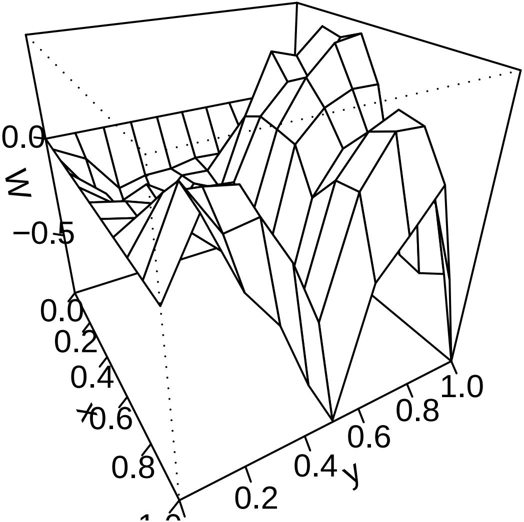

Functional central limit theorem

- Conditions for

$$(n_1 \cdots n_d)^{-1/2} \xi_{\boldsymbol{n}} \xrightarrow{\mathcal{D}} W, \text{ when } m(\boldsymbol{n}) \to \infty.$$

to take place in apropriate space. $m(\boldsymbol{n}) := n_1 \wedge \ldots \wedge n_d$.

- W – Wiener sheet. $W(A)$ – Gaussian with $EW(A) = 0$ and $EW(A)W(B) = |A \cap B|$, for Borels $A, B \subset [0,1]^d$.

Continuous function space

- Bass (1985), Alexander, Pyke (1986), sufficient conditions $EX_1^2 < \infty$ and

$$\int_0^1 \sqrt{\log(N_I(\varepsilon, \mathcal{A}, \delta))} d\varepsilon < \infty$$

  where $N_I(\varepsilon, \mathcal{A}, \delta)$ - metric entropy with inclusion.

Hölder space

- Erickson (1981). Let $E|X_1|^p < \infty$, $r_1(\mathcal{A}) \leq r$, and $q := p/2 - r > 0$.
  Then FCLT in $H_\rho(\mathcal{A}, \mathbb{R})$, with norm $\|x\|_f$, $\rho = o(f)$ holds for $\rho(h)^p = h^q |\log h| |\log\log h|^\beta$, for each $\beta > 1$.

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For $\mathcal{A} := \{[0, t_1] \times \cdots \times [0, t_d]\}$, FCLT holds in space $H^o_\alpha([0,1]^d)$ if $E|X|^q < \infty$, for $q > \frac{d}{1/2-\alpha}$, $0 < \alpha < 1/2$.

Main result

$$(n_1 n_2)^{-1/2}\xi_{\boldsymbol{n}} \xrightarrow[n_1\wedge n_2\to\infty]{\mathcal{D}} W \text{ in the space } H_\alpha^o\big([0,1]^2\big),$$

if and only if

$$n_1 n_2 \mathbf{P}\big(|X_{\mathbf{1}}| > n_1^{1/p} n_2^{1/2}\big) \to 0, \text{ when } n_1 \wedge n_2 \to \infty,$$

where $p = 1/(1/2 - \alpha)$.

Remarks

Erickson's condition: $E|X_1|^q < \infty$, when $q > 2/(1/2 - \alpha)$.
New condition: $t_1 t_2 \mathbf{P}\big(|X_{\mathbf{1}}| > t_1^{1/p} t_2^{1/2}\big) \to 0$, when $t_1 \wedge t_2 \to \infty$.

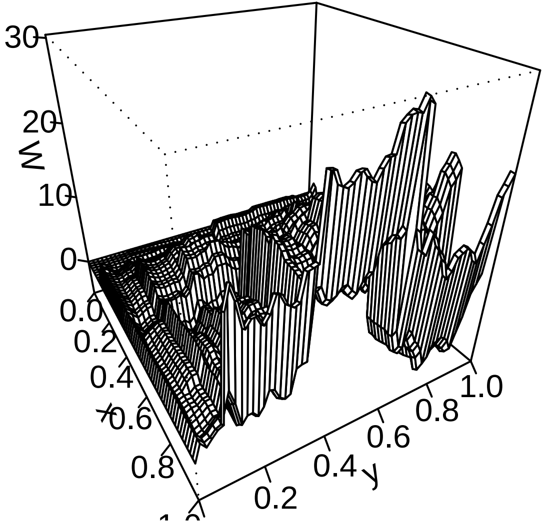
- The result of Erickson is improved for two-dimensional case. You "need" twice as less moments.

$$\lim_{t\to\infty} t^p \mathbf{P}(|X_{\mathbf{1}}| \geq t) = 0.$$

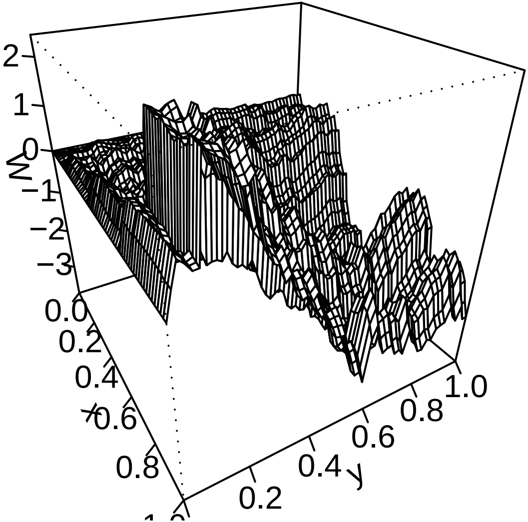
- If path is fixed: $\boldsymbol{n} = (n, n)$ you need less moments than in one dimensional case

$$\lim_{t\to\infty} t^{\frac{2}{1-\alpha}} \mathbf{P}(|X_{\mathbf{1}}| \geq t) = 0.$$

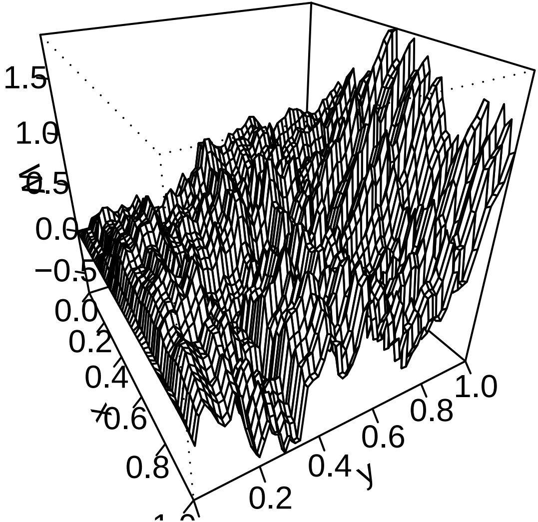
The simulation with one moment



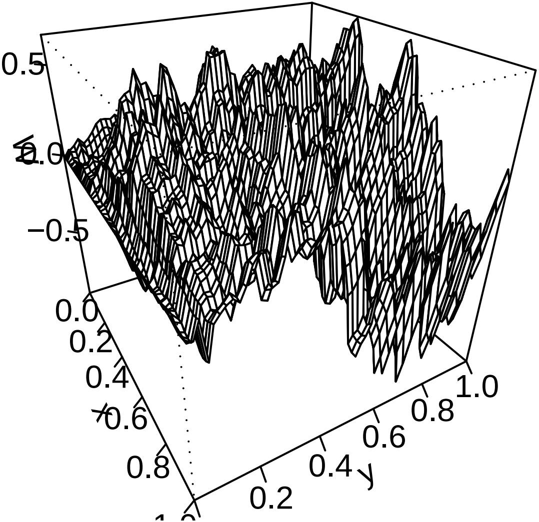
The simulation with two moments



The simulation with three moments



The simulation with all moments



Details of the proof

- $(\xi_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^2)$ is a net with directed set $\mathbb{N}^2$ and relation $\boldsymbol{j} \leq \boldsymbol{n} \Leftrightarrow j_i \leq n_i,\ i = 1, \ldots, d$.

Details of the proof

- $(\xi_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^2)$ is a net with directed set $\mathbb{N}^2$ and relation $\boldsymbol{j} \leq \boldsymbol{n} \Leftrightarrow j_i \leq n_i,\ i = 1, \ldots, d$.
- The finite dimensional distributions of process $(n_1 n_2)^{-1/2}\xi_{\boldsymbol{n}}(\boldsymbol{t})$ converge to those of Wiener sheet, when $n_1 \wedge n_2 \to \infty$. (Erickson (1981))

Convergence criteria

Let $\{\zeta_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^2\}$ be net of $H_\alpha^o([0,1]^2)$ random elements satisfying:

1. Finite-dimensional distributions of $\zeta_{\boldsymbol{n}}$ converges weakly to some $H_\alpha^o([0,1]^d)$ random element ζ finite-dimensional distributions, when $n_1 \wedge n_2 \to \infty$.
2. $\lim_{a\to\infty} \sup_{\boldsymbol{n}} P(\sup_{\boldsymbol{t}\in T} |\zeta_{\boldsymbol{n}}(\boldsymbol{t})| > a) = 0$
3. For each $\varepsilon > 0$

$$\lim_{J\to\infty} \limsup_{\boldsymbol{n}\to\infty} P(\sup_{j\geq J} 2^{\alpha j} \max_{\boldsymbol{v}\in V_j} |\lambda_{j,\boldsymbol{v}}(\zeta_{\boldsymbol{n}})| > \varepsilon) = 0.$$

Definitions

For each $x \in H_\alpha^o([0,1]^d)$:

$$\lambda_{0,\boldsymbol{v}}(x) = x(\boldsymbol{v}), \quad \boldsymbol{v} \in V_0,$$
$$\lambda_{j,\boldsymbol{v}}(x) = x(\boldsymbol{v}) - \frac{1}{2}(x(\boldsymbol{v}^-) + x(\boldsymbol{v}^+)), \quad \boldsymbol{v} \in V_j, \ \ j \geq 1,$$

where $V_j = W_j \setminus W_{j-1}$, $W_j = \{k2^{-j}; 0 \leq k \leq 2^j\}^2$.

$$v_i^{\pm} = \begin{cases} v_i \pm 2^j, & \text{for noneven } k_i; \\ v_i, & \text{for even } k_i. \end{cases}$$

Prokhorov theorem

Let $(X_\alpha, \alpha \in I)$ be a net of separable space random elements. If (X_α) is asymptoticaly tight, i.e. for each $\varepsilon > 0$ there exists a compact K_ε such that

$$\liminf_{\alpha} P(X_\alpha \in K_\varepsilon) > 1 - \varepsilon,$$

then net (X_α) is relatively compact, (1.3.9 theorem, p. 21 psl. van der Vaart ir Wellner (1996))

Arzela-Ascoli theorem

The subset A of H_α^o is relatively compact if and only if

$$\sup_{x \in A} \sup_{t \in T} |x(t)| < \infty$$

and

$$\lim_{J \to \infty} \sup_{x \in A} \sup_{j \geq J} \max_{v \in V_j} 2^{\alpha j} |\lambda_{j,v}(x)| = 0.$$

Necessity

We have

$$X_{\boldsymbol{k}} = (S_{(k_1,k_2)} - S_{(k_1-1,k_2)}) - (S_{(k_1,k_2-1)} - S_{(k_1-1,k_2-1)}),$$

thus

$$\begin{aligned} &P(n_1^{-1/p} n_2^{-1/2} \max_{\boldsymbol{1} \le \boldsymbol{k} \le \boldsymbol{n}} |X_{\boldsymbol{k}}| > t) \\ &\le P\Big(2(n_1 n_2)^{1/2} \max_{|\frac{\boldsymbol{k}-\boldsymbol{l}}{\boldsymbol{n}}| = |\frac{\boldsymbol{1}}{\boldsymbol{n}}|} \frac{|S_{\boldsymbol{k}} - S_{\boldsymbol{l}}|}{|(\boldsymbol{k}-\boldsymbol{l})/\boldsymbol{n}|^{\alpha}} > t\Big) \\ &\le P(w_\alpha((n_1 n_2)^{1/2} \xi_{\boldsymbol{n}}, \delta) > t/2) \end{aligned}$$

Sufficiency

Because of the symmetry it suffices to check that for every $\varepsilon > 0$

$$\lim_{J\to\infty} \limsup_{\boldsymbol{n}\to\infty} P\left(\sup_{j\geq J} 2^{\alpha j} (n_1 n_2)^{-1/2} \max_{\substack{0\leq k\leq 2^j-1 \\ 0\leq \ell\leq 2^j}} \Delta_{\boldsymbol{n}}(t_{k+1}, t_k; s_\ell) > \varepsilon \right) = 0,$$

where $t_k = k2^{-j}$ and $s_\ell = \ell 2^{-j}$.

Sufficiency

For difference $\Delta_{\boldsymbol{n}}(t, t'; s) = |\xi_{(n_1,n_2)}(t', s) - \xi_{(n_1,n_2)}(t, s)|$, we have

$$\sup_{s\in[0,1]} \Delta_{\boldsymbol{n}}(t, t'; s) \leq 3\chi\{t' - t \geq 1/n_1\}\psi_{\boldsymbol{n}}(t', t) + 6 \min\left\{1, n_1(t' - t)\right\}\zeta_{n_1,n_2},$$

where

$$\psi_{\boldsymbol{n}}(t', t) = \max_{1\leq k\leq n_2} \Big| \sum_{i=[n_1 t]+1}^{[n_1 t']} S^{(1)}_{(i,\ k)}\Big|,$$

$$\zeta_{n_1,n_2} = \max_{1\leq i\leq n_1} \max_{1\leq k\leq n_2} |S^{(1)}_{(i,\ k)}|.$$