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Invariance principle for multiparameter summation processes and applications

Doctoral dissertation

Physical sciences, Mathematics (01 P)

Vilnius, 2008

The scientific work was carried out during 2004-2008 at Vilnius University and Université des Sciences et Technologies de Lille

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Invariantiškumo principas daugiamačio indekso sumavimo procesams ir taikymai

Daktaro disertacija

Fiziniai mokslai, matematika (01 P)

Vilnius, 2008

Disertacija rengta 2004-2008 metais Vilniaus universitete bei Lilio universitete, Prancūzijoje.

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Notations

$\boldsymbol{t}$ denotes a real vector $(t_1, \ldots, t_d)$.

$\mathbb{R}^d$ denotes the set of real vectors $\boldsymbol{t}$.

$\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N} = \{1, 2, \ldots\}$.

$\mathbb{Z}$ denotes the set of integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

$\mathbb{H}$ denotes Hilbert space.

$\mathbf{0}$ denotes element $(0, \ldots, 0)$ from the space $\mathbb{R}^d$.

$\mathbf{1}$ denotes element $(1, \ldots, 1)$ from the space $\mathbb{R}^d$.

$\boldsymbol{t}_{k:l}$ denotes the "subvector" $(t_k, t_{k+1}, \ldots, t_l)$.

$\boldsymbol{t}_{-k}$ denotes the "subvector" $(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_d)$.

$\boldsymbol{t}_K$ denotes the subvector $(t_{k_1}, \ldots, t_{k_l})$ with $K = \{k_1, \ldots, k_l\}$ and $1 \le k_1 < k_2 < \cdots < k_l \le d$.

$\boldsymbol{s} \le \boldsymbol{t}$ means $s_k \le t_k$ for all $k = 1, \ldots, d$.

$|\boldsymbol{t}|$ denotes $\max(|t_1|, \ldots, |t_d|)$ for $\boldsymbol{t} \in \mathbb{R}^d$.

$|A|$ denotes Lebesgue measure for the set $A \subset \mathbb{R}^d$.

$\operatorname{card} A$ denotes the cardinality of the set $A \subset \boldsymbol{R}^d$.

$\boldsymbol{st}$ denotes $(s_1 t_1, \ldots, s_d t_d)$, $\boldsymbol{s}/\boldsymbol{t}$ denotes $(s_1/t_1, \ldots, s_d/t_d)$.

$\boldsymbol{\pi}(\boldsymbol{n})$ denotes $n_1 \cdots n_d$ for $\boldsymbol{n} \in \mathbb{N}^d$.

$\mathrm{m}(\boldsymbol{t})$ denotes $\min(t_1, \ldots, t_d)$.

$[x]$ denotes integer part of real number x.

$\{x\}$ denotes the fractional part of real number x, ($\{x\} = x - [x]$).

$[\boldsymbol{t}]$ denotes $([t_1], \dots, [t_d])$, respectively $\{\boldsymbol{t}\}$ denotes $(\{t_1\}, \dots, \{t_d\})$.

$x \wedge y$ denotes $\min(x, y)$ for real numbers x and y.

$x \vee y$ denotes $\max(x, y)$ for real numbers x and y.

$\boldsymbol{t} \wedge \boldsymbol{s}$ denotes $(t_1 \wedge s_1, \dots, t_d \wedge s_d)$, respectively $\boldsymbol{t} \vee \boldsymbol{s}$ denotes $(t_1 \vee s_1, \dots, t_d \vee s_d)$.

$\| \cdot \|$ denotes the norm of Hilbert space $\mathbb{H}$.

$\langle \cdot, \cdot \rangle$ denotes the scalar product of Hilbert space $\mathbb{H}$.

$C([0,1]^d)$ denotes the set of continuous functions $x : [0,1]^d \to \mathbb{R}$.

$\mathrm{H}_\alpha^o([0,1]^d)$ denotes the set of continuous functions $x : [0,1]^d \to \mathbb{R}$ satisfying $\lim_{\delta \to 0} \sup_{|\boldsymbol{t}-\boldsymbol{s}|<\delta} |x(\boldsymbol{t}) - x(\boldsymbol{s})|/|\boldsymbol{t} - \boldsymbol{s}|^\alpha = 0$.

$\mathrm{H}_\alpha^o(\mathbb{H})$ denotes the set of continuous functions $x : [0,1]^d \to \mathbb{H}$ satisfying $\lim_{\delta \to 0} \sup_{|\boldsymbol{t}-\boldsymbol{s}|<\delta} \|x(\boldsymbol{t}) - x(\boldsymbol{s})\|/|\boldsymbol{t} - \boldsymbol{s}|^\alpha = 0$.

$X_\alpha = Y_\alpha + o_P(1)$ iff $\|X_\alpha - Y_\alpha\| \to 0$ in probability.

$\xrightarrow{D}$ denotes weak convergence in the space specified afterwards. If no space is specified it is assumed that weak convergence takes place in $\mathbb{R}$.

For the net $\{X_\alpha\}$ of Banach space valued random variables

$\mathbf{1}(\cdot)$ denotes the indicator function.

$\Delta_s^{(m)}$ denotes the difference operator acting on m-th coordinate, $\Delta_s^{(1)} x(\boldsymbol{t}) = x(\boldsymbol{t}) - x((t_1 - s, t_2, \dots, t_d))$.

ξ_n denotes polygonal summation process.

$\xi_{\boldsymbol{n}}$ denotes continuous multi-parameter summation process.

Introduction

Convergence of stochastic processes to some Brownian motion or related process is an important topic in probability theory and mathematical statistics. The first functional central limit theorem by Donsker and Prokhorov states the $C[0,1]$-weak convergence of $n^{-1/2}\xi_n$ to the standard Brownian motion W. Here ξ_n denotes the random polygonal line process indexed by $[0,1]$:

$$\xi_n(t) = S_{[nt]} + (nt - [nt])X_{[nt]+1}, t \in [0,1],$$

where S_0, $S_k := X_1 + \cdots + X_k$, $k \geq 1$, are the partial sums of a sequence $(X_i)_{i\geq 1}$ of i.i.d. random variables such that $\mathbf{E}\, X_1 = 0$ and $\mathbf{E}\, X_1^2 = 1$. (We say that sequence of random elements Y_n with values in separable metric space B converges weakly to random element Y, if $\mathbf{E}\, f(Y_n) \to \mathbf{E}\, f(Y)$, for every continuous bounded functional f).

This theorem implies via continuous maping the convergence in distribution of $f(n^{-1/2}\xi_n)$ to $f(W)$ for any continuous functional $f : \mathrm{C}[0,1] \to \mathbb{R}$. Clearly this provides many statistical applications. On the other hand, considering that the paths of ξ_n are piecewise linear and that W has roughly speaking, an α-Hölder regularity for any exponent $\alpha < 1/2$, it is tempting to look for a stronger topological framework for the weak convergence of $n^{-1/2}\xi_n$ to W. In addition to the satisfaction of mathematical curiosity, the practical interest of such an investigation is to obtain a richer set of continuous functionals of the paths. For instance, Hölder norms of ξ_n are closely related to some test statistics to detect short "epidemic" changes in the distribution of the X_i's, see [30, 32].

In 1962, Lamperti [19] obtained the first functional central limit theorem in the separable Banach spaces H_α^o, $0 < \alpha < 1/2$, of functions $x : [0,1] \to \mathbb{R}$ such that

$$\|x\|_\alpha := |x(0)| + w_\alpha(x,1) < \infty,$$

with

$$w_\alpha(x,\delta) := \sup_{0<|t-s|\leq\delta} \frac{|x(t) - x(s)|}{|t-s|^\alpha} \xrightarrow[\delta\to 0]{} 0.$$

More precise definitions are given in the section 1.2.

Assuming that $\mathbf{E}\,|X_1|^q < \infty$ for some $q > 2$, he proved the weak convergence of $n^{-1/2}\xi_n$ to W in the Hölder space H_α^o for any $\alpha < 1/2 - 1/q$. Račkauskas and Suquet in [29] (see also [27]) obtained a necessary and sufficient condition for the Lamperti's functional central limit theorem. Namely for $0 < \alpha < 1/2$, $n^{-1/2}\xi_n$ converges weakly in H_α^o to W if and only if

$$\lim_{t\to\infty} t^{p(\alpha)} P(|X_1| > t) = 0, \tag{1}$$

where

$$p(\alpha) := \frac{1}{\frac{1}{2} - \alpha}. \tag{2}$$

Further extensions of Donsker-Prokhorov's functional central limit theorem concern summation processes. Let $|A|$ denote the Lebesgue measure of the Borel subset A of $\mathbb{R}^d$. For a collection $\mathcal{A}$ of Borel subsets of $[0,1]^d$, summation process $\{\xi_{\boldsymbol{n}}(A);\ A \in \mathcal{A}\}$ based on a random field $\{X_{\boldsymbol{j}},\, \boldsymbol{j} \in \mathbb{N}^d\}$, of independent identically distributed real random variables with zero mean is defined by

$$\xi_{\boldsymbol{n}}(A) = \sum_{\boldsymbol{1}\le \boldsymbol{j} \le \boldsymbol{n}} |R_{\boldsymbol{n},\boldsymbol{j}}|^{-1} |R_{\boldsymbol{n},\boldsymbol{j}} \cap A| X_{\boldsymbol{j}},$$

where $\boldsymbol{j} = (j_1, \dots, j_d)$, $\boldsymbol{n} = (n_1, \dots, n_d)$, $R_{\boldsymbol{n},\boldsymbol{j}}$ is the "rectangle"

$$R_{\boldsymbol{n},\boldsymbol{j}} := \left[\frac{j_1 - 1}{n_1}, \frac{j_1}{n_1}\right) \times \cdots \times \left[\frac{j_d - 1}{n_d}, \frac{j_d}{n_d}\right)$$

and the indexation condition "$\boldsymbol{1} \le \boldsymbol{j} \le \boldsymbol{n}$" is understood componentwise : $1 \le j_1 \le n_1, \dots, 1 \le j_d \le n_d$. Of special interest are the partial sum processes based on the collection of sets $\mathcal{A} = \mathcal{Q}_d$ where

$$\mathcal{Q}_d := \Big\{\, [0, t_1] \times \cdots \times [0, t_d];\ \boldsymbol{t} = (t_1, \dots, t_d) \in [0,1]^d \,\Big\}, \tag{3}$$

Note that when $d = 1$ the partial sum process ξ_n based on $\mathcal{Q}_d$ is the random polygonal line of Donsker-Prokhorov's theorem.

By equipping the collection $\mathcal{A}$ with some pseudo-metric δ, one define the space $C(\mathcal{A})$ of real continuous functions on $\mathcal{A}$, endowed with the norm

$$\|f\|_{\mathcal{A}} := \sup_{A\in\mathcal{A}} |f(A)|.$$

The usual semimetrics are $\delta(A,B) = \sqrt{|A \Delta B|}$, or $\delta(A,B) = \sqrt{m(A\Delta B)}$, for $A, B \in \mathcal{A}$, where m is a probability measure on the σ-algebra of Borel subsets of $[0,1]^d$. When $\mathcal{A}$ is totally bounded with respect to δ, $C(\mathcal{A})$ is a separable Banach space.

A Brownian sheet process indexed by $\mathcal{A}$ is a mean zero Gaussian process W with sample paths in $C(\mathcal{A})$ and

$$\mathbf{E}\, W(A)W(B) = |A \cap B|, \quad A, B \in \mathcal{A}. \tag{4}$$

Existence of such process is proved by placing restrictions on collection $\mathcal{A}$ which are usually expressed by some condition on its metric entropy. Dudley [10] and Erickson [12] give conditions for W to exist in a general Hölder space $\mathrm{H}_\rho(\mathcal{A})$. It is defined as the subspace of the space $C(\mathcal{A})$ of the functions satisfying

$$\sup_{0<\delta(A,B)<1} \frac{|x(A) - x(B)|}{\rho(\delta(A,B))} < \infty,$$

with the weight function ρ. For $\rho(h) = h^\alpha$, Erickson [12] proves that for process W, the Hölder exponent α cannot exceed $1/2$ and it decreases as the entropy of $\mathcal{A}$ increases. The functional central limit theorem (FCLT) in $\mathrm{C}(\mathcal{A})$ or in $\mathrm{H}_\rho(\mathcal{A})$ means the weak convergence of the summation process $\{\xi_{\boldsymbol{n}}(A); A \in \mathcal{A}\}$, suitably normalized, to a Brownian sheet process indexed by $\mathcal{A}$.

The first FCLT for $\{\xi_n(A); A \in \mathcal{Q}_d\}$ in $\mathrm{C}(\mathcal{Q}_d)$ was established by Kuelbs [18] under some moment restrictions and by Wichura [40] under finite variance condition. In 1983, Pyke [24] derived a FCLT for summation process in $\mathrm{C}(\mathcal{A})$, provided that the collection $\mathcal{A}$ satisfies the bracketing entropy condition. However, his result required moment conditions which depend on the size of the collection $\mathcal{A}$. Bass [3] and simultaneously Alexander and Pyke [1] extended Pyke's result to i.i.d. random fields with finite variance. Further developments were concerned with relaxing entropy conditions on the collection $\mathcal{A}$, Ziegler [41], and with relaxing i.i.d. condition on the random field $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^d\}$, Dedecker [8], El Machkouri and Ouchti [11] to name a few.

The FCLT for summation process in $\mathrm{H}_\rho(\mathcal{A})$ is not so extensively studied. Most general results are provided by Erickson [12] who shows that if $\mathbf{E}\,|X_{\boldsymbol{j}}|^q < \infty$ for some $q > 2$ then the FCLT holds in $\mathrm{H}_\rho(\mathcal{A})$ for some ρ which depends on q and properties of $\mathcal{A}$. For $d = 1$ and the class $\mathcal{A}$ of intervals $[0,t]$, $0 \le t \le 1$, Erickson's results coincide with Lamperti's ones [19], whereas his case $d > 1$ requires moments of order $q > dp(\alpha)$ with the same $p(\alpha)$ as in (2). In Račkauskas and Zemlys [34], the result by Erickson was improved in the case $d = 2$. In thesis this result was extended for $d > 2$ and Hilbert space valued random variables. Before stating it in full we need some definitions.

With $\mathbb{H}$ as the real separable Hilbert space, define the Hölder space $\mathbb{H}^o_\alpha([0,1]^d)$ of Hilbert-valued multi-parameter functions as the vector space of functions $x : [0,1]^d \to \mathbb{H}$ such that

$$\|x\|_\alpha := \|x(0)\| + w_\alpha(x,1) < \infty,$$

with

$$w_\alpha(x,\delta) := \sup_{0<|\boldsymbol{t}-\boldsymbol{s}|\le\delta} \frac{\|x(\boldsymbol{t}) - x(\boldsymbol{s})\|}{|\boldsymbol{t}-\boldsymbol{s}|^\alpha} \xrightarrow[\delta\to 0]{} 0.$$

Note that for $\mathbb{H} = \mathbb{R}$, the space $\mathbb{H}^o_\alpha([0,1]^d)$ is a subset of $\mathrm{H}_\rho(\mathcal{Q}_d)$ with $\rho(h) = h^\alpha$ and $\mathcal{Q}_d$ defined by (3).

Define $\mathbb{H}$-valued Brownian sheet W with the covariance operator Γ, as a $\mathbb{H}$-valued zero mean Gaussian process indexed by $[0,1]^d$ and satisfying

$$\mathbf{E}\,\langle W(\boldsymbol{t}), x\rangle\langle W(\boldsymbol{s}), y\rangle = (t_1 \wedge s_1)\cdots(t_d \wedge s_d)\langle \Gamma x, y\rangle \tag{5}$$

for $\boldsymbol{t}, \boldsymbol{s} \in [0,1]^d$ and $x, y \in \mathbb{H}$. For $\mathbb{H} = \mathbb{R}$, the space of covariance operators is isomorphic to $\mathbb{R}_+$ and (5) collapses to

$$\mathbf{E}\,W(\boldsymbol{t})W(\boldsymbol{s}) = \sigma^2(t_1 \wedge s_1)\cdots(t_d \wedge s_d).$$

which is the same as (4) for A,$B \in \mathcal{Q}_d$ and $\sigma^2 = 1$.

The following theorem holds.

Theorem 1 *For $0 < \alpha < 1/2$, set $p = p(\alpha) := 1/(1/2 - \alpha)$. For $d \ge 2$, let $\{X_{\boldsymbol{j}};\ \boldsymbol{j} \in \mathbb{N}^d,\ \boldsymbol{j} \ge \mathbf{1}\}$ be an i.i.d. collection of square integrable centered random elements in the separable Hilbert space $\mathbb{H}$ and $\xi_{\boldsymbol{n}}$ be the summation process defined by*

$$\xi_{\boldsymbol{n}}(\boldsymbol{t}) = \sum_{\mathbf{1}\le\boldsymbol{j}\le\boldsymbol{n}} |R_{\boldsymbol{n},\boldsymbol{j}}|^{-1}|R_{\boldsymbol{n},\boldsymbol{j}} \cap [0,\boldsymbol{t}]|X_{\boldsymbol{j}}.$$

Let W be a $\mathbb{H}$-valued Brownian sheet with the same covariance operator as $X_{\mathbf{1}}$. Then the convergence

$$(n_1\cdots n_d)^{-1/2}\xi_{\boldsymbol{n}} \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{\mathbb{H}^o_\alpha([0,1]^d)} W$$

holds if and only if

$$\sup_{t>0} t^{p(\alpha)} P(\|X_{\mathbf{1}}\| > t) < \infty. \tag{6}$$

As we see, condition (6) does not depend on the dimension d provided $d > 1$

and is weaker than necessary and sufficient condition (1) in the extension by Račkauskas and Suquet of Lamperti's functional central limit theorem. Moreover, we show that summation process considered along the diagonal, namely the sequence $n^{-d/2}\xi_n = n^{-d/2}\xi_{n,\dots,n}, n \in \mathbb{N}$, converges in $\mathbb{H}^o_\alpha([0,1]^d)$ if and only if

$$\lim_{t\to\infty} t^{2d/(d-2\alpha)} P(\|X_{\mathbf{1}}\| > t) = 0. \tag{7}$$

As dimension d increases, this condition weakens. For example, (7) is satisfied for any $d > 1$ provided $\mathbf{E}\,\|X_{\mathbf{1}}\|^4 < \infty$. This again shows up a difference between the cases $d = 1$ and $d > 1$ for functional central limit theorems in Hölder spaces.

The result in theorem 1 was obtained together with Račkauskas and Suquet [33]. Its proof and the prerequisites take up a sizeable part of the thesis. Necessary results from Hölder spaces and probability theory are given in the chapter 1. The properties of the summation process $\xi_{\boldsymbol{n}}$ are given in the section 2.1 and the result is proved in the section 3.1.

After i.i.d. case we considered the case of triangular array, when random variables are independent but not identically distributed. For general summation processes, the case of non-identically distributed variables was investigated by Goldie and Greenwood [13], [14]. They used classical construction of summation process, so their result does not coincide with classical Prokhorov [23] result for adaptive polygonal line process Ξ_n indexed by $[0, 1]$ with vertices $(b_n(k), S_n(k))$, where $b_n(k) = \mathbf{E}\,X^2_{n,1} + \cdots + \mathbf{E}\,X^2_{n,k}$, with assumption that $b_n(k_n) = 1$, and $X_{n,k}$ – independent non-identicaly distributed random variables.

The attempt to introduce adaptive construction for general summation processes was made by Bickel and Wichura [5]. However they put some restrictions on variance of random variables in triangular array. For zero mean independent random variables $\{X_{n,ij}, 1 \le i \le I_n, 1 \le j \le J_n\}$ with variances $\mathbf{E}\,X^2_{n,ij} = a_{n,i}b_{n,j}$ satisfying $\sum a_{n,i} = 1 = \sum b_{n,j}$, they defined summation process as

$$\zeta_n(t_1, t_2) = \sum_{i\le A_n(t_1)} \sum_{j\le B_n(t_1)} X_{n,ij},$$

where

$$A_n(t_1) = \max\{k : \sum_{i\le k} a_{n,i} < t_1\}, \quad B_n(t_2) = \max\{l : \sum_{j\le l} b_{n,j} < t_2\}.$$

It is easy to see that this construction is two-dimensional time generalization of jump version of Prokhorov construction. Bickel and Wichura proved that the process ζ_n converges in the space $D([0,1]^2)$ to a Brownian sheet, if $a_{n,i}$ and $b_{n,j}$ are infinitesimally small and the random variables $\{X_{n,ij}\}$ satisfy Lindeberg condition.

In this contribution we introduced new adaptive construction of summation process which reduces to classical construction for triangular arrays in one dimensional case. Sufficient conditions for the weak convergence in Hölder spaces are given. For the case $d = 1$ they coincide with conditions given by Račkauskas and Suquet. The limiting process in general case is not a standard Brownian sheet. It is a mean zero Gaussian process with covariance depending on the limit of $\mathbf{E}\,\Xi_{\boldsymbol{n}}(\boldsymbol{t})^2$. Examples of possible limiting processes are given. In case of special variance structure of triangular array as in Bickel and Wichura it is shown that the limiting process is a standard Brownian sheet.

Finally we provide the application of the theoretical results by constructing statistics for detecting the epidemic change in a given data with multidimensional indexes. Such data naturally arise if for example we measure some property of sample of individuals through time. It is natural then to assign two indexes to observation, the number of the individual and the time period when it was observed. This is so called longitudinal or panel data. First we consider the detection of the change of the mean in the double indexed sample $\{X_{ij}, 1 \le i \le n, 1 \le j \le m\}$. We test the null hypothesis of no change in mean against the alternative hypothesis of the change in a *epidemic rectangle*, i.e. the mean is different for indexes in the rectangle $D^* = [a^*, b^*] \times [c^*, d^*] \cap \mathbb{N}^2$. Our proposed statistic for detecting such change is the classical likelihood ratio statistic of Csörgő and Horváth [7], weighted with the power of diameter of the epidemic rectangle. We show that this statistic is the functional of summation process $\xi_{\boldsymbol{n}}$, with the functional continuous in the Hölder space. Thus using continuous mapping theorem and our theoretical result we find the asymptotic distribution of our statistic. We give the conditions for the consistency of the test and show that division by diameter, improves the ability to detect shorter epidemics, but that the result is not optimal compared to the one-dimensional case considered by Račkauskas and Suquet [31].

Next we turn our attention to panel regression models. We consider classical pooled or ordinary least squares and fixed effects regressions described by Baltagi [2]. We prove functional central limit theorem (FCLT) for the regression residuals under condition that regression disturbances satisfy FCLT and classical conditions on the regressors. This result generalizes the result of Ploberger and Krämer [22] for the time-series regression. Using the FCLT for regression residuals we adapt our statistic for detecting the change of the mean, to detect the change of the regression coefficient in both regression models. We find asymptotic distributions and give conditions for consistency of the statistics. We also investigate the behaviour of these statistics under local alternatives and derive results similar to those of Ploberger and Krämer.

Chapter 1

Weak convergence in Hölder spaces

1.1 General results

1.1.1 Basic definitions

Let us introduce some notation. Vectors $\boldsymbol{t} = (t_1, \dots, t_d)$ of $\mathbb{R}^d$, $d \geq 2$, are typeset in italic bold. In particular,

$$\mathbf{0} := (0, \dots, 0), \quad \mathbf{1} := (1, \dots, 1).$$

For $1 \leq k < l \leq d$, $\boldsymbol{t}_{k:l}$ denotes the "subvector"

$$\boldsymbol{t}_{k:l} := (t_k, t_{k+1}, \dots, t_l),$$

$\boldsymbol{t}_{-k}$ denotes the "subvector"

$$\boldsymbol{t}_{-k} = (t_1, \dots, t_{k-1}, t_{k+1}, \dots, t_d).$$

and $\boldsymbol{t}_K$ denotes the "subvector"

$$\boldsymbol{t}_K = (t_{k_1}, \dots, t_{k_m}),$$

with $K = \{k_1, \dots, k_m\} \subset \{1, \dots, d\}$ and $1 \leq k_1 < k_2 < \dots < k_m \leq d$. The set $\mathbb{R}^d$ is equipped with the partial order

$$\boldsymbol{s} \leq \boldsymbol{t} \quad \text{if and only if} \quad s_k \leq t_k, \text{ for all } k = 1, \dots, d. \tag{1.1}$$

As a vector space, $\mathbb{R}^d$ is endowed with the norm

$$|\boldsymbol{t}| = \max(|t_1|, \ldots, |t_d|), \quad \boldsymbol{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d.$$

Together with the usual addition of vectors and multiplication by a scalar, we use also the componentwise multiplication and division of vectors $\boldsymbol{s} = (s_1, \ldots, s_d)$, $\boldsymbol{t} = (t_1, \ldots, t_d)$ in $\mathbb{R}^d$ defined whenever it makes sense, by

$$\boldsymbol{st} := (s_1t_1, \ldots, s_dt_d), \quad \boldsymbol{s}/\boldsymbol{t} := (s_1/t_1, \ldots, s_d/t_d).$$

Partial order as well as all these operations are also intended componentwise when one of the two involved vectors is replaced by a scalar. So for $c \in \mathbb{R}$ and $\boldsymbol{t} \in \mathbb{R}^d$, $c \leq \boldsymbol{t}$ means $c \leq t_k$ for $k = 1, \ldots, d$, $\boldsymbol{t} + c := (t_1 + c, \ldots, t_d + c)$, $c/\boldsymbol{t} := (c/t_1, \ldots, c/t_d)$.

For $\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$, we write

$$\boldsymbol{\pi}(\boldsymbol{n}) := n_1 \cdots n_d,$$

and for $\boldsymbol{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d$,

$$\mathrm{m}(\boldsymbol{t}) := \min(t_1, \ldots, t_d).$$

For any real number x, denote by $[x]$ and $\{x\}$ its integer part and fractional part defined respectively by

$$[x] \leq x < [x] + 1, \quad [x] \in \mathbb{Z} \quad \text{and} \quad \{x\} := x - [x].$$

When applied to vectors $\boldsymbol{t}$ of $\mathbb{R}^d$, these operations are defined componentwise:

$$[\boldsymbol{t}] := ([t_1], \ldots, [t_d]), \quad \{\boldsymbol{t}\} := (\{t_1\}, \ldots, \{t_d\}).$$

The context should dispel any notational confusion between the fractional part of x (or $\boldsymbol{t}$) and the set having x (or $\boldsymbol{t}$) as unique element.

We denote by $\mathbb{H}$ a separable real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$.

1.1.2 Nets and asymptotic tightness

Let A be a set with a partial order and let A be a directed set. For a general topological space X, a map from A to X is called a net and denoted by $\{x_\alpha, \alpha \in A\} \subset X$. We say that this net has a limit x if and only if for every neighborhood U of x there exists $\alpha_0 \in A$ such that $x_\alpha \in U$ for each $\alpha_0 \leq \alpha$. When the space X is Hausdorff, any net in X has *at most* one limit. All the spaces we are dealing with are Banach, therefore Hausdorff, soit is always implicit that if the limit of the net exists, it is unique.

We are mainly interested in the nets $\{x_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^d\}$, where $\mathbb{N}^d$ is a directed set with partial order $\boldsymbol{s} \leq \boldsymbol{t}$ defined in (1.1). Note that if we have $\boldsymbol{n}_0 \leq \boldsymbol{n}$, then $\mathrm{m}(\boldsymbol{n}) \geq \mathrm{m}(\boldsymbol{n}_0)$ and if $\mathrm{m}(\boldsymbol{n}) \geq N$, then $(N, \dots, N) \leq \boldsymbol{n}$. Thus if the net $\{x_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^d\}$ has the limit x it makes sense to write

$$\lim_{\mathrm{m}(\boldsymbol{n}) \to \infty} x_{\boldsymbol{n}} = x.$$

We will use this notation throughout the thesis.

Let B be some separable Banach space and $(Y_\alpha)_{\alpha \in A}$ be a net of random elements in B. We write

$$Y_\alpha \xrightarrow{B} Y,$$

for weak convergence in the space B to the random element Y, i.e. $\mathbf{E} f(Y_\alpha) \to \mathbf{E} f(Y)$ for any continuous and bounded $f : B \to \mathbb{R}$.

For proving weak convergence of the nets we use some variant of Prokhorov's theorem (see e.g. van der Vaart and Wellner [39] p.21 theorem 1.3.9) which asserts that the net $\{Y_\alpha\}$ has a weakly convergent subnet if it is asymptoticaly tight, i. e. for each $\varepsilon > 0$ there exists a compact set $K_\varepsilon \in B$ such that

$$\liminf_{\alpha} P(Y_\alpha \in K_\varepsilon) > 1 - \varepsilon. \tag{1.2}$$

Thus weak convergence of the net Y_α can be proved by classical approach, by checking the property of asymptotical tightness and proving the convergence of the finite-dimensional distributions.

1.1.3 Schauder decomposition

To check the property of asymptotical tightness we need some way of characterizing compact subsets of the paths space. Suquet [37] gives us a criteria exploiting the notion of Schauder decomposition.

Definition 1 *An infinite sequence $(\mathcal{B}_j, j \in \mathbb{N})$ of closed linear subspaces of a Banach space B such that $\mathcal{B}_j \neq 0$ $(j \in \mathbb{N})$ is called a Schauder decomposition of B if for every $x \in B$ there exists an unique sequence $(y_n, n \in \mathbb{N})$ with $y_j \in \mathcal{B}_j$ $(j \in \mathbb{N})$ such that:*

$$x = \sum_{j=0}^{\infty} y_j$$

and if the coordinate projections defined by $v_n(x) = y_n$, are continuous on B.

Let us denote $\mathcal{Z}_j = \bigoplus_{i \le j} \mathcal{B}_i$ and $E_j = \sum_{i \le j} v_i$ the continuous projections of B onto $\mathcal{Z}_j$. Operation $\bigoplus$ here means the direct sum of vector subspaces, i.e. if $U = V \bigoplus W$ then for each $u \in U$ there exists an unique decomposition $u = v + w$, with $v \in V$ and $w \in W$.

Relatively compact subsets (whose closure are compacts) of separable Banach spaces with Schauder decomposition are then characterized by the following theorem.

Theorem 2 (Suquet, [37]) *Let B be a separable Banach space having a Schauder decomposition $(\mathcal{B}_j, j \in \mathbb{N})$. A subset K is relatively compact in B if and only if:*

i) for each $j \in \mathbb{N}$, $E_j K$ is relatively compact in $\mathcal{Z}_j := \bigoplus_{i \le j} \mathcal{B}_i$;

ii) $\sup_{x \in K} \|x - E_j x\| \to 0$ as $j \to \infty$.

1.2 Hölder space and its properties

The functional framework for our study of convergence of random fields is a certain class of Hölder spaces whose definition and some useful properties are gathered in this section.

1.2.1 Definition

For $0 < \alpha < 1$, define the Hölder space $\mathbb{H}_\alpha^o([0,1]^d)$ as the vector space of functions $x : [0,1]^d \to \mathbb{H}$ such that

$$\|x\|_\alpha := \|x(0)\| + w_\alpha(x, 1) < \infty,$$

with

$$w_\alpha(x, \delta) := \sup_{0 < |\boldsymbol{t} - \boldsymbol{s}| \le \delta} \frac{\|x(\boldsymbol{t}) - x(\boldsymbol{s})\|}{|\boldsymbol{t} - \boldsymbol{s}|^\alpha} \xrightarrow[\delta \to 0]{} 0.$$

Endowed with the norm $\|.\|_\alpha$, $\mathbb{H}_\alpha^o([0,1]^d)$ is a separable Banach space. In the special case $\mathbb{H} = \mathbb{R}$, we write $\mathrm{H}_\alpha^o([0,1]^d)$ instead of $\mathbb{H}_\alpha^o([0,1]^d)$. For other Hilbert spaces H we write $\mathrm{H}_\alpha^o([0,1]^d, H)$.

1.2.2 Pyramidal functions

From works of Račkauskas and Suquet [26, 31] we know that the space $\mathbb{H}_\alpha^o([0,1]^d)$ has a Schauder decomposition. The linear subspaces featuring in the decomposition contain linear combinations of certain pyramidal functions. We give now the precise definitions.

For convenience we write $T = [0,1]^d$ in this section. If A is a convex subset of T, the function $f : T \to \mathbb{H}$ is said to be affine on A if it preserves the barycenter, i.e. for any finite sequence $\boldsymbol{u}_1, \dots, \boldsymbol{u}_m$ in A and non negative scalars $r_1, \dots, r_m$ such that $\sum_{i=1}^m r_i = 1$, $f(\sum_{i=1}^m r_i \boldsymbol{u}_i) = \sum_{i=1}^m r_i f(\boldsymbol{u}_i)$.

Let us define the *standard triangulation* of the unit cube $T = [0,1]^d$. Write Π_d for the set of permutations of the indexes $1, \dots, d$. For any $\pi = (i_1, \dots, i_d) \in \Pi_d$, let $\Delta_\pi(T)$ be the convex hull of the $d+1$ points

$$\mathbf{0},\ \boldsymbol{e}_{i_1},\ (\boldsymbol{e}_{i_1} + \boldsymbol{e}_{i_2}), \dots, \sum_{k=1}^d \boldsymbol{e}_{i_k},$$

where the $\boldsymbol{e}_i$'s are the vectors of the canonical basis of $\mathbb{R}^d$. So, each simplex $\Delta_\pi(T)$ corresponds to one path from 0 to $(1, \dots, 1)$ via vertices of T and such that along each segment of the path, only one coordinate increases while the others remain constants. Thus T is divided into $d!$ simplexes with disjoint interiors. The standard triangulation of T is the family T_0 of simplexes $\{\Delta_\pi(T),\ \pi \in \Pi_d\}$.

Next, we divide T into 2^{jd} dyadic cubes with edge 2^{-j}. By dyadic translations and change of scale, each of them is equipped with a triangulation similar to T_0. And T_j is the set of the $2^{jd}d!$ simplexes so constructed. For $j \geq 1$ the set $W_j := \mathrm{vert}(T_j)$ of vertices of the simplexes in T_j is

$$W_j = \{k2^{-j};\ 0 \leq k \leq 2^j\}^d.$$

In what follows we put $V_0 := W_0$ and $V_j := W_j \setminus W_{j-1}$. So V_j is the set of new vertices born with the triangulation T_j. More explicitly, V_j is the set of dyadic points $\boldsymbol{v} = (k_1 2^{-j}, \dots, k_d 2^{-j})$ in W_j with at least one k_i odd.

The T_j-pyramidal function $\Lambda_{j,\boldsymbol{v}}$ with peak vertex $\boldsymbol{v} \in V_j$ is the *real valued* function defined on T by three conditions

i. $\Lambda_{j,\boldsymbol{v}}(\boldsymbol{v}) = 1$;

ii. $\Lambda_{j,\boldsymbol{v}}(\boldsymbol{w}) = 0$ if $\boldsymbol{w} \in \mathrm{vert}(T_j)$ and $\boldsymbol{w} \neq \boldsymbol{v}$;

iii. $\Lambda_{j,\boldsymbol{v}}$ is affine on each simplex Δ in T_j.

It follows clearly due to affinity from above definition that the support of $\Lambda_{j,\boldsymbol{v}}$ is the union of all simplexes in T_j containing the peak vertex $\boldsymbol{v}$. By [35](Prop. 3.4.5) the functions $\Lambda_{j,\boldsymbol{v}}$ are obtained by dyadic translations and changes of scale:

$$\Lambda_{j,\boldsymbol{v}}(\boldsymbol{t}) = \Lambda(2^j(\boldsymbol{t} - \boldsymbol{v})), \quad \boldsymbol{t} \in T,$$

from the same function Λ:

$$\Lambda(\boldsymbol{t}) := \max\Big(0,\ 1 - \max_{t_i<0}|t_i| - \max_{t_i>0} t_i\Big), \quad \boldsymbol{t} = (t_1, \dots, t_d) \in [-1, 1]^d.$$

Since $\Lambda_{j,\boldsymbol{v}}$ are affine on each simplex Δ in T_j, it is clear that $\Lambda_{j,\boldsymbol{v}} \in \mathrm{H}^o_\alpha([0,1]^d)$ for $0 < \alpha < 1$.Thus linear combinations $\sum_{\boldsymbol{v}\in V_j} h_{\boldsymbol{v}}\Lambda_{j,\boldsymbol{v}}$ with $h_{\boldsymbol{v}} \in \mathbb{H}$ are elements from $\mathbb{H}^o_\alpha([0,1]^d)$. For each j such sums are continuous on T, affine on each simplex Δ of T_i and vanishing at the vertices of W_{j-1}. Račkauskas and Suquet[31] prove that subspaces containning such functions form Shauder decomposition of $\mathbb{H}^o_\alpha([0,1]^d)$.

Theorem 3 (Račkauskas and Suquet [31]) *The space $\mathbb{H}^o_\alpha([0,1]^d)$ has the Schauder decomposition*

$$\mathbb{H}^o_\alpha([0,1]^d) = \bigoplus_{i=0}^{\infty} \boldsymbol{V}_i,$$

where $\boldsymbol{V}_0$ is the space of $\mathbb{H}$-valued functions continuous on T and affine on each simplex Δ of T_0 and for $i \geq 1$, $\boldsymbol{V}_i$ is the space of $\mathbb{H}$-valued functions continuous on T, affine on each simplex Δ of T_i and vanishing at the vertices of W_{i-1}.

Each element $x \in \mathbb{H}^o_\alpha([0,1]^d)$ then has unique representation

$$x = \sum_{i=0}^{\infty} \sum_{\boldsymbol{v}\in V_i} \lambda_{i,\boldsymbol{v}}(x)\Lambda_{i,\boldsymbol{v}},$$

with the $\mathbb{H}$-valued coefficients $\lambda_{j,v}(x)$ defined as

$$\begin{aligned} \lambda_{0,\boldsymbol{v}}(x) &= x(\boldsymbol{v}), \quad \boldsymbol{v} \in V_0; \\ \lambda_{j,\boldsymbol{v}}(x) &= x(\boldsymbol{v}) - \frac{1}{2}\Big(x(\boldsymbol{v}^-) + x(\boldsymbol{v}^+)\Big), \quad \boldsymbol{v} \in V_j,\ j \geq 1, \end{aligned}$$

where $\boldsymbol{v}^-$ and $\boldsymbol{v}^+$ are defined as follows. Each $\boldsymbol{v} \in V_j$ admits a unique representation $\boldsymbol{v} = (v_1, \dots, v_d)$ with $v_i = k_i/2^j$, $(1 \leq i \leq d)$. The points $\boldsymbol{v}^- = (v_1^-, \dots, v_d^-)$ and $\boldsymbol{v}^+ = (v_1^+, \dots, v_d^+)$ are defined by

$$v_i^- = \begin{cases} v_i - 2^{-j}, & \text{if } k_i \text{ is odd;} \\ v_i, & \text{if } k_i \text{ is even} \end{cases} \qquad v_i^+ = \begin{cases} v_i + 2^{-j}, & \text{if } k_i \text{ is odd;} \\ v_i, & \text{if } k_i \text{ is even.} \end{cases}$$

Having specified Schauder decomposition of the space $\mathbb{H}^o_\alpha([0,1]^d)$ we adapt theorem 2 specifically for space $\mathbb{H}^o_\alpha([0,1]^d)$. Define $\boldsymbol{W}_j = \bigoplus_{i=0}^{j} \boldsymbol{V}_j$. Note that

$\boldsymbol{W}_j$ corresponds to $\mathcal{Z}_j$ in theorem 2. Define the projection operators E_j $(j \geq 0)$ onto subspaces $\boldsymbol{W}_j$ by

$$E_j x := \sum_{i=0}^{j} \sum_{\boldsymbol{v} \in V_i} \lambda_{i,\boldsymbol{v}}(x) \Lambda_{i,\boldsymbol{v}}, \quad x \in \mathbb{H}_\alpha^o([0,1]^d). \tag{1.3}$$

Note that E_j is actually the operator of affine interpolation at the vertices of W_j, i.e. value of $E_j x$ depends only on values of $x(w)$ for $w \in W_j$. Since W_j is a finite set, $\boldsymbol{W}_j$ is clearly isomorphic to the Cartesian product of $\operatorname{card}(W_j)$ copies of $\mathbb{H}$, where $\operatorname{card}(w_j)$ is the number of elements in W_j. We exploit this fact later in proving tightness.

Having defined operators E_j, we give now some alternative representation of $\|x - E_j x\|_\alpha$. For any function $x \in H_\alpha(\mathbb{H})$, define its sequential seminorm by

$$\|x\|_\alpha^{\text{seq}} := \sup_{j \geq 0} 2^{\alpha j} \max_{\boldsymbol{v} \in V_j} \|\lambda_{j,\boldsymbol{v}}(x)\|.$$

Račkauskas and Suquet [31] show that this seminorm is actually a norm and that it is equivalent to the norm $\|x\|_\alpha$ on $\mathbb{H}_\alpha^o([0,1]^d)$. Furthermore

$$\|x - E_J x\|_\alpha = \|x - E_J x\|_\alpha^{\text{seq}} := \sup_{j > J} 2^{\alpha j} \max_{\boldsymbol{v} \in V_j} \|\lambda_{j,\boldsymbol{v}}(x)\|, \tag{1.4}$$

is non increasing in J. Now we can state the adaptation of theorem 2 for the space $\mathbb{H}_\alpha^o([0,1]^d)$.

Theorem 4 *A subset K is relatively compact in $\mathbb{H}_\alpha^o([0,1]^d)$ if and only if:*

i. for each $j \in \mathbb{N}$, $E_j K$ is relatively compact in $\boldsymbol{W}_j$;

ii. $\sup_{x \in K} \sup_{j > J} 2^{\alpha j} \max_{\boldsymbol{v} \in V_j} \|\lambda_{j,\boldsymbol{v}}(x)\| \to 0$, as $J \to \infty$.

1.2.3 Tightness criteria

Now we prove the tightness criteria. Note that this is an adaptation of the theorem 2 in [31] for nets $\{\zeta_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^d\}$.

Theorem 5 *Let $\{\zeta_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^d\}$ and ζ be random elements with values in the space $\mathbb{H}_\alpha^o([0,1]^d)$. Assume that the following conditions are satisfied.*

i) For each dyadic $\boldsymbol{t} \in [0,1]^d$, the net of $\mathbb{H}$-valued random elements $\zeta_{\boldsymbol{n}}(\boldsymbol{t})$ is asymptoticaly tight on $\mathbb{H}$.

ii) For each $\varepsilon > 0$

$$\lim_{J\to\infty} \limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} P(\sup_{j\geq J} 2^{\alpha j} \max_{\boldsymbol{v}\in V_j} \|\lambda_{j,\boldsymbol{v}}(\zeta_{\boldsymbol{n}})\| > \varepsilon) = 0.$$

Then the net $\zeta_{\boldsymbol{n}}$ is asymptoticaly tight in the space $\mathbb{H}^o_\alpha([0,1]^d)$.

Proof.

For fixed positive η, put $\eta_l = 2^{-l}$, $l = 1, 2, ..$ and choose a sequence (ε_l) decreasing to zero. By (ii) there is and integer J_l and index $\boldsymbol{n}_0 \in \mathbb{N}^d$ such that for set

$$A_l := \{x : \sup_{j\geq J_l} 2^{\alpha j} \max_{\boldsymbol{v}\in V_j} \|\lambda_{j,\boldsymbol{v}}(x)\| < \varepsilon_l\}$$

$P(\zeta_{\boldsymbol{n}} \in A_l) > 1 - \eta_l$, for all $\boldsymbol{n}$, $\boldsymbol{n}_0 \leq \boldsymbol{n}$.

Recall now from subsection 1.2.2 that $\boldsymbol{W}_j$ is isomorphic to the Cartesian product of card(W_j) copies of $\mathbb{H}$. Thus from (i) there exists for all $\boldsymbol{n}$, $\boldsymbol{n}_0 \leq \boldsymbol{n}$ a compact $K_l \subset \mathbb{H}^o_\alpha([0,1]^d)$, such that for set B_l

$$B_l := \{x \in \mathbb{H}^o_\alpha([0,1]^d) : E_{J_l}x \in K_l\}$$

$P(\zeta_{\boldsymbol{n}} \in B_l) > 1 - \eta_l$. Take K the closure of $\cap_{l=1}^\infty (A_l \cap B_l)$. Then $P(K) > 1 - 2\eta$, and K is compact due to theorem 4.

Since in $\mathbb{R}$ closed bounded sets are compact and vice versa we have following corollary for space $\mathrm{H}^o_\alpha([0,1]^d)$

Corollary 6 *Let $\{\zeta_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^d\}$ and ζ be random elements with values in the space $\mathrm{H}^o_\alpha([0,1]^d)$. Assume that the following conditions are satisfied.*

i) $\lim_{a\to\infty} P(\sup_{\boldsymbol{t}\in[0,1]^d} |\zeta_{\boldsymbol{n}}| > a) = 0$

ii) For each $\varepsilon > 0$

$$\lim_{J\to\infty} \limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} P(\sup_{j\geq J} 2^{\alpha j} \max_{\boldsymbol{v}\in V_j} |\lambda_{j,\boldsymbol{v}}(\zeta_{\boldsymbol{n}})| > \varepsilon) = 0.$$

Then the net $\zeta_{\boldsymbol{n}}$ is asymptoticaly tight in the space $\mathrm{H}^o_\alpha([0,1]^d)$.

1.3 Results in probability

1.3.1 Gaussian processes

Limiting random fields considered in this work are mainly Gaussian ones. Recall that a real valued random field $\{G(\boldsymbol{t}), \boldsymbol{t} \in [0,1]^d\}$ is called Gaussian if its finite-dimensional distributions are multivariate normal. Mean zero real valued Gaussian processes can be uniquely defined by their covariance function $\mathbf{E}\, G(\boldsymbol{t})G(\boldsymbol{s})$. The reverse problem is also of interest: when is a given function $g : [0,1]^d \times [0,1]^d \to \mathbb{R}$ the covariance function of a certain Gaussian random field ? The answer is a classical result which can be found in Khoshnevisan [17] for example. For convenience we state it here.

Theorem 7 *If the function $g : [0,1]^d \times [0,1]^d \to \mathbb{R}$ is symmetric and positive definite, i.e. for n-tuple of reals $x_1, \dots, x_n$ and n-tuple of vectors $\boldsymbol{t}_1, \dots, \boldsymbol{t}_n$ from $[0,1]^d$, expression $\sum_{i,j=1}^n x_i g(\boldsymbol{t}_i, \boldsymbol{t}_j) x_j \geq 0$, then there exists a zero mean Gaussian random field $\{G(\boldsymbol{t}), \boldsymbol{t} \in [0,1]^d\}$ with covariance function $\mathbf{E}\, G(\boldsymbol{t})G(\boldsymbol{s}) = g(\boldsymbol{t}, \boldsymbol{s})$.*

For $d = 1$ and $g(s,t) = s \wedge t$, we get a Brownian motion. Its generalization for $d > 1$ is called Brownian sheet, a zero mean Gaussian process with covariance function $g(\boldsymbol{s}, \boldsymbol{t}) = (s_1 \wedge t_1) \dots (s_d \wedge t_d)$. As Brownian motion is usual limiting process in invariance principles for one parameter summation processes, Brownian sheet is limiting process for multiparameter summation processes.

We now define Hilbert space valued Brownian sheet. Recall that zero mean Gaussian random variables in Hilbert space are uniquely defined through their covariance operator. Covariance operator of $\mathbb{H}$-valued random variable X is linear operator $\Gamma : \mathbb{H} \to \mathbb{H}$ satisfying

$$\langle \Gamma x, y \rangle = \mathbf{E}\, \langle X, x \rangle \langle X, y \rangle, \quad x, y \in \mathbb{H}.$$

Linear operator Γ is the covariance operator of some $\mathbb{H}$-valued random variable if it is

1. Symmetric: $\langle \Gamma x, y \rangle = \langle \Gamma y, x \rangle$ for all $x, y \in \mathbb{H}$.
2. Positive: $\langle \Gamma x, x \rangle \geq 0$, for all $x \in \mathbb{H}$.
3. Nuclear: operator Γ is compact and for every orthonormal base $\{e_n\} \subset \mathbb{H}$

$$\sum_n \langle \Gamma e_n, e_n \rangle < \infty.$$

Note that if $\mathbf{E}\, \|X\|^2 < \infty$ we have

$$\sum_n \langle \Gamma e_n, e_n \rangle = \sum_n \mathbf{E}\, \langle X, e_n \rangle^2 = \mathbf{E}\, \|X\|^2 < \infty \tag{1.5}$$

It is well known that in the Hilbert space $\mathbb{H}$, every random element X such that $\mathbf{E}\,\|X\|^2 < \infty$ is *pregaussian*, i.e. there is a Gaussian random element G in $\mathbb{H}$ with the same covariance operator as X, see [21, Prop. 9.24]. Let the X_i's be i.i.d. copies of X. If moreover $\mathbf{E}\,X = 0$, then $n^{-1/2}\sum_{i=1}^n X_i$ converges weakly to G in $\mathbb{H}$, in other words X satisfies the CLT in $\mathbb{H}$ [21, Th. 10.5].

Existence of real valued Gaussian processes is given by Kolmogorov theorem. Since it applies also for Cartesian products of Polish spaces it is natural to call $\mathbb{H}$-valued random field $\{G(\boldsymbol{t}), \boldsymbol{t} \in [0,1]^d\}$ Gaussian if for every m-tuple $\boldsymbol{t}_1, \dots, \boldsymbol{t}_m$ vector $(G(\boldsymbol{t}_1), \dots, G(\boldsymbol{t}_m))$ is $\mathbb{H}^m$ valued Gaussian random variable.

Define then $\mathbb{H}$-valued Brownian sheet with covariance operator Γ as a $\mathbb{H}$-valued zero mean Gausian process indexed by $[0,1]^d$ and satisfying

$$\mathbf{E}\,\langle W(\boldsymbol{t}), x\rangle\langle W(\boldsymbol{s}), y\rangle = (t_1 \wedge s_1)\dots(t_d \wedge s_d)\langle \Gamma x, y\rangle \tag{1.6}$$

for $\boldsymbol{t}, \boldsymbol{s} \in [0,1]^d$ and $x, y \in \mathbb{H}$. To check that this definition is valid note at first that for each $\boldsymbol{t} \in [0,1]^d$, $W(\boldsymbol{t})$ is $\mathbb{H}$-valued zero mean Gaussian random variable with covariance operor $\pi(\boldsymbol{t})\Gamma$. Denote by $\langle\,\cdot\,,\,\cdot\,\rangle_m$ the scalar product in $\mathbb{H}^m$ which is defined by

$$\langle h, g\rangle_m := \sum_{i=1}^m \langle h_i, g_i\rangle, \quad h = (h_1, \dots, h_m),\ g = (g_1, \dots, g_m) \in \mathbb{H}^m.$$

Denote by $\Gamma_{\boldsymbol{t}_1,\dots,\boldsymbol{t}_m}$ the covariance operator of $(W(\boldsymbol{t}_1), \dots, W(\boldsymbol{t}_m))$. For $x = (x_1, \dots, x_m) \in \mathbb{H}$ and $y = (y_1, \dots, y_m) \in \mathbb{H}$ from (1.6) we get

$$\begin{aligned}\langle \Gamma_{\boldsymbol{t}_1,\dots,\boldsymbol{t}_m} x, y\rangle_m &= \mathbf{E}\sum_{i=1}^m \langle W(\boldsymbol{t}_i), x_i\rangle \sum_{j=1}^m \langle W(\boldsymbol{t}_j), y_j\rangle \\ &= \sum_i \sum_j g(\boldsymbol{t}_i, \boldsymbol{t}_j)\langle \Gamma x_i, y_j\rangle\end{aligned}$$

with $g(\boldsymbol{t}_i, \boldsymbol{t}_j) = \prod_{k=1}^d t_{ik} \wedge t_{jk}$. Since Γ is symmetric we get that $\Gamma_{\boldsymbol{t}_1,\dots,\boldsymbol{t}_m}$ is symmetric also. Denote by X, a $\mathbb{H}$-valued random variable with covariance operator Γ. Then for $x = (x_1, \dots, x_m) \in \mathbb{H}$

$$\begin{aligned}\langle \Gamma_{\boldsymbol{t}_1,\dots,\boldsymbol{t}_m} x, x\rangle_m &= \sum_i \sum_j g(\boldsymbol{t}_i, \boldsymbol{t}_j)\langle \Gamma x_i, x_j\rangle \\ &= \sum_i \sum_j g(\boldsymbol{t}_i, \boldsymbol{t}_j)\mathbf{E}\,\langle X, x_i\rangle\langle X, x_j\rangle \\ &= \mathbf{E}\left(\sum_i \sum_j \langle X, x_i\rangle g(\boldsymbol{t}_i, \boldsymbol{t}_j)\langle X, x_j\rangle\right) \geq 0.\end{aligned}$$

Thus $\Gamma_{\boldsymbol{t}_1,\dots,\boldsymbol{t}_m}$ is positive. Now since

$$\mathbf{E}\,\|W(\boldsymbol{t}_1),\dots,W(\boldsymbol{t}_d))\|^2 = \sum_{i=1}^{m} \mathbf{E}\,\|W(\boldsymbol{t}_i)\|^2 < \infty$$

(1.5) implies that $\Gamma_{\boldsymbol{t}_1,\dots,\boldsymbol{t}_m}$ is nuclear. Thus there exists $\mathbb{H}^m$-valued Gaussian random variable with covariance function $\Gamma_{\boldsymbol{t}_1,\dots,\boldsymbol{t}_m}$ and our definition of $\mathbb{H}$-valued Brownian sheet is valid.

It is well known that trajectories of the real valued Brownian sheet are in $\mathrm{H}^o_\alpha([0,1]^d)$ for $0<\alpha<1/2$. As the following estimate

$$\mathbf{E}\,\|W(\boldsymbol{t}+\boldsymbol{h})+W(\boldsymbol{t}-\boldsymbol{h})-2W(\boldsymbol{t})\|^2 \le c|h|\operatorname{tr}\Gamma,$$

is valid for all $\boldsymbol{t}-\boldsymbol{h}$, $\boldsymbol{t}$, $\boldsymbol{t}+\boldsymbol{h} \in [0,1]^d$, it follows from Račkauskas and Suquet [26] that $W(\boldsymbol{t})$ has a version in $\mathbb{H}^o_\alpha([0,1]^d)$ for any $0<\alpha<1/2$.

1.3.2 A variant of continuous mapping theorem

Classical continuous mapping theorem states that if the sequence (or net) of random elements X_α converges weakly to X, then for any continuous functional g, real random variable $g(X_\alpha)$ converges weakly to $g(X)$. This result is widely applied in statistics to obtain limiting distributions. Sometimes though it is too restrictive, since sometimes it is more convenient to use the converging sequence (or net) of continuous functionals. Recall that the net of continuous functionals $g_\alpha : B \to R$ where $(B,\|\cdot\|)$ is a normed Banach space is called equicontinuous if for every $\varepsilon>0$ and any $x,y\in B$ such that $\|x-y\|<\delta$ we have

$$\sup_\alpha |g_\alpha(x)-g_\alpha(y)| < \varepsilon.$$

Then following theorem holds (it is stated as lemma in [30], we restate it for the case of nets).

Lemma 8 *Let $\{\eta_\alpha\}$ be a asymptotically tight net of random elements in the separable Banach space B, and g_α, g be a continuous functionals $B\to R$. Assume that g_α converges pointwise to g on B, and that g_α is equicontinuous. Then*

$$g_\alpha(\eta_\alpha) = g(\eta_\alpha) + o_P(1).$$

Proof. By the asymptotic tightness assumption there is for every $\varepsilon>0$ a compact subset K in B and α_0 such that for every $\alpha_0\le\alpha$, $P(\eta_\alpha\notin K)<\varepsilon$.

Now by a Arzela-Ascoli theorem the net g_α is totaly bounded on the compact K with respect to norm of uniform convergence. Since g_α converges pointwise to g, this gives us uniform convergence of g_α to g on K. Then for every $\delta > 0$ there is some α_1 depending on δ and K, such that

$$\sup_{x \in K} |g_\alpha(x) - g(x)| \leq \delta, \quad \alpha_1 \leq \alpha.$$

Take now α_2 such that $\alpha_1 \leq \alpha_2$ and $\alpha_0 \leq \alpha_2$. Then for $\alpha_2 \leq \alpha$ we have

$$P(|g_\alpha(\eta_\alpha) - g(\eta_\alpha)| \geq \delta) \leq P(\eta_\alpha \notin K) < \varepsilon,$$

which gives us the proof.

The following lemma from [30] provides some practical sufficient conditions to check the equicontinuity of some families of functionals.

Lemma 9 *Let $(B, \|\cdot\|)$ be a vector normed space and $q : B \to \mathbb{R}$ such that*

(a). q is subadditive: $q(x+y) \leq q(x) + q(y))$, $x, y \in B$.

(b). q is symmetric: $q(x) = q(-x)$, $x \in B$.

(c). For some constant C, $q(x) \leq C\|x\|$, $x \in B$.

Then q satisfies the Lipschitz condition

$$|q(x+y) - q(x)| \leq C\|y\|, \; x, y \in B \tag{1.7}$$

If $\mathcal{F}$ is any set of functionals q fulfilling (a), (b), (c) with the same constant C, then (a), (b), (c) are inherited by $g(x) := \sup\{q(x), q \in \mathcal{F}\}$ which therefore satisfies (1.7).

1.3.3 Rosenthal inequality

Since the Hilbert space $\mathbb{H}$ has cotype 2, it satisfies the following vector valued version of Rosenthal's inequality for every $q \geq 2$, see [20, Th. 2.6]. For any finite set $(Y_i)_{i \in I}$ of independent random elements in $\mathbb{H}$ with zero mean and such that $\mathbf{E}\,\|Y_i\|^q < \infty$ for every $i \in I$,

$$\mathbf{E}\left\|\sum_{i \in I} Y_i\right\|^q \leq C'_q \left(\mathbf{E}\left\|\sum_{i \in I} G(Y_i)\right\|^q + \sum_{i \in I} \mathbf{E}\,\|Y_i\|^q\right), \tag{1.8}$$

where the constant C'_q depends only on q and the $G(Y_i)$ are centered Gaussian independant random elements in $\mathbb{H}$ such that for every $i \in I$, $G(Y_i)$ has the same covariance structure as Y_i. For real random variables Rosenthal inequality simply reads

$$\mathbf{E}\left|\sum_{i\in I} Y_i\right|^q \le c\left(\left(\sum_{i\in I}\sigma_i^2\right)^{q/2} + \sum_{i\in I}\mathbf{E}\,|Y_i|^q\right), \tag{1.9}$$

where $\sigma_i^2 = \mathbf{E}\,Y_i^2$.

In the i.i.d. case with $N = \mathrm{card}(I)$, we note that $\sum_{i\in I} G(Y_i)$ is Gaussian with the same distribution as $N^{1/2}G(Y_1)$ and using the equivalence of moments for Gaussian random elements, see [21, Cor. 3.2], we obtain

$$\mathbf{E}\left\|\sum_{i\in I} Y_i\right\|^q \le C_q''\left(N^{q/2}\Big(\mathbf{E}\,\|G(Y_1)\|^2\Big)^{q/2} + N\mathbf{E}\,\|Y_1\|^q\right),$$

where C_q'' depends on q and does not depend on the distribution of Y_1. Since $\mathbb{H}$ has also the type 2, there is a constant a depending only on $\mathbb{H}$ such that $\mathbf{E}\,\|G(Y_1)\|^2 \le a\mathbf{E}\,\|Y_1\|^2$, see [21, Prop. 9.24]. Finally there is a constant C_q depending on $\mathbb{H}$, q, but not on the distribution of the Y_i's, such that

$$\mathbf{E}\left\|\sum_{i\in I} Y_i\right\|^q \le C_q\left(N^{q/2}\Big(\mathbf{E}\,\|Y_1\|^2\Big)^{q/2} + N\mathbf{E}\,\|Y_1\|^q\right), \quad (N = \mathrm{card}(I)). \tag{1.10}$$

1.3.4 Doob inequality

We shall need a generalization of maximal Doob inequality for multiparameter martingales. We use definitions and results from Khoshnevisan [17].

Definition 2 *Let $d \in \mathbb{N}$ and consider d (one parameter) filtrations $\mathcal{F}^1, \dots, \mathcal{F}^d$, where $\mathcal{F}^i = \{\mathcal{F}_k^i, k \ge 1\}$ $(1 \le i \le d)$. A stochastic process $M = (M_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^d)$ is an orthosubmartingale if for each $1 \le i \le d$ $(M_{\boldsymbol{n}}, n_i \ge 1)$ is a one parameter submartingale with respect to the one parameter filtration $\mathcal{F}^i$ with other coordinates $n_j \ne n_i$ fixed.*

The classical example of orthosubmartingale is the multiparameter random walk $\{S_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^d\}$ defined as

$$S_{\boldsymbol{n}} = \sum_{\boldsymbol{1}\le\boldsymbol{j}\le\boldsymbol{n}} X_{\boldsymbol{j}},$$

where $\{X_{\boldsymbol{j}}, \boldsymbol{j} \in \mathbb{N}^d\}$ is the collection of mean-zero random variables. The process $\{S_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^d\}$ is an orthosubmartingale with respect to filtrations

$$\mathcal{F}_k^i = \sigma(X_{\boldsymbol{j}}, j_i \leq k)$$

where $\sigma(\dots)$ denotes the σ-field generated by the random variables in parenthesis.

For nonnegative orthosubmartingales the following so called Cairoli's strong (p,p) inequality holds.

Theorem 10 (Th. 2.3.1 in Khoshnevisan [17]) *Suppose that $M = (M_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^d)$ is a nonnegative orthosubmartingale with respect to one-parameter filtrations $\mathcal{F}^1, \dots, \mathcal{F}^d$. Then for all $\boldsymbol{n} \in \mathbb{N}^d$ and $p > 1$*

$$\mathbf{E}\left[\max_{0 \leq \boldsymbol{k} \leq \boldsymbol{n}} M_{\boldsymbol{k}}^p\right] \leq \left(\frac{p}{p-1}\right)^{dp} \mathbf{E}\left[M_{\boldsymbol{n}}^p\right].$$

Following lemma is useful

Theorem 11 (Lemma 2.1.1 in Khoshnevisan [17]) *Suppose that $M = (M_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^d)$ is a nonnegative orthosubmartingale with respect to one-parameter filtrations $\mathcal{F}^1, \dots, \mathcal{F}^d$, that $\Psi : [0, \infty) \to [0, \infty)$ is convex nondecreasing, and that for all $\boldsymbol{n} \in \mathbb{N}^d$, $\mathbf{E}\,\Psi(M_{\boldsymbol{n}}) < \infty$. Then $(\Psi(M_{\boldsymbol{n}}), \boldsymbol{n} \in \mathbb{N}^d)$ is an orthosubmartingale.*

For independent zero mean real random variables $\{X_{\boldsymbol{j}}, \boldsymbol{j} \in \mathbb{N}^d\}$ introduce one parameter filtrations $\mathcal{F}^i = \mathcal{F}_k^i = \sigma(X_{\boldsymbol{j}}, j_i \leq k)$. Then process $S_{\boldsymbol{n}} = \sum_{\boldsymbol{j} \leq \boldsymbol{n}} X_{\boldsymbol{j}}$ is orthosubmartingale with respect to filtrations $\mathcal{F}^i$ and process $|S_{\boldsymbol{n}}|$ is nonnegative orthosubmartingale with respect to the same filtrations. Thus we have

$$\mathbf{E} \max_{\boldsymbol{1} \leq \boldsymbol{j} \leq \boldsymbol{n}} |S_{\boldsymbol{j}}|^p \leq \left(\frac{p}{p-1}\right)^{dp} \mathbf{E}\,|S_{\boldsymbol{n}}|^p. \tag{1.11}$$

For i.i.d. Hilbert space valued random field $\{X_{\boldsymbol{j}}, \boldsymbol{j} \in \mathbb{N}^d\}$ introduce d one parameter filtrations, $\mathcal{F}^i = (\mathcal{F}_k^i, k = 0, 1, \dots)$, $i = 1, \dots, d$, where $\mathcal{F}_k^i = \sigma(X_{\boldsymbol{j}}, \boldsymbol{j} \in \mathbb{N}^d, j_i \leq k)$. Since Cairoli's inequality applies for real valued orthosubmartingales we cannot use it directly for $S_{\boldsymbol{n}} = \sum_{\boldsymbol{j} \leq \boldsymbol{n}} X_{\boldsymbol{j}}$, since $S_{\boldsymbol{n}}$ is $\mathbb{H}$ valued stochastic process. On the other hand stochastic process $(\|S_{\boldsymbol{n}}\|, \boldsymbol{n} \in \mathbb{N}^d)$ is real valued so to apply theorem 10 we must show that $\|S_{\boldsymbol{n}}\|$ is orthosubmartingale with respect to filtrations $\mathcal{F}^1, \dots, \mathcal{F}^d$. Since norm is a continuous functional, the map $n_i \to \|S_{\boldsymbol{n}}\|$ is $\mathcal{F}_{n_i}^i$-measurable for each $i = 1, \dots, d$.

Assume that $\mathbf{E}\,\|X_1\| < \infty$. Then the X_j's are Bochner integrable and according to [38] we can introduce conditional expectations with respect to $\mathcal{F}^i$, $i = 1, \ldots, d$. Let $\mathbf{E}\,X_{\boldsymbol{j}} = 0$. From properties of conditional expectation we have for $i = 1, \ldots, d$, $\boldsymbol{n} \in \mathbb{N}^d$ and $k = 0, 1, \ldots$

$$\mathbf{E}\,(\|S_{\boldsymbol{n}}\| | \mathcal{F}^i_k) \geq \|\mathbf{E}\,(S_{\boldsymbol{n}} | \mathcal{F}^i_k)\| = \left\| \sum_{\boldsymbol{j} \leq \boldsymbol{n}} \mathbf{E}\,(X_{\boldsymbol{j}} | \mathcal{F}^i_k) \right\| = \|S_{(n_1, \ldots, n_{i-1}, k, n_{i+1}, \ldots, n_d)}\|.$$

Hence for each $i = 1, \ldots, d$, $n_i \to S_{\boldsymbol{n}}$ is a one parameter submartingale with respect to the filtration $\mathcal{F}^i$. Applying then theorem 10 we have

$$\mathbf{E} \max_{\boldsymbol{1} \leq \boldsymbol{j} \leq \boldsymbol{n}} \|S_{\boldsymbol{j}}\|^p \leq \left(\frac{p}{p-1} \right)^{dp} \mathbf{E}\,\|S_{\boldsymbol{n}}\|^p, \tag{1.12}$$

for all $\boldsymbol{n} \in \mathbb{N}^d$ and $p > 1$.

Chapter 2

Summation processes

We call random process a summation process if its values are defined only by the values of partial sums $S_k = X_1 + \cdots + X_k$, where X_k, $k = 1, \ldots, n$ are random variables. Usualy summation process is defined using interpolation arguments. Classical example is polygonal line process indexed by $[0, 1]$ with vertices $(k/n, S_k)$, $k = 0, 1 \ldots, n$ and $S_0 = 0$. This process has continuous and Hölderian trajectories. Sometimes it is convenient to drop the requirement of continuity and to analyze jump process defined as $\sum_{k=1}^{[nt]} X_k$. In this section we define summation processes indexed by $[0, 1]^d$ and give some useful representations. For reasons explained in section 2.2 we analyze separately summation processes based on random variables with the same variance and summation processes based on random variables with different variances. Though the results in this chapter are presented in a context of probability theory, they are derived without using any results from it. This chapter can be viewed as investigation of properties of certain interpolation schema of functions with domain $[0, 1]^d$. To improve readability, more technical and longer proofs are given at the end of each subsection.

2.1 Uniform variance case

2.1.1 Differences of partial sums

In this and following chapters we deal a lot with differences of partial sums $S_{\boldsymbol{j}} = \sum_{\boldsymbol{1} \leq \boldsymbol{i} \leq \boldsymbol{j}} X_{\boldsymbol{i}}$. Let us introduce the notation

$$\Delta_k^{(i)} S_{\boldsymbol{j}} = S_{(j_1, \ldots, j_{i-1}, k, j_{i+1}, \ldots, j_d)} - S_{(j_1, \ldots, j_{i-1}, k-1, j_{i+1}, \ldots, j_d)}$$

Since $S_{\boldsymbol{k}}$ is really a function with domain $\mathbb{N}^d$, we can say that $\Delta^{(i)}_{\boldsymbol{k}}$ is a difference operator acting on i-th coordinate of the argument of function $S_{\boldsymbol{k}}$. Note that superposition of operators $\Delta^{(i)}_{\boldsymbol{k}}$ commute

$$\Delta^{(i)}_{j_i}\Delta^{(l)}_{j_l}S_{\boldsymbol{j}} = \Delta^{(l)}_{j_l}\Delta^{(i)}_{j_i}S_{\boldsymbol{j}}.$$

In particular for any $\boldsymbol{j} \in \mathbb{N}^d$ we have

$$X_{\boldsymbol{j}} = \Delta^{(1)}_{j_1}\dots\Delta^{(d)}_{j_d}S_{\boldsymbol{j}}.$$

2.1.2 Definitions and representations

For $d = 1$ polygonal line and jump processes are given as

$$\xi_n(t) = S_{[nt]} + (nt - [nt])X_{[nt]+1}, \tag{2.1}$$
$$\zeta_n(t) = S_{[nt]}. \tag{2.2}$$

It is natural then to define $[0,1]^d$ indexed jump summation process as

$$\zeta_{\boldsymbol{n}}(\boldsymbol{t}) = S_{[\boldsymbol{nt}]}$$

It is not possible to do this for continuous version of $[0,1]^d$ indexed summation process, since the relation

$$S_{\boldsymbol{k}+\boldsymbol{1}} - S_{\boldsymbol{k}} = X_{\boldsymbol{k}+\boldsymbol{1}}$$

holds only for $\boldsymbol{k} = \boldsymbol{0}$. The continuous version of summation process for $d > 1$ still can be defined using analogy. Note that for $d = 1$ we can write

$$\xi_n(t) = \sum_{1\le i\le n} n\left|\left[\frac{i-1}{n}, \frac{i}{n}\right) \cap [0,t]\right| X_i, \tag{2.3}$$

where $|A|$ denotes Lebesgue measure of the set A. Define then continuous $[0,1]^d$ indexed summation process as

$$\xi_{\boldsymbol{n}}(\boldsymbol{t}) = \sum_{\boldsymbol{i}\le\boldsymbol{n}} \boldsymbol{\pi}(\boldsymbol{n})\left|\left[\frac{\boldsymbol{i}-\boldsymbol{1}}{\boldsymbol{n}}, \frac{\boldsymbol{i}}{\boldsymbol{n}}\right) \cap [0,\boldsymbol{t}]\right| X_{\boldsymbol{i}}. \tag{2.4}$$

where we write

$$[\boldsymbol{a},\boldsymbol{b}) = [a_1,b_1) \times \dots \times [a_d,b_d) \tag{2.5}$$

for $\boldsymbol{a}, \boldsymbol{b} \in [0,1]^d$. As in one dimensional case we can see that $\xi_{\boldsymbol{n}}(\boldsymbol{k}/\boldsymbol{n}) = S_{\boldsymbol{k}}$, i. e. the process is a continuous interpolation of the grid $(\boldsymbol{k}/\boldsymbol{n}, S_{\boldsymbol{k}})$.

To prove tightness of summation process we must control the difference of the process when the distance between points is small. For case $d = 1$ if s and t are close together, they fall into interval $[i/n, (i+1)/n]$ for some i. In this interval we have

$$\begin{aligned}\xi_n(t) &= S_i + (nt - i)X_{i+1}\\ \xi_n(s) &= S_i + (ns - i)X_{i+1}\end{aligned}$$

thus

$$\xi_n(t) - \xi_n(s) = n(t-s)X_{i+1}. \tag{2.6}$$

Thus it is of interest to investigate properties of the summation process in rectangles $[(\boldsymbol{i}-1)/\boldsymbol{n}, \boldsymbol{i}/\boldsymbol{n}]$. For the case $d = 1$ the summation process has the property that in interval $[i/n, (i+1)/n]$ it is the affine interpolation of its values at interval endpoints:

$$\begin{aligned}\xi_n(t) &= S_i + (nt-i)X_{i+1} = S_i + (nt-i)(S_{i+1} - S_i)\\ &= (1-(nt-i))S_i + (nt-i)S_{i+1} = (1-\{nt\})S_{[nt]} + \{nt\}S_{[nt]+1}\\ &= (1-\{nt\})\xi_n(i/n) + \{nt\}\xi((i+1)/n),\end{aligned}$$

with the weights coming from

$$nt = (1-\{nt\})[nt] + \{nt\}([nt]+1) = [nt] + \{nt\}. \tag{2.7}$$

The summation process $\xi_{\boldsymbol{n}}(\boldsymbol{t})$ retains this property. We show this directly for the case $d = 2$ and then prove it for general case. Fix $\boldsymbol{t} \in [0,1]^2$ and choose $\boldsymbol{i}$ so that $\boldsymbol{t} \in [(\boldsymbol{i}-\boldsymbol{1})/\boldsymbol{n}, \boldsymbol{i}/\boldsymbol{n})$. Necessarily then $[\boldsymbol{nt}] = \boldsymbol{i} - \boldsymbol{1}$. In this case the expression $\boldsymbol{\pi}(\boldsymbol{n})|[(\boldsymbol{j}-1)/\boldsymbol{n}, \boldsymbol{j}/\boldsymbol{n}] \cap [\boldsymbol{0}, \boldsymbol{t}]|$ assumes only following possible values:

$$\boldsymbol{\pi}(\boldsymbol{n})\left|\left[\frac{\boldsymbol{j}-\boldsymbol{1}}{\boldsymbol{n}}, \frac{\boldsymbol{j}}{\boldsymbol{n}}\right) \cap [0, \boldsymbol{t}]\right| = \begin{cases}1, & \text{for } \boldsymbol{j} \le \boldsymbol{i} - 1\\ 0, & \text{for } \boldsymbol{j}, \text{ such that either } j_1 \ge i_1, \text{ or } j_2 \ge i_2\\ \{n_1 t_1\}, & \text{for } \boldsymbol{j}, \text{ such that } j_1 = i_1 \text{ and } j_2 < i_2\\ \{n_2 t_2\}, & \text{for } \boldsymbol{j}, \text{ such that } j_2 = i_2 \text{ and } j_1 < i_1\\ \boldsymbol{\pi}(\{\boldsymbol{nt}\}), & \text{for } \boldsymbol{j} = \boldsymbol{i}.\end{cases}$$

Thus

$$\xi_{\boldsymbol{n}}(\boldsymbol{t}) = S_{\boldsymbol{i-1}} + \{n_1t_1\}\sum_{j=1}^{i_2-1} X_{i_1j} + \{n_2t_2\}\sum_{i=1}^{i_1-1} X_{ii_2} + \boldsymbol{\pi}(\{\boldsymbol{nt}\})X_{\boldsymbol{i}}. \tag{2.8}$$

We can rewrite this expression using difference operators as

$$\xi_{\boldsymbol{n}}(\boldsymbol{t}) = S_{\boldsymbol{i-1}} + \{n_1t_1\}\Delta_{i_1}^{(1)}S_{\boldsymbol{i-1}} + \{n_2t_2\}\Delta_{i_2}^{(2)}S_{\boldsymbol{i-1}} + \boldsymbol{\pi}(\{\boldsymbol{nt}\})\Delta_{i_1}^{(1)}\Delta_{i_2}^{(2)}S_{\boldsymbol{i-1}}, \tag{2.9}$$

or alternatively

$$\begin{aligned}\xi_{\boldsymbol{n}}(\boldsymbol{t}) &= \boldsymbol{\pi}(\boldsymbol{1}-\{\boldsymbol{nt}\})S_{\boldsymbol{i-1}} + \boldsymbol{\pi}(\{\boldsymbol{nt}\})S_{\boldsymbol{i}} \\ &\quad + \{n_1t_1\}(1-\{n_2t_2\})S_{i_1,i_2-1} + (1-\{n_1t_1\})\{n_2t_2\}S_{i_1-1,i_2}.\end{aligned} \tag{2.10}$$

Note that by doing so we expressed $\xi_{\boldsymbol{n}}(\boldsymbol{t})$ as linear combination of its values at vertices of rectangle $[(\boldsymbol{i}-\boldsymbol{1})/\boldsymbol{n}, \boldsymbol{i}/\boldsymbol{n}]$. Furthermore the weights in this combination sum to one and

$$\begin{aligned}\boldsymbol{nt} &= \boldsymbol{\pi}(\boldsymbol{1}-\{\boldsymbol{nt}\})[\boldsymbol{nt}] + \boldsymbol{\pi}(\{\boldsymbol{nt}\})([\boldsymbol{nt}]+\boldsymbol{1}) \\ &\quad + \{n_1t_1\}(1-\{n_2t_2\})([n_1t_1]+1,[n_2t_2]) \\ &\quad + (1-\{n_1t_1\})\{n_2t_2\}([n_1t_1],[n_2t_2]+1) \\ &= [\boldsymbol{nt}] + \{\boldsymbol{nt}\}.\end{aligned}$$

Thus in the point in the grid rectangle our summation process is weighted sum of its values on rectangle vertexes with the weights comming from barycentric decomposition of the point as it is in the case $d = 1$. Note that though we derived this decomposition for real valued random variables it holds for Banach space valued random variables also. We extend now (2.9) and (2.10) for general d.

Proposition 12 *For $\boldsymbol{t} \in [0,1)^d$, denote $\boldsymbol{s} = \{\boldsymbol{nt}\}$ and represent vertices of the rectangle $R_{\boldsymbol{n},[\boldsymbol{nt}]+\boldsymbol{1}}$ as*

$$V(\boldsymbol{u}) := \frac{[\boldsymbol{nt}]}{\boldsymbol{n}} + \frac{\boldsymbol{u}}{\boldsymbol{n}}, \quad \boldsymbol{u} \in \{0,1\}^d. \tag{2.11}$$

It is possible to express $\boldsymbol{t}$ as a barycenter of these 2^d vertices with weights $w(\boldsymbol{u}) \geq 0$ depending on $\boldsymbol{t}$, i.e.,

$$\boldsymbol{t} = \sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u})V(\boldsymbol{u}), \quad \textit{with} \quad \sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u}) = 1, \tag{2.12}$$

where

$$w(\boldsymbol{u}) = \prod_{l=1}^{d} s_l^{u_l}(1-s_l)^{1-u_l}. \tag{2.13}$$

Using this representation, define the random field $\xi_{\boldsymbol{n}}^*$ by

$$\xi_{\boldsymbol{n}}^*(\boldsymbol{t}) = \sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u}) S_{[\boldsymbol{nt}]+\boldsymbol{u}}, \quad \boldsymbol{t} \in [0,1]^d.$$

Then $\xi_{\boldsymbol{n}}^*$ coincides with the summation process defined by (2.4), where $\{X_{\boldsymbol{i}}, \boldsymbol{1} \leq \boldsymbol{i} \leq \boldsymbol{n}\}$ is a collection of Banach space valued random variables. Furthermore $\xi_{\boldsymbol{n}}$ admits representation

$$\xi_{\boldsymbol{n}}(\boldsymbol{t}) = S_{[\boldsymbol{nt}]} + \sum_{l=1}^{d} \sum_{1\leq i_1<i_2<\cdots<i_l\leq d} \left(\prod_{k=1}^{l}\{n_{i_k}t_{i_k}\}\right)\left(\prod_{k=1}^{l}\Delta^{(i_k)}_{[n_{i_k}t_{i_k}]+1}\right) S_{[\boldsymbol{nt}]}. \tag{2.14}$$

Proof of proposition 12

For notational convenience write

$$R_{\boldsymbol{n},\boldsymbol{j}} := \left[\frac{\boldsymbol{j}-1}{\boldsymbol{n}}, \frac{\boldsymbol{j}}{\boldsymbol{n}}\right). \tag{2.15}$$

For fixed $\boldsymbol{n} \geq \boldsymbol{1} \in \mathbb{N}^d$, any $\boldsymbol{t} \neq \boldsymbol{1} \in [0,1)^d$ belongs to a unique rectangle $R_{\boldsymbol{n},\boldsymbol{j}}$, defined by (2.15), namely $R_{\boldsymbol{n},[\boldsymbol{nt}]+\boldsymbol{1}}$. Recalling definition $\boldsymbol{s} = \{\boldsymbol{nt}\}$, note that

$$\boldsymbol{t} = \frac{[\boldsymbol{nt}]}{\boldsymbol{n}} + \frac{\boldsymbol{s}}{\boldsymbol{n}}. \tag{2.16}$$

For any non empty subset L of $\{1,\dots,d\}$, we denote by $\{0,1\}^L$ the set of binary vectors indexed by L. In particular $\{0,1\}^d$ is an abriged notation for $\{0,1\}^{\{1,\dots,d\}}$. Now define the non negative weights

$$w_L(\boldsymbol{u}) := \prod_{l\in L} s_l^{u_l}(1-s_l)^{1-u_l}, \quad \boldsymbol{u} \in \{0,1\}^L$$

and note that when $L = \{1,\dots,d\}$ these weights coincide with weights $w(\boldsymbol{u})$ defined in (2.13), hence we will not write subscript L in this case. For fixed L, the sum of all these weigths is one since

$$\sum_{\boldsymbol{u}\in\{0,1\}^L} w_L(\boldsymbol{u}) = \prod_{l\in L}\Big(s_l + (1-s_l)\Big) = 1. \tag{2.17}$$

The special case $L = \{1, \dots, d\}$ gives the second equality in (2.12). From (2.17) we immediately deduce that for any K non empty and strictly included in $\{1, \dots, d\}$, with $L := \{1, \dots, d\} \setminus K$,

$$\sum_{\substack{\boldsymbol{u}\in\{0,1\}^d,\\ \forall k\in K,\, u_k=1}} w(\boldsymbol{u}) = \prod_{k\in K} s_k \sum_{\boldsymbol{u}\in\{0,1\}^L} s_l^{u_l}(1-s_l)^{1-u_l} = \prod_{k\in K} s_k. \tag{2.18}$$

Formula (2.18) remains obviously valid in the case where $K = \{1, \dots, d\}$.

Now let us observe that

$$\sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u})V(\boldsymbol{u}) = \sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u})\Big(\frac{[\boldsymbol{nt}]}{\boldsymbol{n}} + \frac{\boldsymbol{u}}{\boldsymbol{n}}\Big) = \frac{[\boldsymbol{nt}]}{\boldsymbol{n}} + \sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u})\frac{\boldsymbol{u}}{\boldsymbol{n}}.$$

Comparing with the expression of $\boldsymbol{t}$ given by (2.16), we see that the first equality in (2.12) will be established if we check that

$$\boldsymbol{s}' := \sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u})\boldsymbol{u} = \boldsymbol{s}. \tag{2.19}$$

This is easily seen componentwise using (2.18) because for any fixed $l \in \{1, \dots, d\}$,

$$s'_l = \sum_{\substack{\boldsymbol{u}\in\{0,1\}^d,\\ u_l=1}} w(\boldsymbol{u}) = \prod_{k\in\{l\}} s_k = s_l.$$

Next we check that $\xi_{\boldsymbol{n}}(\boldsymbol{t}) = \xi_{\boldsymbol{n}}^*(\boldsymbol{t})$ for every $\boldsymbol{t} \in [0,1]^d$. Introduce the notation

$$D_{\boldsymbol{t},\boldsymbol{u}} := \mathbb{N}^d \cap \Big(\Big[0, [\boldsymbol{nt}] + \boldsymbol{u}\Big] \setminus \Big[0, [\boldsymbol{nt}]\Big]\Big).$$

Then we have

$$\begin{aligned}
\xi_{\boldsymbol{n}}^*(\boldsymbol{t}) &= \sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u})\Big(S_{[\boldsymbol{nt}]} + (S_{[\boldsymbol{nt}]+\boldsymbol{u}} - S_{[\boldsymbol{nt}]})\Big)\\
&= S_{[\boldsymbol{nt}]} \sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u}) \sum_{\boldsymbol{i}\in D_{\boldsymbol{t},\boldsymbol{u}}} X_{\boldsymbol{i}}.
\end{aligned}$$

Now in view of (2.4) the proof of $\xi_{\boldsymbol{n}}(\boldsymbol{t}) = \xi_{\boldsymbol{n}}^*(\boldsymbol{t})$ reduces clearly to that of

$$\sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u}) \sum_{\boldsymbol{i}\in D_{\boldsymbol{t},\boldsymbol{u}}} X_{\boldsymbol{i}} = \sum_{\boldsymbol{i}\in I} \boldsymbol{\pi}(\boldsymbol{n})|R_{\boldsymbol{n},\boldsymbol{i}} \cap [0, \boldsymbol{t}]|X_{\boldsymbol{i}}, \tag{2.20}$$

where

$$I := \{\boldsymbol{i} \le \boldsymbol{n};\, \forall k \in \{1, \dots, d\},\ i_k \le [n_k t_k] + 1 \text{ and } \exists l \in \{1, \dots, d\},\ i_l = [n_l t_l] + 1\}. \tag{2.21}$$

Clearly I is the union of all $D_{\boldsymbol{t},\boldsymbol{u}}$, $\boldsymbol{u} \in \{0,1\}^d$, so we can rewrite the left hand side of (2.20) under the form $\sum_{\boldsymbol{i}\in I} a_{\boldsymbol{i}} X_{\boldsymbol{i}}$. For $\boldsymbol{i} \in I$, put

$$K(\boldsymbol{i}) := \Big\{k \in \{1, \dots, d\};\ \boldsymbol{i}_k = [n_k t_k] + 1\Big\}. \tag{2.22}$$

Then observe that for $\boldsymbol{i} \in I$, the $\boldsymbol{u}$'s such that $\boldsymbol{i} \in D_{\boldsymbol{t},\boldsymbol{u}}$ are exactly those which satisfy $u_k = 1$ for every $k \in K(\boldsymbol{i})$. Using (2.18), this gives

$$\forall \boldsymbol{i} \in I, \qquad a_{\boldsymbol{i}} = \sum_{\substack{\boldsymbol{u}\in\{0,1\}^d,\\ \forall k \in K(\boldsymbol{i}),\, u_k = 1}} w(\boldsymbol{u}) = \prod_{k\in K(\boldsymbol{i})} s_k. \tag{2.23}$$

On the other hand we have for every $\boldsymbol{i} \in I$,

$$|R_{\boldsymbol{n},\boldsymbol{i}} \cap [0, \boldsymbol{t}]| = \prod_{k\in K(\boldsymbol{i})} \left(t_k - \frac{[n_k t_k]}{n_k}\right) \prod_{k\notin K(\boldsymbol{i})} \frac{1}{n_k} = \frac{1}{\boldsymbol{\pi}(\boldsymbol{n})} \prod_{k\in K(\boldsymbol{i})} s_k = \frac{a_{\boldsymbol{i}}}{\boldsymbol{\pi}(\boldsymbol{n})}. \tag{2.24}$$

Thus (2.20) follows. To prove (2.14) note that

$$\xi_{\boldsymbol{n}}(\boldsymbol{t}) = S_{[\boldsymbol{n}\boldsymbol{t}]} + \sum_{\boldsymbol{i}\in I} \boldsymbol{\pi}(\boldsymbol{n}) |R_{\boldsymbol{n},\boldsymbol{i}} \cap [0, \boldsymbol{t}]| X_{\boldsymbol{i}} = S_{[\boldsymbol{n}\boldsymbol{t}]} + \sum_{\boldsymbol{i}\in I} \left(\prod_{k\in K(\boldsymbol{i})} s_k\right) X_{\boldsymbol{i}},$$

which can be recast as

$$\xi_{\boldsymbol{n}}(\boldsymbol{t}) = S_{[\boldsymbol{n}\boldsymbol{t}]} + \sum_{l=1}^{d} T_l(\boldsymbol{t}) \tag{2.25}$$

with

$$T_l(\boldsymbol{t}) := \sum_{\substack{\boldsymbol{i}\in I\\ \operatorname{card}(K(\boldsymbol{i}))=l}} \left(\prod_{k\in K(\boldsymbol{i})} s_k\right) X_{\boldsymbol{i}}. \tag{2.26}$$

Let $K \subset \{1, \dots, d\}$ and $I_K = \{\boldsymbol{i} \in I; K(i) = K\}$. Then

$$I_K = \{\boldsymbol{i} \le \boldsymbol{n};\, i_k = [n_k t_k] + 1, \text{ for } k \in K \text{ and } i_k \le [n_k t_k], \text{for } k \in \{1, \dots, d\} \backslash K\},$$

Take for example $K = \{1\}$ and notice that

$$\sum_{\boldsymbol{i}\in I_K} X_{\boldsymbol{i}} = \sum_{i_2=1}^{[nt_2]} \dots \sum_{i_d=1}^{[nt_d]} X_{([nt_1]+1,i_2,\dots,i_d)} = \Delta^{(1)}_{[nt_1]+1} S_{[\boldsymbol{nt}]}.$$

Then it should be clear that

$$\sum_{\boldsymbol{i}\in I_K} X_{\boldsymbol{i}} = \left(\prod_{k\in K} \Delta^{(k)}_{[n_k t_k]+1}\right) S_{[\boldsymbol{nt}]}.$$

Now observe that

$$T_l(\boldsymbol{t}) = \sum_{\substack{K\subset\{1,\dots,d\}\\ \operatorname{card}(K)=l}} \sum_{\boldsymbol{i}\in I_K} \left(\prod_{k\in K} s_k\right) X_{\boldsymbol{i}} = \sum_{\substack{K\subset\{1,\dots,d\}\\ \operatorname{card}(K)=l}} \left(\prod_{k\in K} s_k\right) \sum_{\boldsymbol{i}\in I_K} X_{\boldsymbol{i}}.$$

Recalling that $s_k = \{n_k t_k\}$, this leads to

$$T_l(\boldsymbol{t}) = \sum_{\substack{K\subset\{1,\dots,d\}\\ \operatorname{card}(K)=l}} \left(\prod_{k\in K} \{n_k t_k\}\right) \left(\prod_{k\in K} \Delta^{(k)}_{[n_k t_k]+1}\right) S_{[\boldsymbol{nt}]}. \tag{2.27}$$

Finally we obtain the representation (2.14) and complete the proof.

2.1.3 Estimate of sequential Hölder norm

Using the results from previous sections we give now the estimate of sequential norm of $\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}\xi_{\boldsymbol{n}}$ in terms of m-indexed sums. We use this result later for proving the tightness of process $\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}\xi_{\boldsymbol{n}}$. Recall that sequential norm of $x \in \mathbb{H}^o_\alpha([0,1]^d)$ is defined as

$$\|x\|^{\text{seq}}_\alpha = \sup_{j\geq 0} 2^{\alpha j} \max_{\boldsymbol{v}\in V_j} \|\lambda_{j,\boldsymbol{v}}(x)\|.$$

Recall from 1.2.2 that a dyadic point $\boldsymbol{v} \in V_j$ can be expressed as $\boldsymbol{v} = (k_1 2^{-j}, \dots, k_d 2^{-j})$ with at least one k_i odd. Denote by $K = \{j_1, \dots, j_l\}$ the set of indices for which coordinates of $2^j\boldsymbol{v}$ are odd. Then

$$\xi_{\boldsymbol{n}}(\boldsymbol{v}) - \xi_{\boldsymbol{n}}(\boldsymbol{v}^+) = \sum_{i=1}^{\operatorname{card}(K)} \xi_{\boldsymbol{n}}\left(\boldsymbol{v} + 2^{-j}\sum_{k=1}^{i-1} e_{j_k}\right) - \xi_{\boldsymbol{n}}\left(\boldsymbol{v} + 2^{-j}\sum_{k=1}^{i} e_{j_k}\right)$$

similarly

$$\xi_{\boldsymbol{n}}(\boldsymbol{v}) - \xi_{\boldsymbol{n}}(\boldsymbol{v}^-) = \sum_{i=1}^{\text{card}(K)} \xi_{\boldsymbol{n}}\left(\boldsymbol{v} - 2^{-j}\sum_{k=1}^{i-1} e_{j_k}\right) - \xi_{\boldsymbol{n}}\left(\boldsymbol{v} - 2^{-j}\sum_{k=1}^{i} e_{j_k}\right)$$

thus we can express double difference

$$\lambda_{j,\boldsymbol{v}}(\xi_{\boldsymbol{n}}) = \xi_{\boldsymbol{n}}(\boldsymbol{v}) - \frac{1}{2}(\xi_{\boldsymbol{n}}(\boldsymbol{v}^+) + \xi_{\boldsymbol{n}}(\boldsymbol{v}^-))$$

as differences of process $\xi_{\boldsymbol{n}}$ where only one coordinate is changing. Define

$$\Delta_{\boldsymbol{n}}^{(1)}(t, t'; \boldsymbol{s}) := \|\xi_{\boldsymbol{n}}(t', s_2, \dots, s_d) - \xi_{\boldsymbol{n}}(t, s_2, \dots, s_d)\|,$$

for the change in the first coordinate and similarly $\Delta_{\boldsymbol{n}}^{(j)}(t, t'; \boldsymbol{s})$ for the change in the j-th coordinate. Then

$$\max_{\boldsymbol{v}\in V_j} \|\lambda_{j,\boldsymbol{v}}(\xi_{\boldsymbol{n}})\| \le \sum_{m=1}^{d} \max_{\substack{0\le k<2^j \\ \boldsymbol{0}\le\boldsymbol{\ell}\le\boldsymbol{2}^j}} \Delta_{\boldsymbol{n}}^{(m)}(t_{k+1}, t_k; \boldsymbol{s}_{\boldsymbol{\ell}}) \tag{2.28}$$

with $t_k = k2^{-j}$, $\boldsymbol{\ell} = (l_2, \dots, l_d)$, $\boldsymbol{2}^j = (2^j, \dots, 2^j)$ (vector of dimension $d-1$) and $\boldsymbol{s}_\ell = \boldsymbol{\ell}\boldsymbol{2}^{-j}$.

Let us first examine the the case $d = 1$. Then

$$\max_{\boldsymbol{v}\in V_j} \|\lambda_{j,\boldsymbol{v}}(\xi_{\boldsymbol{n}})\| \le \max_{0\le k<2^j} |\xi_n(t_{k+1}) - \xi_n(t_k)|.$$

Denote by "log" the logarithm *with basis* 2 ($\log 2 = 1$). If $j > \log n$ then $t_{k+1} - t_k = 2^{-j} < 1/n$ and from definition (2.1) we get

$$\xi_n(t_{k+1}) - \xi_n(t_k) = n2^{-j} X_{[nt_k]+1}$$

if $[nt_{k+1}] = [nt_k]$. For $j \le \log n$ we get

$$\xi_n(t_{k+1}) - \xi_n(t_k) = \sum_{i=[nt_k]+1}^{[nt_{k+1}]} X_i$$

if $n = 2^l$ with $l > j$. With little additional work it is possible to refine these expressions to get the estimate

$$\max_{0\le k<2^j} |\xi_n(t_{k+1}) - \xi_n(t_k)| \le \mathbf{1}\,(j \le \log n) \max_{0\le k<2^j} \sum_{i=[nt_k]+1}^{[nt_{k+1}]} |X_i| \\ + \mathbf{1}\,(j > \log n)\, n2^{-j+1} \max_{1\le i\le n} |X_i|.$$

For the case $d > 1$ it is possible to get similar estimate given by the following lemma.

Lemma 13 *For $m = 1, \ldots, d$ and any $t', t \in [0,1], t' > t$,*

$$\sup_{\boldsymbol{s}\in[0,1]^{d-1}} \Delta_{\boldsymbol{n}}^{(m)}(t, t'; \boldsymbol{s}) \le 3^d \mathbf{1}\left(t' - t \ge \frac{1}{n_m}\right) \psi_{\boldsymbol{n}}^{(m)}(t', t) \\ + 3^d \min\Big(1, n_m(t'-t)\Big) Z_{\boldsymbol{n}}^{(m)},$$

where

$$\psi_{\boldsymbol{n}}^{(m)}(t', t) := \max_{\mathbf{1}_{-m}\le \boldsymbol{k}_{-m}\le \boldsymbol{n}_{-m}} \Bigg\| \sum_{k_m=[n_m t]+1}^{[n_m t']} \Delta_{k_m}^{(m)} S_{\boldsymbol{k}} \Bigg\|, \tag{2.29}$$

$$Z_{\boldsymbol{n}}^{(m)} := \max_{\mathbf{1}\le \boldsymbol{k}\le \boldsymbol{n}} \|\Delta_{k_1}^{(m)} S_{\boldsymbol{k}}\|. \tag{2.30}$$

Thus

$$\max_{\substack{0\le k<2^j\\ \mathbf{0}\le \boldsymbol{\ell}\le \mathbf{2}^j}} \Delta_{\boldsymbol{n}}^{(m)}(t_{k+1}, t_k; \boldsymbol{s_\ell}) \le \max_{0\le k<2^j} \Big[3^d \mathbf{1}\,(t_{k+1} - t_k \ge 1/n_m)\, \psi_{\boldsymbol{n}}^{(m)}(t_{k+1}, t_k) \\ + 3^d \min\{1, n_m(t_{k+1} - t_k)\} Z_{\boldsymbol{n}}^{(m)} \Big]$$

It is possible to further refine this expression by considering certain values of j. For $j > \log n_m$, we have $2^j > n_m$, whence $(t_{k+1} - t_k) = 2^{-j} < 1/n_m$ thus in this case

$$\max_{\substack{0\le k<2^j\\ \mathbf{0}\le \boldsymbol{\ell}\le \mathbf{2}^j}} \Delta_{\boldsymbol{n}}^{(m)}(t_{k+1}, t_k; \boldsymbol{s_\ell}) \le 3^d \max_{0\le k<2^j} n_m(t_{k+1} - t_k) Z_{\boldsymbol{n}}^{(m)}.$$

Now since $2^{-j} < n_m^{-1}$ we have

$$2^{j\alpha} n_m(t_{k+1} - t_k) = 2^{-j(1-\alpha)} n_m < n_m^{\alpha},$$

giving us

$$2^{j\alpha} \max_{\substack{0\le k<2^j \\ \mathbf{0}\le \boldsymbol{\ell}\le \mathbf{2}^j}} \Delta_{\boldsymbol{n}}^{(m)}(t_{k+1}, t_k; \boldsymbol{s}_{\boldsymbol{\ell}}) \le 3^d n_m^\alpha Z_{\boldsymbol{n}}^{(m)} \tag{2.31}$$

for $j > \log n_m$. On the other hand, for $j \le \log n_m$, we have $2^{\alpha j} \le n_m^\alpha$, whence

$$2^{j\alpha} \max_{\substack{0\le k<2^j \\ \mathbf{0}\le \boldsymbol{\ell}\le \mathbf{2}^j}} \Delta_{\boldsymbol{n}}^{(m)}(t_{k+1}, t_k; \boldsymbol{s}_{\boldsymbol{\ell}}) \le 3^d \Big[\max_{j\le \log n_m} \max_{0\le k<2^j} \psi_{\boldsymbol{n}}^{(m)}(t_{k+1}, t_k) + n_m^\alpha Z_{\boldsymbol{n}}^{(m)} \Big] \tag{2.32}$$

for $j \le \log n_m$. Reporting now (2.31) and (2.32) to (2.28) we get

$$\|\xi_{\boldsymbol{n}}\|_\alpha^{\mathrm{seq}} \le 3^d \sum_{m=1}^d \left(\max_{j\le \log n_m} 2^{\alpha j} \max_{0\le k<2^j} \psi_{\boldsymbol{n}}^{(m)}(t_{k+1}, t_k) + n_m^\alpha Z_{\boldsymbol{n}}^{(m)} \right). \tag{2.33}$$

Recalling the definition of E_J from (1.3) and noting that we can restrict the domain of j in inequality (2.32) we also get the estimate for $\|\xi_{\boldsymbol{n}} - E_J \xi_{\boldsymbol{n}}\|$

$$\|\xi_{\boldsymbol{n}} - E_J\xi_{\boldsymbol{n}}\|_\alpha^{\mathrm{seq}} \le 3^d \sum_{m=1}^d \left(\max_{J\le j\le \log n_m} 2^{\alpha j} \max_{0\le k<2^j} \psi_{\boldsymbol{n}}^{(m)}(t_{k+1}, t_k) + n_m^\alpha Z_{\boldsymbol{n}}^{(m)} \right). \tag{2.34}$$

From these inequalities we see that the sequential norm of process $\xi_{\boldsymbol{n}}$ can be controlled by only first differences of the process when only one coordinate changes.

Proof of lemma 13

We prove this lemma in case of $m = 1$, proof for other cases is identical. Put $\boldsymbol{u} := (t, \boldsymbol{s})$, $\boldsymbol{u}' := (t', \boldsymbol{s})$, so $u_1 = t$, $u_1' = t'$ and $\boldsymbol{u}_{2:d} = \boldsymbol{u}'_{2:d} = \boldsymbol{s}$. Denote

$$T_l(\boldsymbol{t}) = \sum_{1\le i_1<i_2<\cdots<i_l\le d} \left(\prod_{k=1}^l \{n_{i_k} t_{i_k}\} \right) \left(\prod_{k=1}^l \Delta_{[n_{i_k} t_{i_k}]+1}^{(i_k)} \right) S_{[\boldsymbol{n}\boldsymbol{t}]}.$$

Then from representation 2.14 we have

$$\xi_{\boldsymbol{n}}(\boldsymbol{u}') - \xi_{\boldsymbol{n}}(\boldsymbol{u}) = S_{[\boldsymbol{n}\boldsymbol{u}']} - S_{[\boldsymbol{n}\boldsymbol{u}]} + \sum_{l=1}^d \Big(T_l(\boldsymbol{u}') - T_l(\boldsymbol{u}) \Big). \tag{2.35}$$

To estimate this $\xi_{\boldsymbol{n}}$'s increment we discuss according to the different possible configurations.

Case 1. $0 < t' - t < 1/n_1$.

Case 1.a. $[n_1 t'] = [n_1 t]$, whence $[\boldsymbol{nu'}] = [\boldsymbol{nu}]$. Consider first the increment $T_1(\boldsymbol{u'}) - T_1(\boldsymbol{u})$ and note that by (2.27) with $l = 1$,

$$T_1(\boldsymbol{u}) = \sum_{1\le k\le d} \{n_k u_k\} \Delta^{(k)}_{[n_k u_k]+1} S_{[\boldsymbol{nu}]}.$$

Because $\boldsymbol{u}_{2:d} = \boldsymbol{u}'_{2:d}$ and $[\boldsymbol{nu'}] = [\boldsymbol{nu}]$, all the terms indexed by $k \ge 2$ disappear in the difference $T_1(\boldsymbol{u'}) - T_1(\boldsymbol{u})$. Note also that $\{n_1 t'\} - \{n_1 t\} = n_1(t' - t)$. This leads to the factorization

$$T_1(\boldsymbol{u'}) - T_1(\boldsymbol{u}) = n_1(t' - t)\Delta^{(1)}_{[n_1 t]+1} S_{[\boldsymbol{nu}]}.$$

For $l \ge 2$, $T_l(\boldsymbol{u})$ is expressed by (2.27) as

$$T_l(\boldsymbol{u}) = \sum_{1\le i_1<\dots<i_l\le d} \{n_{i_1} u_{i_1}\} \dots \{n_{i_l} u_{i_l}\} \Delta^{(i_1)}_{[n_{i_1} u_{i_1}]+1} \dots \Delta^{(i_l)}_{[n_{i_l} u_{i_l}]+1} S_{[\boldsymbol{nu}]}.$$

As above, all the terms for which $i_1 \ge 2$ disappear in the difference $T_l(\boldsymbol{u'}) - T_l(\boldsymbol{u})$ and we obtain

$$T_l(\boldsymbol{u'}) - T_l(\boldsymbol{u}) = n_1(t' - t) \sum_{1<i_2<\dots<i_l\le d} \{n_{i_2} s_{i_2}\} \dots \{n_{i_l} s_{i_l}\} \Delta^{(1)}_{[n_1 t]+1} \Delta^{(i_2)}_{[n_{i_2} s_{i_2}]+1} \dots \Delta^{(i_l)}_{[n_{i_l} s_{i_l}]+1} S_{[\boldsymbol{nu}]}.$$

Since $\{n_{i_2} s_{i_2}\} \dots \{n_{i_l} s_{i_l}\} < 1$ and

$$\left\|\Delta^{(1)}_{[n_1 t]+1} \Delta^{(i_2)}_{[n_{i_2} s_{i_2}]+1} \dots \Delta^{(i_l)}_{[n_{i_l} s_{i_l}]+1} S_{[\boldsymbol{nu}]}\right\| = \left\|\Delta^{(1)}_{[n_1 t]+1} \sum_{\boldsymbol{i}\in I} \varepsilon_{\boldsymbol{i}} S_{\boldsymbol{i}}\right\| \le \sum_{\boldsymbol{i}\in I} \left\|\Delta^{(1)}_{[n_1 t]+1} S_{\boldsymbol{i}}\right\|,$$

where $\varepsilon_{\boldsymbol{i}} = \pm 1$ and I is some appropriate subset of $[0, \boldsymbol{n}] \cap \mathbb{N}^d$ with 2^{l-1} elements. Hence with $Z_{\boldsymbol{n}}$ defined by (2.30), we obtain for $l \ge 2$

$$\|T_l(\boldsymbol{u'}) - T_l(\boldsymbol{u})\| \le n_1(t' - t)\binom{d-1}{l-1} 2^{l-1} Z_{\boldsymbol{n}}.$$

Clearly this estimate holds true also for $l = 1$, so going back to (2.35) and recalling that in the case under consideration $[\boldsymbol{nu'}] = [\boldsymbol{nu}]$, we obtain

$$\|\xi_{\boldsymbol{n}}(\boldsymbol{u'}) - \xi_{\boldsymbol{n}}(\boldsymbol{u})\| \leq \sum_{l=1}^{d} n_1(t'-t)\binom{d-1}{l-1}2^{l-1}Z_{\boldsymbol{n}} = 3^{d-1}n_1(t'-t)Z_{\boldsymbol{n}}. \quad (2.36)$$

Case 1.b. $n_1 t < [n_1 t'] \leq n_1 t'$. Using chaining to exploit the result of case 1.a, we obtain

$$\begin{aligned}\|\xi_{\boldsymbol{n}}(\boldsymbol{u'}) - \xi_{\boldsymbol{n}}(\boldsymbol{u})\| &\leq \left\|\xi_{\boldsymbol{n}}(\boldsymbol{u'}) - \xi_{\boldsymbol{n}}\Big(\frac{[n_1t']}{n_1}, \boldsymbol{s}\Big)\right\| + \left\|\xi_{\boldsymbol{n}}\Big(\frac{[n_1t']}{n_1}, \boldsymbol{s}\Big) - \xi_{\boldsymbol{n}}(\boldsymbol{u})\right\| \\ &\leq 3^{d-1}(n_1t' - [n_1t'])Z_{\boldsymbol{n}} + 3^{d-1}([n_1t'] - n_1t)Z_{\boldsymbol{n}} \\ &= 3^{d-1}n_1(t'-t)Z_{\boldsymbol{n}}. \end{aligned} \quad (2.37)$$

Case 2. $t' - t \geq 1/n_1$. Then $[n_1t] \leq n_1t < [n_1t] + 1 \leq [n_1t'] \leq n_1t'$ and putting

$$t_1 := \frac{[n_1t]}{n_1},\ t_1' := \frac{[n_1t']}{n_1}, \quad \boldsymbol{v} := (t_1, \boldsymbol{s}),\ \boldsymbol{v'} := (t_1', \boldsymbol{s}),$$

we get the upper bound

$$\begin{aligned}\|\xi_{\boldsymbol{n}}(\boldsymbol{u'}) - \xi_{\boldsymbol{n}}(\boldsymbol{u})\| \leq \|\xi_{\boldsymbol{n}}(\boldsymbol{u'}) - \xi_{\boldsymbol{n}}(\boldsymbol{v'})\| + \|\xi_{\boldsymbol{n}}(\boldsymbol{v'}) - \xi_{\boldsymbol{n}}(\boldsymbol{v})\| \\ + \|\xi_{\boldsymbol{n}}(\boldsymbol{v}) - \xi_{\boldsymbol{n}}(\boldsymbol{u})\|,\end{aligned}$$

where the first and third terms fall within the case 1 since $t' - t_1' < 1/n_1$ and $t - t_1 < 1/n_1$. As $n_1v_1 = n_1t_1 = [n_1t]$, we have

$$[\boldsymbol{nv}] = ([n_1t_1], [\boldsymbol{n}_{2:d}\boldsymbol{s}]) = [\boldsymbol{nu}] \quad \text{and} \quad \{n_1v_1\} = \{[n_1t]\} = 0,$$

so the representation (2.14) for $\xi_{\boldsymbol{n}}(\boldsymbol{v})$ may be recast as

$$\xi_{\boldsymbol{n}}(\boldsymbol{v}) = S_{[\boldsymbol{nu}]} + \sum_{l=1}^{d-1} \sum_{2\leq i_1<i_2<\cdots<i_l\leq d} \left(\prod_{k=1}^{l}\{n_{i_k}v_{i_k}\}\right)\left(\prod_{k=1}^{l}\Delta^{(i_k)}_{[n_{i_k}v_{i_k}]+1}\right)S_{[\boldsymbol{nu}]}.$$

Clearly the same representation holds for $\xi_{\boldsymbol{n}}(\boldsymbol{v'})$, by just replacing $\boldsymbol{u}$ by $\boldsymbol{u'}$. Now since Δ's are interchangable and

$$S_{[\boldsymbol{nu'}]} - S_{[\boldsymbol{nu}]} = \sum_{i=[nt]+1}^{[nt']} \Delta_i^{(1)} S_{(i,[\boldsymbol{n}_{2:d}\boldsymbol{s}])},$$

we get

$$\|\xi_{\boldsymbol{n}}(\boldsymbol{v'}) - \xi_{\boldsymbol{n}}(\boldsymbol{v})\| \leq \psi_{\boldsymbol{n}}(t',t)\sum_{l=0}^{d-1}\binom{d-1}{l}2^l = 3^{d-1}\psi_{\boldsymbol{n}}(t',t),$$

with $\psi_{\boldsymbol{n}}(t',t)$ defined by (2.29). Using case 1 to bound $\|\xi_{\boldsymbol{n}}(\boldsymbol{u'}) - \xi_{\boldsymbol{n}}(\boldsymbol{v'})\|$ and $\|\xi_{\boldsymbol{n}}(\boldsymbol{v}) - \xi_{\boldsymbol{n}}(\boldsymbol{u})\|$, we obtain

$$\begin{aligned}\|\xi_{\boldsymbol{n}}(t',s) - \xi_{\boldsymbol{n}}(t,s)\| &\leq 3^{d-1}\{n_1t'\}Z_{\boldsymbol{n}} + 3^{d-1}\psi_{\boldsymbol{n}}(t',t) + 3^{d-1}\{n_1t\}Z_{\boldsymbol{n}} \\ &\leq 3^{d-1}\psi_{\boldsymbol{n}}(t',t) + 2\cdot 3^{d-1}Z_{\boldsymbol{n}}. \end{aligned} \tag{2.38}$$

Combining (2.36), (2.37) and (2.38) we complete the proof of lemma 13. □

2.2 Unequal variance

Let us examine a simple example. Take collection $\{X_{n,k}, k = 1,\dots,2n\}$ of random variables i.i.d. for each n with zero mean and variance $\mathbf{E}\,X_{n,k}^2 = 1/(2n)$. Then the sum $S_{n,2n} = X_{n,1}+\dots+X_{n,2n}$ converges to standard normal and the polygonal line process $\xi_n(t) = S_{n,[2nt]} + (2nt-[2nt])X_{[2nt]+1}$ converges to Brownian motion. Now introduce collection $\{Y_{n,k}, k = 1,\dots,2n\}$ by taking $Y_{n,k} = 1/\sqrt{2}X_{n,k}$ for $k \leq n$, and $Y_{n,k} = \sqrt{3/2}X_{n,k}$ for $k = n+1,\dots,2n$. The sum $S'_{n,2n} = Y_{n,1} + \dots + Y_{n,2n}$ still converges to standard normal. But for polygonal line process $\xi'_n(t) = S'_{n,[2nt]} + (2nt-[2nt])Y_{[2nt]+1}$ we then have $\mathbf{E}\,(\xi_n(1/2)')^2 = 1/4$ for all n. Thus if functional central limit theorem holds the variance of limiting process at $1/2$ is $1/4$. Yet Brownian motion variance at $1/2$ is $1/2$, thus the limiting process (if it exists) in this case is not the Brownian motion. From classical result of Prokhorov [23] we know that for triangular arraysit is possible to use different construction of summation process so that the functional central limit theorem holds and the limiting process is always Brownian motion. Furthermore in the case of i.i.d. random variables both definitions coincide. We propose similar definition for $[0,1]^d$ indexed summation processes, which though does not solve the problem completely as in case $d = 1$, is nevertheless an improvement on using the definition (2.4).

2.2.1 Definitions and representations

Let us first review the case of $[0,1]$ indexed summation process, i.e. the classical result of Prokhorov [23] for triangular arrays. Suppose we have collection $\{X_{n,k}, 1 \leq k \leq k_n, k_n, n \in \mathbb{N}\}$ of random variables. Let $\mathbf{E}\,X_{n,k}^2 =$

$\sigma^2_{n,k}$ and assume that $\sum_{k=1}^{k_n} \sigma^2_{n,k} = 1$. Define

$$S_n(k) = X_{n,1} + \cdots + X_{n,k}$$

and

$$b_n(k) = \sigma^2_{n,1} + \cdots + \sigma^2_{n,k}.$$

Then classical definition of summation process is

$$\xi_n(t) = S_n(k) + (t - b_n(k))\sigma^{-2}_{n,k+1} X_{n,k+1}, \text{ for } b_n(k) \le t < b_n(k+1). \quad (2.39)$$

Define

$$u_n(t) = \min(k : b_n(k) < t),$$

then

$$\mathbf{E}\,\xi_n(t)^2 = \sum_{k=1}^{u_n(t)} \sigma^2_{n,k} + \frac{(t - b_n(k))^2}{\sigma^2_{n,k+1}}.$$

If we assume

$$\max_{1 \le k \le k_n} \sigma^2_{n,k} \to 0, \text{ as } n \to \infty$$

which is necessary for central limit theorem to apply, we have

$$\mathbf{E}\,\xi_n(t)^2 \to t. \quad (2.40)$$

Define triangular array with multidimensional index as

$$(X_{\boldsymbol{n},\boldsymbol{k}},\ \mathbf{1} \le \boldsymbol{k} \le \boldsymbol{k_n}),\ \boldsymbol{n} \in \mathbb{N}^d,$$

where for each $\boldsymbol{n}$ the random variables $X_{\boldsymbol{n},\boldsymbol{k}}$ are independent. The expression $\boldsymbol{k_n}$ is the element from $\mathbb{N}^d$ with multidimensional index: $\boldsymbol{k_n} = (k^1_{\boldsymbol{n}}, \ldots, k^d_{\boldsymbol{n}})$. Assume that $\mathbf{E}\,X_{\boldsymbol{n},\boldsymbol{k}} = 0$ and that $\sigma^2_{\boldsymbol{n},\boldsymbol{k}} := \mathbf{E}\,X^2_{\boldsymbol{n},\boldsymbol{k}} < \infty$, for $\mathbf{1} \le \boldsymbol{k} \le \boldsymbol{k_n}$, $\boldsymbol{n} \in \mathbb{N}^d$. Define for each $\mathbf{1} \le \boldsymbol{k} \le \boldsymbol{k_n}$

$$S_{\boldsymbol{n}}(\boldsymbol{k}) := \sum_{\boldsymbol{j} \le \boldsymbol{k}} X_{\boldsymbol{n},\boldsymbol{j}}, \quad b_{\boldsymbol{n}}(\boldsymbol{k}) := \sum_{\boldsymbol{j} \le \boldsymbol{k}} \sigma^2_{\boldsymbol{n},\boldsymbol{j}}.$$

We require that the sum of all variances is one, i.e. $b_{\boldsymbol{n}}(\boldsymbol{k_n}) = 1$ and that $\mathrm{m}(\boldsymbol{k_n}) \to \infty$, as $\mathrm{m}(\boldsymbol{n}) \to \infty$. Note that these requirements are the same as for one-dimensional triangular array.

If $\boldsymbol{\pi}(\boldsymbol{k}) = 0$, let $S_{\boldsymbol{n}}(\boldsymbol{k}) = 0$, $b_{\boldsymbol{n}}(\boldsymbol{k}) = 0$. For $i = 1, \ldots, d$ introduce the notations

$$
\begin{aligned}
b_i(k) &:= b_{\boldsymbol{n}}(k_{\boldsymbol{n}}^1, \dots, k_{\boldsymbol{n}}^{i-1}, k, k_{\boldsymbol{n}}^{i+1}, \dots, k_{\boldsymbol{n}}^d), \\
\Delta b_i(k) &:= b_i(k) - b_i(k-1)
\end{aligned} \tag{2.41}
$$

and

$$
\boldsymbol{B}(\boldsymbol{k}) = (b_1(k_1), \dots, b_d(k_d)), \quad \Delta\boldsymbol{B}(\boldsymbol{k}) := (\Delta b_1(k_1), \dots, \Delta b_d(k_d)) \tag{2.42}
$$

for its vector counterparts. Note that these variables depend on $\boldsymbol{n}$ and $\boldsymbol{k_n}$. For $\boldsymbol{1} \leq \boldsymbol{k} \leq \boldsymbol{k_n}$ let

$$
Q_{\boldsymbol{n},\boldsymbol{k}} := \Big[b_1(k_1-1),\ b_1(k_1)\Big) \times \cdots \times \Big[b_d(k_d-1),\ b_d(k_d)\Big). \tag{2.43}
$$

Due to definition of $b_i(k)$ we have $Q_{\boldsymbol{n},\boldsymbol{j}} \cap Q_{\boldsymbol{n},\boldsymbol{k}} = \varnothing$, if $\boldsymbol{k} \neq \boldsymbol{j}$, Also $\cup_{\boldsymbol{k} \leq \boldsymbol{k_n}} Q_{\boldsymbol{n},\boldsymbol{k}} = [0,1)^d$ and $\sum_{\boldsymbol{k} \leq \boldsymbol{k_n}} |Q_{\boldsymbol{n},\boldsymbol{k}}| = 1$ with $|Q_{\boldsymbol{n},\boldsymbol{k}}| = \boldsymbol{\pi}(\Delta\boldsymbol{B}(\boldsymbol{k}))$. Thus any $\boldsymbol{t} \in [0,1)^d$ falls into unique rectangle $Q_{\boldsymbol{n},\boldsymbol{k}}$, for some $\boldsymbol{k}$. In that case trivial equality

$$
t_i = b_i(k_i - 1) + \frac{t_i - b_i(k_i-1)}{\Delta b_i(k_i)} \Delta b_i(k_i)
$$

gives

$$
\boldsymbol{t} = \boldsymbol{B}(\boldsymbol{k}-\boldsymbol{1}) + \frac{\boldsymbol{t} - \boldsymbol{B}(\boldsymbol{k}-\boldsymbol{1})}{\Delta\boldsymbol{B}(\boldsymbol{k})} \Delta\boldsymbol{B}(\boldsymbol{k})
$$

with

$$
\boldsymbol{0} \leq \frac{\boldsymbol{t} - \boldsymbol{B}(\boldsymbol{k}-\boldsymbol{1})}{\Delta\boldsymbol{B}(\boldsymbol{k})} < \boldsymbol{1}.
$$

This corresponds to decomposition

$$
\boldsymbol{t} = \frac{[\boldsymbol{nt}]}{\boldsymbol{n}} + \frac{\{\boldsymbol{nt}\}}{\boldsymbol{n}}.
$$

It is natural then that summation process defined on the grid $Q_{\boldsymbol{n},\boldsymbol{k}}$ as

$$
\Xi_{\boldsymbol{n}}(\boldsymbol{t}) = \sum_{\boldsymbol{1} \leq \boldsymbol{j} \leq \boldsymbol{n}} |Q_{\boldsymbol{n},\boldsymbol{j}}|^{-1} |Q_{\boldsymbol{n},\boldsymbol{j}} \cap [0, \boldsymbol{t}]| X_{\boldsymbol{n},\boldsymbol{j}}, \tag{2.44}
$$

similar to (2.4) assumes the same representations as process $\xi_{\boldsymbol{n}}(\boldsymbol{t})$. For $t \in$

$[0,1]$ and $\boldsymbol{t} \in [0,1]^d$, write

$$u_i(t) := \max\{j \geq 0 : b_i(j) \leq t\}, \quad \boldsymbol{U}(\boldsymbol{t}) := (u_1(t_1), \dots, u_d(t_d)).$$

Then following proposition holds.

Proposition 14 *For $\boldsymbol{t} \in [0,1)^d$, denote*

$$\boldsymbol{s} = \frac{\boldsymbol{t} - \boldsymbol{B}(\boldsymbol{U}(\boldsymbol{t}))}{\Delta \boldsymbol{B}(\boldsymbol{U}(\boldsymbol{t}) + 1)}$$

and write vertices of the rectangle $R_{\boldsymbol{n},\boldsymbol{U}(\boldsymbol{t})+\mathbf{1}}$ as

$$V(\boldsymbol{u}) := \boldsymbol{B}(\boldsymbol{U}(\boldsymbol{t})) + \boldsymbol{u}\Delta \boldsymbol{B}(\boldsymbol{U}(\boldsymbol{t}) + \mathbf{1}). \quad \boldsymbol{u} \in \{0,1\}^d. \tag{2.45}$$

It is possible to express $\boldsymbol{t}$ as a barycenter of these 2^d vertices with weights $w(\boldsymbol{u}) \geq 0$ depending on $\boldsymbol{t}$, i.e.,

$$\boldsymbol{t} = \sum_{\boldsymbol{u} \in \{0,1\}^d} w(\boldsymbol{u}) V(\boldsymbol{u}), \quad \text{where} \quad \sum_{\boldsymbol{u} \in \{0,1\}^d} w(\boldsymbol{u}) = 1, \tag{2.46}$$

where

$$w(\boldsymbol{u}) = \prod_{l=1}^{d} s_l^{u_l} (1 - s_l)^{1-u_l}.$$

Using this representation, define the random field Ξ_n^* by

$$\Xi_{\boldsymbol{n}}^*(\boldsymbol{t}) = \sum_{\boldsymbol{u} \in \{0,1\}^d} w(\boldsymbol{u}) S_{\boldsymbol{n}}(\boldsymbol{U}(\boldsymbol{t}) + \boldsymbol{u}), \quad \boldsymbol{t} \in [0,1]^d.$$

Then $\Xi_{\boldsymbol{n}}^*$ coincides with the summation process defined by (2.44). Furthermore $\Xi_{\boldsymbol{n}}(\boldsymbol{t})$ admits representation

$$\Xi_{\boldsymbol{n}}(\boldsymbol{t}) = S_{\boldsymbol{n}}(\boldsymbol{U}(\boldsymbol{t})) + \sum_{l=1}^{d} \sum_{1 \leq i_1 < i_2 < \cdots < i_l \leq d} \left(\prod_{k=1}^{l} \frac{t_{i_k} - b_{i_k}(u_{i_k})}{\Delta b_{i_k}(u_{i_k}(t_{i_k}) + 1)} \right) \left(\prod_{k=1}^{l} \Delta_{u_{i_k}(t_{i_k})+1}^{(i_k)} \right) S_{\boldsymbol{n}}(\boldsymbol{U}(\boldsymbol{t})). \tag{2.47}$$

Proof. The proof is the same as in proposition 12 with the change of notation: $[\boldsymbol{nt}]$ changed to $\boldsymbol{U}(\boldsymbol{t})$ and $\{\boldsymbol{nt}\}$ to $(\boldsymbol{t} - \boldsymbol{B}(\boldsymbol{U}(\boldsymbol{t})))/\Delta \boldsymbol{B}(\boldsymbol{U}(\boldsymbol{t}) + 1)$.

2.2.2 Estimate of sequential norm

We give now the estimate of sequential norm of $\Xi_{\boldsymbol{n}}$. The estimate is similar to the one given in section 2.1.3 for process $\xi_{\boldsymbol{n}}$. As in section 2.1.3 we can write

$$\max_{\boldsymbol{v}\in V_j} \|\lambda_{j,\boldsymbol{v}}(\Xi_{\boldsymbol{n}})\| \leq \sum_{m=1}^{d} \max_{\substack{0\leq k<2^j \\ \boldsymbol{0}\leq\boldsymbol{\ell}\leq\boldsymbol{2}^j}} \Delta_{\boldsymbol{n}}^{(m)}(t_{k+1}, t_k; \boldsymbol{s}_{\boldsymbol{\ell}}) \tag{2.48}$$

with $t_k = k2^{-j}$, $\boldsymbol{\ell} = (l_2, \dots, l_d)$, $\boldsymbol{2}^j = (2^j, \dots, 2^j)$ (vector of dimension $d-1$), $\boldsymbol{s}_\ell = \boldsymbol{\ell}\boldsymbol{2}^{-j}$, and $\Delta_{\boldsymbol{n}}^{(m)}(t, t'; \boldsymbol{s})$ defined for $m = 1$ as

$$\Delta_{\boldsymbol{n}}^{(1)}(t, t'; \boldsymbol{s}) := |\Xi_{\boldsymbol{n}}(t', \boldsymbol{s}) - \Xi_{\boldsymbol{n}}(t, \boldsymbol{s})|$$

and similarly for other coordinates for $m > 1$. Introduce set $D_j = \{2(l-1)2^{-j}; 1 \leq l \leq 2^{j-1}\}$ and notation $r^- = r - 2^{-j}$ and $r^+ = r + 2^{-j}$ for $r \in D_j$. Then

$$\max_{\boldsymbol{v}\in V_j} \|\lambda_{j,\boldsymbol{v}}(\Xi_{\boldsymbol{n}})\| \leq \sum_{m=1}^{d} \max\{ \max_{\substack{r\in D_j \\ \boldsymbol{0}\leq\boldsymbol{\ell}\leq\boldsymbol{2}^j}} \Delta_{\boldsymbol{n}}^{(m)}(r, r^-; \boldsymbol{s}_{\boldsymbol{\ell}}), \max_{\substack{r\in D_j \\ \boldsymbol{0}\leq\boldsymbol{\ell}\leq\boldsymbol{2}^j}} \Delta_{\boldsymbol{n}}^{(m)}(r^+, r; \boldsymbol{s}_{\boldsymbol{\ell}})\}. \tag{2.49}$$

At first glance this separation seems unnecessary, especially since the treatment of both $\Delta_{\boldsymbol{n}}^{(m)}(r, r^-; \boldsymbol{s}_{\boldsymbol{\ell}})$ and $\Delta_{\boldsymbol{n}}^{(m)}(r, r^+; \boldsymbol{s}_{\boldsymbol{\ell}})$ is identical, but this simplifies the proofs later on. Similar lemma to 13 holds.

Lemma 15 *For $m = 1, \dots, d$ and any $r \in D_j$*

$$\sup_{\boldsymbol{s}\in[0,1]^{d-1}} \Delta_{\boldsymbol{n}}^{(m)}(r, r^-; \boldsymbol{s}) \leq 3^d \mathbf{1}\left(u_m(r) > u_m(r^-) + 1\right) \psi_{\boldsymbol{n}}^{(m)}(r, r^-) + 3^d 2^{-j\alpha} Z_{\boldsymbol{n}}^{(m)},$$

where

$$\psi_{\boldsymbol{n}}(r, r^-)^{(m)} := \max_{\boldsymbol{k}_{-m}\leq(\boldsymbol{k}_{\boldsymbol{n}})_{-m}} \left| \sum_{k_m=u_m(r^-)+2}^{u_m(r)} \Delta_{k_m}^{(m)} S_{\boldsymbol{n}}(\boldsymbol{k}) \right| \tag{2.50}$$

$$Z_{\boldsymbol{n}}^{(m)} := \max_{\boldsymbol{1}\leq\boldsymbol{k}\leq\boldsymbol{k}_{\boldsymbol{n}}} \frac{|\Delta_{k_m}^{(m)} S_{\boldsymbol{n}}(\boldsymbol{k})|}{(\Delta b_m(k_m))^\alpha}. \tag{2.51}$$

Similarly

Lemma 16 *For $m = 1, \ldots, d$ and any $r \in D_j$*

$$\sup_{\boldsymbol{s}\in[0,1]^{d-1}} \Delta_{\boldsymbol{n}}^{(m)}(r, r^+; \boldsymbol{s}) \leq 3^d \mathbf{1}\left(u_m(r) > u_m(r^-) + 1\right) \psi_{\boldsymbol{n}}^{(m)}(r, r^+) + 3^d 2^{-j\alpha} Z_{\boldsymbol{n}}^{(m)},$$

where

$$\psi_{\boldsymbol{n}}(r, r^+)^{(m)} := \max_{\boldsymbol{k}_{-m}\leq(\boldsymbol{k_n})_{-m}} \left| \sum_{k_m=u_m(r)+2}^{u_m(r^+)} \Delta_{k_m}^{(m)} S_{\boldsymbol{n}}(\boldsymbol{k}) \right|. \tag{2.52}$$

with $Z_{\boldsymbol{n}}^{(m)}$ as in (2.51).

Note that only definitions of $\psi_{\boldsymbol{n}}$ differs and $Z_{\boldsymbol{n}}^{(m)}$ does not depend on j. We make no distinction for different j and derive immediately estimate of sequential norm

$$\|\Xi_{\boldsymbol{n}}\|_\alpha^{\mathrm{seq}} \leq 3^d \sum_{m=1}^d \left(\max_{j\geq 0} 2^{j\alpha} \max_{r\in D_j} [\psi_{\boldsymbol{n}}^{(m)}(r, r^-) + \psi_{\boldsymbol{n}}^{(m)}(r, r^+)] + Z_{\boldsymbol{n}}^{(m)} \right) \tag{2.53}$$

and the tail

$$\|\Xi_{\boldsymbol{n}} - E_J \Xi_{\boldsymbol{n}}\|_\alpha^{\mathrm{seq}} \leq 3^d \sum_{m=1}^d \left(\max_{j\geq J} 2^{j\alpha} \max_{r\in D_j} [\psi_{\boldsymbol{n}}^{(m)}(r, r^-) + \psi_{\boldsymbol{n}}^{(m)}(r, r^+)] + Z_{\boldsymbol{n}}^{(m)} \right). \tag{2.54}$$

Proof of lemma 15

We prove this for $m = 1$ since the proof is the same for other m, subsequently we drop the superscript in definitions $Z_{\boldsymbol{n}}$ and $\psi_{\boldsymbol{n}}$. The proof is similar to proof of lemma 13. Denote by $\mathbf{v} = (r, \boldsymbol{s})$, and $\mathbf{v}^- = (r^-, \boldsymbol{s})$. Recall representation (2.47) and write

$$T_l(\boldsymbol{t}) = \sum_{1\leq i_1<i_2<\cdots<i_l\leq d} \left(\prod_{k=1}^l \frac{t_{i_k} - b_{i_k}(u_{i_k})}{\Delta b_{i_k}(u_{i_k}(t_{i_k}) + 1)} \right) \left(\prod_{k=1}^l \Delta_{u_{i_k}(t_{i_k})+1}^{(i_k)} \right) S_{\boldsymbol{n}}(\boldsymbol{U}(\boldsymbol{t}))$$

for $l = 1, \dots, d$. Then

$$\begin{aligned}\Xi_{\boldsymbol{n}}(r, s_{\boldsymbol{\ell}}) - \Xi_{\boldsymbol{n}}(r^-, s_{\boldsymbol{\ell}}) = S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{v})) - S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{v}^-)) \\ + \sum_{l=1}^{d}(T_l(\mathbf{v}) - T_l(\mathbf{v}^-)).\end{aligned}$$

To estimate this increment we discuss according to following configurations *Case 1.* $u_1(r) = u_1(r^-)$. Consider first the increment $T_1(\mathbf{v}) - T_1(\mathbf{v}^-)$ and note that by (2.27) with $l = 1$,

$$T_1(\mathbf{v}) = \sum_{1 \le k \le d} \frac{\mathrm{v}_k - b_k(u_k(\mathrm{v}_k))}{\Delta b_k(u_k(\mathrm{v}_k) + 1)} \Delta^{(k)}_{u_k(\mathrm{v}_k)+1} S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{v})).$$

Because $\mathbf{v}_{2:d} = \mathbf{v}^-_{2:d}$ and $\boldsymbol{U}(\mathbf{v}) = \boldsymbol{U}(\mathbf{v}^-)$, all terms indexed by $k \ge 2$ disappear in difference $T_1(\mathbf{v}) - T_1(\mathbf{v}^-)$. This leads to the factorisation

$$T_1(\mathbf{v}) - T_1(\mathbf{v}^-) = \frac{r - r^-}{\Delta b_1(u_1(r) + 1)} \Delta^{(1)}_{u_1(r)+1} S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{v})). \tag{2.55}$$

For $l \ge 2$, $T_l(\mathbf{v})$ is expressed by (2.27) as

$$\begin{aligned}T_l(\mathbf{v}) = \sum_{1 \le i_1 < \dots < i_l \le d} \frac{\mathrm{v}_{i_1} - b_{i_1}(u_{i_1}(\mathrm{v}_{i_1}))}{\Delta b_{i_1}(u_{i_1}(\mathrm{v}_{i_1}) + 1)} \cdots \frac{\mathrm{v}_{i_l} - b_{i_l}(u_{i_l}(\mathrm{v}_{i_l}))}{\Delta b_{i_l}(u_{i_l}(\mathrm{v}_{i_l}) + 1)} \\ \Delta^{(i_1)}_{u_{i_1}(\mathrm{v}_{i_1})+1} \cdots \Delta^{(i_l)}_{u_{i_l}(\mathrm{v}_{i_l})+1} S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{v})).\end{aligned}$$

In the difference $T_1(\mathbf{v}) - T_1(\mathbf{v}^-)$ all the terms for which $i_1 \ge 2$ again disappear and we obtain

$$\begin{aligned}T_l(\mathbf{v}) - T_l(\mathbf{v}^-) = \frac{r - r^-}{\Delta b_1(u_1(r) + 1)} \\ \sum_{1 < i_2 < \dots < i_l \le d} \frac{\mathrm{v}_{i_2} - b_{i_2}(u_{i_2}(\mathrm{v}_{i_2}))}{\Delta b_{i_1}(u_{i_1}(\mathrm{v}_{i_1}) + 1)} \cdots \frac{\mathrm{v}_{i_l} - b_{i_l}(u_{i_l}(\mathrm{v}_{i_l}))}{\Delta b_{i_l}(u_{i_l}(\mathrm{v}_{i_l}) + 1)} \\ \Delta^{(1)}_{u_1(r)+1} \Delta^{(i_2)}_{u_{i_2}(\mathrm{v}_{i_2})+1} \cdots \Delta^{(i_l)}_{u_{i_l}(\mathrm{v}_{i_l})+1} S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{v})).\end{aligned} \tag{2.56}$$

Since $u_1(r) = u_1(r^-)$, we have $b_1(u_1(r)) \le r < r^- < b_1(u_1(r) + 1)$, thus

$$\frac{r - r^-}{\Delta b_1(u_1(r) + 1)} \le \left(\frac{r - r^-}{\Delta b_1(u_1(r) + 1)} \right)^{\alpha}.$$

Now

$$\frac{\mathrm{v}_{i_2} - b_{i_2}(u_{i_2}(\mathrm{v}_{i_2}))}{\Delta b_{i_1}(u_{i_1}(\mathrm{v}_{i_1}) + 1)} \cdots \frac{\mathrm{v}_{i_l} - b_{i_l}(u_{i_l}(\mathrm{v}_{i_l}))}{\Delta b_{i_l}(u_{i_l}(\mathrm{v}_{i_l}) + 1)} < 1$$

and

$$\begin{aligned} |\Delta^{(1)}_{u_1(r)+1} \Delta^{(i_2)}_{u_{i_2}(\mathrm{v}_{i_2})+1} \dots \Delta^{(i_l)}_{u_{i_l}(\mathrm{v}_{i_l})+1} S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{v}))| &= |\Delta^{(1)}_{u_1(r)+1} \sum_{\boldsymbol{i} \in I} \varepsilon_{\boldsymbol{i}} S_{\boldsymbol{n}}(\boldsymbol{i})| \\ &\leq \sum_{\boldsymbol{i} \in I} |\Delta^{(1)}_{u_1(r)+1} S_{\boldsymbol{n}}(\boldsymbol{i})|, \end{aligned} \tag{2.57}$$

where $\varepsilon_{\boldsymbol{i}} = \pm 1$ and I is some appropriate subset of $[0, \boldsymbol{n}] \cap \mathbb{N}^d$ with 2^{l-1} elements. Recall that $Z_{\boldsymbol{n}}$ is defined as

$$Z_{\boldsymbol{n}} = \max_{1 \leq \boldsymbol{k} \leq \boldsymbol{k}_{\boldsymbol{n}}} \frac{|\Delta^{(1)}_{k_1} S_{\boldsymbol{n}}(\boldsymbol{k})|}{(\Delta b_1(k_1))^\alpha}.$$

Now noting that $r - r^- = 2^{-j}$ and $\Delta b_1(k_1)$ depends only on k_1, we obtain for $l \geq 2$

$$|T_l(\boldsymbol{u'}) - T_l(\boldsymbol{u})| \leq 2^{-j\alpha} \binom{d-1}{l-1} 2^{l-1} Z_{\boldsymbol{n}}.$$

Thus

$$|\Xi_{\boldsymbol{n}}(\mathbf{v}) - \Xi_{\boldsymbol{n}}(\mathbf{v}^-)| \leq \sum_{l=1}^{d} 2^{-j\alpha} \binom{d-1}{l-1} 2^{l-1} Z_{\boldsymbol{n}} = 3^{d-1} 2^{-j\alpha} Z_{\boldsymbol{n}}. \tag{2.58}$$

Case 2. $u_1(r) = u_1(r^-) + 1$. In this case we have $b_1(u_1(r^-)) \leq r^- < b_1(u_1(r)) \leq r$. Using previous definitions we can write

$$\begin{aligned} |\Xi_{\boldsymbol{n}}(\mathbf{v}) - \Xi_{\boldsymbol{n}}(\mathbf{v}^-)| &\leq |\Xi_{\boldsymbol{n}}(\mathbf{v}) - \Xi_{\boldsymbol{n}}(b_1(u_1(r)), \boldsymbol{s}_\ell)| \\ &\quad + |\Xi_{\boldsymbol{n}}(b_1(u_1(r)), \boldsymbol{s}_\ell) - \Xi_{\boldsymbol{n}}(\mathbf{v}^-)|. \end{aligned}$$

Now

$$\frac{r - b_1(u_1(r))}{\Delta b_1(u_1(r) + 1)} \leq \left(\frac{r - b_1(u_1(r))}{\Delta b_1(u_1(r) + 1)} \right)^\alpha \leq \frac{2^{-j\alpha}}{(\Delta b_1(u_1(r) + 1))^\alpha}$$

and similarly

$$\frac{b_1(u_1(r)) - r^-}{\Delta b_1(u_1(r^-) + 1)} \leq \frac{2^{-j\alpha}}{(\Delta b_1(u_1(r^-) + 1))^\alpha}.$$

Combining these inequalities with (2.55) and (2.56) we get as in(2.36)

$$|\Xi_{\boldsymbol{n}}(\mathbf{v}) - \Xi_{\boldsymbol{n}}(\mathbf{v}^-)| \le 2 \cdot 3^{d-1} 2^{-j\alpha} Z_{\boldsymbol{n}}.$$

Case 3. $u_1(r) > u_1(r^-) + 1$. Put

$$\mathbf{u} = (b_1(u_1(r)), \boldsymbol{s}_\ell), \quad \mathbf{u}^- = (b_1(u_1(r^-)) + 1, \boldsymbol{s}_\ell).$$

Then

$$|\Xi_{\boldsymbol{n}}(\mathbf{v}) - \Xi_{\boldsymbol{n}}(\mathbf{v}^-)| \le |\Xi_{\boldsymbol{n}}(\mathbf{v}) - \Xi_{\boldsymbol{n}}(\mathbf{u})| + |\Xi_{\boldsymbol{n}}(\mathbf{u}) - \Xi_{\boldsymbol{n}}(\mathbf{u}^-)| \\ + |\Xi_{\boldsymbol{n}}(\mathbf{u}^-) - \Xi_{\boldsymbol{n}}(\mathbf{v}^-)|.$$

Since $\boldsymbol{U}(\mathbf{u})_{2:d} = \boldsymbol{U}(\mathbf{u}^-)_{2:d} = \boldsymbol{U}(\mathbf{v})_{2:d}$, we have

$$\Xi_{\boldsymbol{n}}(\mathbf{u}) = S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{u})) + \sum_{l=1}^{d-1} \sum_{2 \le i_1 < i_2 < \cdots < i_l \le d} \left(\prod_{k=1}^{l} \frac{\mathrm{v}_{i_k} - b_{i_k}(u_{i_k}(\mathrm{v}_{i_k}))}{\Delta b_{i_k} u_{i_k}(\mathrm{v}_{i_k}) + 1)} \right) \\ \left(\prod_{k=1}^{l} \Delta^{(i_k)}_{u_{i_k}(\mathrm{v}_{i_k})+1} \right) S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{u}))$$

and similar representation holds for $\Xi_n(\mathbf{u}^-)$. We have

$$S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{u})) - S_{\boldsymbol{n}}(\boldsymbol{U}(\mathbf{u}^-)) = \sum_{i=u_1(r^-)+2}^{u_1(r)} \Delta_i^{(1)} S_{\boldsymbol{n}}((i, \boldsymbol{U}(\boldsymbol{s}_\ell))).$$

Recalling the definition

$$\psi_{\boldsymbol{n}}(r, r^-) = \max_{\boldsymbol{k}_{2:d} \le \boldsymbol{k}_{\boldsymbol{n},2:d}} \left| \sum_{i=u_1(r^-)+2}^{u_1(r)} \Delta_i^{(1)} S_{\boldsymbol{n}}((i, \boldsymbol{k}_{2:d})) \right|,$$

similar to (2.57) and (2.58) we get

$$|\Xi_{\boldsymbol{n}}(\mathbf{u}) - \Xi_{\boldsymbol{n}}(\mathbf{u}^-)| \le \psi_{\boldsymbol{n}}(r, r^-) \sum_{l=0}^{d-1} 2^l \le 3^{d-1} \psi_{\boldsymbol{n}}(r, r^-).$$

We can bound $|\Xi_{\boldsymbol{n}}(\mathbf{v}) - \Xi_{\boldsymbol{n}}(\mathbf{u})|$ and $|\Xi_{\boldsymbol{n}}(\mathbf{u}^-) - \Xi_{\boldsymbol{n}}(\mathbf{v})|$ as in case 2. Thus we get

$$|\Xi_{\boldsymbol{n}}(r, \boldsymbol{s}_\ell) - \Xi_{\boldsymbol{n}}(r^-, \boldsymbol{s}_\ell)| \le 3^{d-1} \psi_{\boldsymbol{n}}(r, r^-) + 2 \cdot 3^{d-1} 2^{-j\alpha} Z_{\boldsymbol{n}}, \tag{2.59}$$

which gives us the proof of the lemma.

Proof of the lemma 16

Proof is identical to the proof of lemma 15. Instead of analyzing configurations when $u_1(r) \geq u_1(r^-)$, analyze configurations when $u_1(r^+) \geq u_1(r)$.

Chapter 3

Functional central limit theorems

Functional central limit theorems deal with weak convergence of summation processes. Classical approach is to prove first the convergence of finite-dimensional distributions usually using central limit theorem, and then to show that the summation process is tight. Due to Prokhorov theorem this then gives the weak convergence and we say that functional central limit theorem is proved. Functional central limit theorem is called the invariance principle if necessary and sufficient conditions for the convergence are given. In this section we prove invariance principle for i.i.d. Hilbert space valued random variables in Hölder space. We also prove functional central limit theorem for real valued independent but non-identically distributed random variables. Usually proving tightness is harder task, but that is not necessarily so as we show for the triangular array.

For better readability shorter proofs are given straight after theorems in this chapter. The end of the proof is noted by the symbol $\square$.

3.1 Invariance principle

3.1.1 Finite dimensional distributions

Recall the definition of summation process $\xi_{\boldsymbol{n}}$:

$$\xi_{\boldsymbol{n}}(\boldsymbol{t}) = \sum_{\boldsymbol{i} \leq \boldsymbol{n}} \boldsymbol{\pi}(\boldsymbol{n}) \left| \left[\frac{\boldsymbol{i}-\boldsymbol{1}}{\boldsymbol{n}}, \frac{\boldsymbol{i}}{\boldsymbol{n}} \right) \cap [0, \boldsymbol{t}] \right| X_{\boldsymbol{i}}.$$

The so called jump summation process is defined by $\zeta_{\boldsymbol{n}}(\boldsymbol{t}) = S_{[\boldsymbol{n}\boldsymbol{t}]}$ which can be put alternatively as

$$\zeta_{\boldsymbol{n}}(\boldsymbol{t}) = \sum_{\boldsymbol{j}\le\boldsymbol{n}} \mathbf{1}\left(\boldsymbol{j}/\boldsymbol{n} \in [\mathbf{0},\boldsymbol{t}]\right) X_{\boldsymbol{j}}.$$

The following theorem holds.

Theorem 17 *Let $\{X_{\boldsymbol{i}}, \mathbf{1} \le \boldsymbol{i} \le \boldsymbol{n}\}$ be a collection of $\mathbb{H}$-valued random variables. Assume that all variables have finite second moment and uniform variance $\sigma^2 = \mathbf{E}\|X_{\boldsymbol{i}}\|^2 < \infty$ for all $\boldsymbol{i} \le \boldsymbol{n}$. Then if $\mathbf{E}\langle X_{\boldsymbol{i}}, X_{\boldsymbol{j}}\rangle = 0$ for $\mathbf{1} \le \boldsymbol{i} \ne \boldsymbol{j} \le \boldsymbol{n}$,*

$$\|\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}(\xi_{\boldsymbol{n}}(\boldsymbol{t}) - \zeta_{\boldsymbol{n}}(\boldsymbol{t}))\| \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{\Pr} 0, \tag{3.1}$$

for each $\boldsymbol{t} \in [0,1]^d$.

Proof. For each $\boldsymbol{t}$ we can write $\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}(\xi_{\boldsymbol{n}}(\boldsymbol{t}) - \zeta_{\boldsymbol{n}}(\boldsymbol{t})) = \sum_{\boldsymbol{i}\le\boldsymbol{n}} \alpha_{\boldsymbol{i}} X_{\boldsymbol{i}}$, where

$$\alpha_{\boldsymbol{i}} := \boldsymbol{\pi}(\boldsymbol{n})^{1/2}(|[(\boldsymbol{i}-\mathbf{1})/\boldsymbol{n}, \boldsymbol{i}/\boldsymbol{n}] \cap [\mathbf{0},\boldsymbol{t}]| - \boldsymbol{\pi}(\boldsymbol{n})^{-1}\mathbf{1}\left(\boldsymbol{i}/\boldsymbol{n} \in [\mathbf{0},\boldsymbol{t}]\right)).$$

Then

$$\mathbf{E}\|\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}(\xi_{\boldsymbol{n}} - \zeta_{\boldsymbol{n}})\|^2 = \sum_{\boldsymbol{i}\le\boldsymbol{n}}\sum_{\boldsymbol{j}\le\boldsymbol{n}} \alpha_{\boldsymbol{i}}\alpha_{\boldsymbol{j}} \mathbf{E}\langle X_{\boldsymbol{i}}, X_{\boldsymbol{j}}\rangle = \sigma^2 \sum_{\boldsymbol{i}\le\boldsymbol{n}} \alpha_{\boldsymbol{i}}^2.$$

Now $|\alpha_{\boldsymbol{i}}| < 1$, and vanishes if $[(\boldsymbol{i}-1)/\boldsymbol{n}, \boldsymbol{i}/\boldsymbol{n}) \subset [0,\boldsymbol{t}]$, or $[(\boldsymbol{i}-1)/\boldsymbol{n}, \boldsymbol{i}/\boldsymbol{n}) \cap [0,\boldsymbol{t}] = \varnothing$. Actually $\alpha_{\boldsymbol{i}} \ne 0$ if and only if $\boldsymbol{i} \in I$, where I is defined as

$$I := \{\boldsymbol{i} \le \boldsymbol{n};\ \forall k \in \{1,\ldots,d\},\ i_k \le [n_k t_k] + 1 \text{ and } \exists l \in \{1,\ldots,d\},\ i_l = [n_l t_l] + 1\}.$$

For any Borel set $A \subset [0,1]^d$ define for $\varepsilon > 0$

$$A^{\varepsilon} := \left\{ y \in \mathbb{R}^d,\ \exists x \in A;\ |x-y| < \varepsilon \right\}, \quad A^{-\varepsilon} := \mathbb{R}^d \setminus (\mathbb{R}^d \setminus A)^{\varepsilon}.$$

Put $\varepsilon_{\boldsymbol{n}} := \mathrm{m}(\boldsymbol{n})^{-1}$ and $\beta_{\boldsymbol{n}}(\boldsymbol{t}) := |[\mathbf{0},\boldsymbol{t}]^{\varepsilon_{\boldsymbol{n}}} \setminus [\mathbf{0},\boldsymbol{t}]^{-\varepsilon_{\boldsymbol{n}}}|$. Then

$$\sum_{\boldsymbol{i}\le\boldsymbol{n}} \alpha_{\boldsymbol{i}}^2 = \sum_{\boldsymbol{i}\in I} \alpha_{\boldsymbol{i}}^2 \le \beta_{\boldsymbol{n}}(\boldsymbol{t}),$$

and this upper bound tends to zero since the Lebesgue measure of $[\mathbf{0},\boldsymbol{t}]^{\varepsilon_{\boldsymbol{n}}} \setminus [\mathbf{0},\boldsymbol{t}]^{-\varepsilon_{\boldsymbol{n}}}$ is clearly $O(\varepsilon_{\boldsymbol{n}}) = O(\mathrm{m}(\boldsymbol{n})^{-1})$. Combined with the estimate $P(\|Y\| > r) \le r^{-2}\mathbf{E}\|Y\|^2$, for any random variable Y, the theorem follows. □

This theorem coupled with Slutsky's lemma, implies then that for i.i.d. $\mathbb{H}$-valued random variables the limits of finite dimensional distributions of both processes $\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}\xi_{\boldsymbol{n}}$ and $\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}\zeta_{\boldsymbol{n}}$ coincide. Note that for fixed $\boldsymbol{t}$

$$\zeta_{\boldsymbol{n}}(\boldsymbol{t}) = \sum_{\boldsymbol{j}\in J(\boldsymbol{n})} X_{\boldsymbol{j}},$$

where

$$J(\boldsymbol{n}) := \{\boldsymbol{j} \in \mathbb{N}^d : \boldsymbol{j}/\boldsymbol{n} \in [\boldsymbol{0}, \boldsymbol{t}]\}.$$

If $X_{\boldsymbol{j}}$ are zero mean i.i.d. $\mathbb{H}$-valued random variables, satisfying $\mathbf{E}\,\|X_{\boldsymbol{1}}\|^2 < \infty$ and G is the Gaussian random variable with the same covariance operator as $X_{\boldsymbol{1}}$, $\{X_{\boldsymbol{j}}\}$ satisfy CLT in $\mathbb{H}$ [21, Th. 10.5]., i.e.

$$\boldsymbol{\pi}(\boldsymbol{n})^{-1/2} \sum_{\boldsymbol{j}\le\boldsymbol{n}} X_{\boldsymbol{j}} \to G, \text{ as } \boldsymbol{\pi}(\boldsymbol{n}) \to \infty.$$

By denoting $l(\boldsymbol{n})$ the number of elements in the set $J(\boldsymbol{n})$, we then get

$$l(\boldsymbol{n})^{-1/2} \sum_{\boldsymbol{j}\in J(\boldsymbol{n})} X_{\boldsymbol{j}} \xrightarrow{\mathbb{H}} G, \text{ as } l(\boldsymbol{n}) \to \infty.$$

So it is easier to deal with the limits of finite-dimensional distributions of $\zeta_{\boldsymbol{n}}$. Now

$$\begin{aligned}\frac{l(\boldsymbol{n})}{\boldsymbol{\pi}(\boldsymbol{n})} &= \frac{1}{\boldsymbol{\pi}(\boldsymbol{n})} \sum_{\boldsymbol{j}\le\boldsymbol{n}} \mathbf{1}\left(\boldsymbol{j}/\boldsymbol{n} \in [\boldsymbol{0}, \boldsymbol{t}]\right) \\ &= P(U_{\boldsymbol{n}} \in [0, \boldsymbol{t}]) \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{} |[\boldsymbol{0}, \boldsymbol{t}]| = t_1 \dots t_d,\end{aligned}$$

with $U_{\boldsymbol{n}}$ - random variable uniformly distributed on the points $\boldsymbol{j}/\boldsymbol{n}$. Recalling definition of $\mathbb{H}$-valued Brownian sheet we get

$$\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}\zeta_{\boldsymbol{n}}(\boldsymbol{t}) \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{\mathbb{H}} W(\boldsymbol{t}). \tag{3.2}$$

It turns out that this convergence also holds for any vector $\Big(\zeta_{\boldsymbol{n}}(\boldsymbol{t}_1), \dots, \zeta_{\boldsymbol{n}}(\boldsymbol{t}_q)\Big)$ under the same conditions.

Theorem 18 *The convergence*

$$\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}\Big(\zeta_{\boldsymbol{n}}(\boldsymbol{t}_1), \dots, \zeta_{\boldsymbol{n}}(\boldsymbol{t}_q)\Big) \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{D} \Big(W(\boldsymbol{t}_1), \dots, W(\boldsymbol{t}_q)\Big)$$

holds for each $q \ge 1$ *and each* $\boldsymbol{t}_1, \dots, \boldsymbol{t}_q \in [0,1]^d$, *if* $X_{\boldsymbol{j}}$ *are zero mean i.i.d.*

$\mathbb{H}$*-valued random variables, satisfying* $\mathbf{E}\,\|X_{\mathbf{1}}\|^2 < \infty$*.*

This in view of theorem 17 gives the following.

Theorem 19 *Let* $\{X_{\boldsymbol{i}}, \mathbf{1} \le \boldsymbol{i} \le \boldsymbol{n}\}, \boldsymbol{n} \in \mathbb{N}$ *be a collection of* $\mathbb{H}$*-valued i.i.d. random variables. Assume that* $\mathbf{E}\,X_1 = 0$*,* $\mathbf{E}\,\|X_{\mathbf{1}}\|^2 < \infty$*. Then for each* q*-tuple* $\boldsymbol{t}_1, \dots, \boldsymbol{t}_q \in [0,1]^d$

$$\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}\Big(\xi_{\boldsymbol{n}}(\boldsymbol{t}_1), \dots, \xi_{\boldsymbol{n}}(\boldsymbol{t}_q)\Big) \xrightarrow{D} \Big(W(\boldsymbol{t}_1), \dots, W(\boldsymbol{t}_q)\Big).$$

Proof of theorem 18

For convenience write $\widetilde{\zeta}_{\boldsymbol{n}} := \boldsymbol{\pi}(\boldsymbol{n})^{-1/2}\zeta_{\boldsymbol{n}}$. Equip $\mathbb{H}^q$ with product topology. Then the net $(\widetilde{\zeta}_{\boldsymbol{n}}(\boldsymbol{t}_1), \dots, \widetilde{\zeta}_{\boldsymbol{n}}(\boldsymbol{t}_q))$ is tight in $\mathbb{H}^q$ since the nets $(\widetilde{\zeta}_{\boldsymbol{n}}(\boldsymbol{t}_i))$ are tight in $\mathbb{H}$ due to (3.2).

Denote by $\langle\,\cdot\,,\,\cdot\,\rangle_q$ the scalar product in $\mathbb{H}^q$ which is defined by

$$\langle h, g\rangle_q := \sum_{i=1}^{q}\langle h_i, g_i\rangle, \quad h = (h_1, \dots, h_q),\ g = (g_1, \dots, g_q) \in \mathbb{H}^q.$$

Accounting the above mentionned tightness, to prove the theorem we have to check that for each $h \in \mathbb{H}^q$, the following weak convergence holds

$$V_{\boldsymbol{n}} := \Big\langle\Big(\widetilde{\zeta}_{\boldsymbol{n}}(\boldsymbol{t}_1), \dots, \widetilde{\zeta}_{\boldsymbol{n}}(\boldsymbol{t}_q)\Big), h\Big\rangle_q \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{\mathbb{R}} \Big\langle\Big(W(\boldsymbol{t}_1), \dots, W(\boldsymbol{t}_q)\Big), h\Big\rangle_q. \quad (3.3)$$

This will be done through Lindeberg theorem. The first step is to establish the convergence of the variance $b_{\boldsymbol{n}} := \mathbf{E}\,V_{\boldsymbol{n}}^2$ using the decomposition

$$V_{\boldsymbol{n}} = \sum_{k=1}^{q}\langle\widetilde{\zeta}_{\boldsymbol{n}}(\boldsymbol{t}_k), h_k\rangle = \boldsymbol{\pi}(\boldsymbol{n})^{-1/2}\sum_{\boldsymbol{i}\le\boldsymbol{n}}\sum_{k=1}^{q}\mathbf{1}\left(\boldsymbol{i}/\boldsymbol{n} \in [0, \boldsymbol{t}_k]\right)\langle X_{\boldsymbol{i}}, h_k\rangle.$$

Denoting by Γ the covariance operator of $X_{\mathbf{1}}$, we get

$$\begin{aligned} b_{\boldsymbol{n}} &= \frac{1}{\boldsymbol{\pi}(\boldsymbol{n})}\sum_{\boldsymbol{i}\le\boldsymbol{n}}\sum_{\boldsymbol{j}\le\boldsymbol{n}}\sum_{k=1}^{q}\sum_{l=1}^{q}\mathbf{1}\left(\boldsymbol{i}/\boldsymbol{n} \in [0,\boldsymbol{t}_k]\right)\mathbf{1}\left(\boldsymbol{j}/\boldsymbol{n} \in [0,\boldsymbol{t}_l]\right)\mathbf{E}\Big(\langle X_{\boldsymbol{i}}, h_k\rangle\langle X_{\boldsymbol{j}}, h_l\rangle\Big) \\ &= \sum_{k=1}^{q}\sum_{l=1}^{q}\langle\Gamma h_k, h_l\rangle\frac{1}{\boldsymbol{\pi}(\boldsymbol{n})}\sum_{\boldsymbol{i}\le\boldsymbol{n}}\mathbf{1}\left(\boldsymbol{i}/\boldsymbol{n} \in [0,\boldsymbol{t}_k]\cap[0,\boldsymbol{t}_l]\right) \\ &= \sum_{k=1}^{q}\sum_{l=1}^{q}\langle\Gamma h_k, h_l\rangle P(U_{\boldsymbol{n}} \in [0,\boldsymbol{t}_k]\cap[0,\boldsymbol{t}_l]), \end{aligned}$$

where the discrete random variable $U_{\boldsymbol{n}}$ is uniformly distributed on the grid $\boldsymbol{i}/\boldsymbol{n}$, $\mathbf{1} \le \boldsymbol{i} \le \boldsymbol{n}$. Under this form it is clear that when $\mathrm{m}(\boldsymbol{n})$ goes to infinity,

$b_{\boldsymbol{n}}$ converges to b given by

$$b := \sum_{k=1}^{q}\sum_{l=1}^{q}\langle \Gamma h_k, h_l\rangle |[0,\boldsymbol{t}_k]\cap[0,\boldsymbol{t}_l]| = \mathbf{E}\left(\sum_{k=1}^{q}\langle W(\boldsymbol{t}_k), h_k\rangle\right)^2.$$

When $b=0$, the convergence (3.3) is obvious. When $b>0$, let us introduce the real random variables

$$Y_{\boldsymbol{n},\boldsymbol{i}} := \sum_{k=1}^{q}\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}\mathbf{1}\left(\boldsymbol{i}/\boldsymbol{n}\in[0,\boldsymbol{t}_k]\right)\langle X_{\boldsymbol{i}}, h_k\rangle,$$

which have both zero mean and finite variance and note that $V_{\boldsymbol{n}} = \sum_{\boldsymbol{i}\le\boldsymbol{n}} Y_{\boldsymbol{n},\boldsymbol{i}}$. To obtain (3.3) we have to check, by Lindeberg theorem, that for each $\varepsilon>0$,

$$L(\boldsymbol{n}) := \frac{1}{b_{\boldsymbol{n}}}\sum_{\boldsymbol{i}\le\boldsymbol{n}}\mathbf{E}\left(Y_{\boldsymbol{n},\boldsymbol{i}}^2\mathbf{1}\left(|Y_{\boldsymbol{n},\boldsymbol{i}}|>\varepsilon b_{\boldsymbol{n}}^{1/2}\right)\right)\xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{}0. \tag{3.4}$$

Now we have

$$\begin{aligned}
Y_{\boldsymbol{n},\boldsymbol{i}}^2 &= \frac{1}{\boldsymbol{\pi}(\boldsymbol{n})}\sum_{k=1}^{q}\sum_{l=1}^{q}\mathbf{1}\left(\boldsymbol{i}/\boldsymbol{n}\in[0,\boldsymbol{t}_k]\right)\mathbf{1}\left(\boldsymbol{i}/\boldsymbol{n}\in[0,\boldsymbol{t}_l]\right)\langle X_{\boldsymbol{i}}, h_k\rangle\langle X_{\boldsymbol{i}}, h_l\rangle\\
&\le \frac{1}{\boldsymbol{\pi}(\boldsymbol{n})}\sum_{k=1}^{q}\sum_{l=1}^{q}\|X_{\boldsymbol{i}}\|^2\|h_k\|\|h_l\|\\
&= \frac{1}{\boldsymbol{\pi}(\boldsymbol{n})}\left(\sum_{k=1}^{q}\|h_k\|\right)^2\|X_{\boldsymbol{i}}\|^2 = \frac{c_h}{\boldsymbol{\pi}(\boldsymbol{n})}\|X_{\boldsymbol{i}}\|^2.
\end{aligned}$$

Recalling that the number of terms in $\sum_{\boldsymbol{i}\le\boldsymbol{n}}$ is exactly $\boldsymbol{\pi}(\boldsymbol{n})$ and choosing $\mathrm{m}(\boldsymbol{n})$ large enough to have $b_{\boldsymbol{n}} > b/2$, we obtain :

$$L(\boldsymbol{n}) \le \frac{2}{b}\mathbf{E}\left(\|X_{\mathbf{1}}\|^2\mathbf{1}\left(\|X_{\mathbf{1}}\|^2 > \frac{b\varepsilon^2}{2c_h}\boldsymbol{\pi}(\boldsymbol{n})\right)\right),$$

which gives (3.4) by square integrability of $X_{\mathbf{1}}$.

3.1.2 Necessity

Suppose we have weak convergence of summation process $\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}\xi_{\boldsymbol{n}}$ to $\mathbb{H}$-valued Brownian sheet W_d. Since the function $w_\alpha(\,\cdot\,,\delta)$ is continuous on $\mathbb{H}_\alpha^o([0,1]^d)$, by continuous mapping theorem it follows that

$$\lim_{\mathrm{m}(\boldsymbol{n})\to\infty} P(w_\alpha((n_1\dots n_d)^{1/2}\xi_{\boldsymbol{n}},\delta)>a) = P(w_\alpha(W_d,\delta)>a) \tag{3.5}$$

for each continuity point a of distribution function of the random variable $w_\alpha(W_d, \delta)$. Since paths of W_d lie in $\mathbb{H}^o_\alpha([0,1]^d)$,

$$P(w_\alpha(W_d, \delta) > t) \to 0 \quad \text{as} \quad \delta \to 0. \tag{3.6}$$

Thus oscillations of process $\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}\xi_{\boldsymbol{n}}$ should be small. Recall that $\xi_{\boldsymbol{n}}(\boldsymbol{k}/\boldsymbol{n}) = S_{\boldsymbol{k}}$. For arbitrary $\delta > 0$ and $\boldsymbol{n}$ such that $|\mathbf{1}/\boldsymbol{n}| = \mathrm{m}(\boldsymbol{n})^{-1} < \delta$, we have

$$\begin{aligned} &P\Big(w_\alpha((n_1 \dots n_d)^{-1/2}\xi_{\boldsymbol{n}}, \delta) > t\Big) \\ &\geq P\Big((n_1 \dots n_d)^{-1/2} \max_{|\frac{\boldsymbol{k}-\boldsymbol{l}}{\boldsymbol{n}}| = |\frac{\mathbf{1}}{\boldsymbol{n}}|} \frac{\|S_{\boldsymbol{k}} - S_{\boldsymbol{l}}\|}{|(\boldsymbol{k}-\boldsymbol{l})/\boldsymbol{n}|^\alpha} > t\Big). \end{aligned}$$

On the other hand since

$$X_{\boldsymbol{k}} = \Delta^{(1)}_{k_1} \dots \Delta^{(d)}_{k_d} S_{\boldsymbol{k}},$$

we get

$$\|X_{\boldsymbol{k}}\| = \Big\|\Delta^{(1)}_{k_1} \sum_{\boldsymbol{i} \in I} \varepsilon_{\boldsymbol{i}} S_{\boldsymbol{i}}\Big\| \leq \Big\|\sum_{\boldsymbol{i} \in I} \Delta^{(1)}_{k_1} S_{\boldsymbol{i}}\Big\|$$

where $\varepsilon_i = \pm 1$ and I is some apropriate subset of $[0, \boldsymbol{n}] \cap \mathbb{N}^d$ with 2^{d-1} elements. Thus

$$\max_{\mathbf{1} \leq \boldsymbol{k} \leq \boldsymbol{n}} \frac{\|X_{\boldsymbol{k}}\|}{n_1^{-\alpha}} \leq 2^{d-1} \max_{|\frac{\boldsymbol{k}-\boldsymbol{l}}{\boldsymbol{n}}| = |\frac{\mathbf{1}}{\boldsymbol{n}}|} \frac{\|S_{\boldsymbol{k}} - S_{\boldsymbol{l}}\|}{|(\boldsymbol{k}-\boldsymbol{l})/\boldsymbol{n}|^\alpha}.$$

Now with $p = (1/2 - \alpha)^{-1}$

$$\begin{aligned} &P\Big((n_1 \dots n_d)^{-1/2} \max_{\mathbf{1} \leq \boldsymbol{k} \leq \boldsymbol{n}} \frac{\|X_{\boldsymbol{k}}\|}{n_1^{-\alpha}} > t\Big) \\ &= P\Big(n_1^{-1/p} n_2^{-1/2} \dots n_d^{-1/2} \max_{\mathbf{1} \leq \boldsymbol{k} \leq \boldsymbol{n}} \|X_{\boldsymbol{k}}\| > t\Big) \end{aligned}$$

and we see that (3.6) gives us

$$n_1^{-1/p} n_2^{-1/2} \dots n_d^{-1/2} \max_{\mathbf{1} \leq \boldsymbol{k} \leq \boldsymbol{n}} \|X_{\boldsymbol{k}}\| \xrightarrow[\mathrm{m}(\boldsymbol{n}) \to \infty]{\Pr} 0. \tag{3.7}$$

If we assume that $\{X_{\boldsymbol{k}},\ \boldsymbol{k} \le \boldsymbol{n}\}$ are independent and identically distributed, we have for each $t > 0$

$$P\Big(n_1^{-1/p} n_2^{-1/2} \dots n_d^{-1/2} \max_{\boldsymbol{1} \le \boldsymbol{k} \le \boldsymbol{n}} \|X_{\boldsymbol{k}}\| > t\Big) = \\ = 1 - \Big(1 - P\Big(\|X_{\boldsymbol{1}}\| > t n_1^{1/p} n_2^{1/2} \dots n_d^{1/2}\Big)\Big)^{n_1 n_2 \dots n_d}. \tag{3.8}$$

Thus (3.7) is equivalent to

$$n_1 \cdots n_d P\Big(\|X_{\boldsymbol{1}}\| > n_1^{1/p} n_2^{1/2} \cdots n_d^{1/2}\Big) \xrightarrow[\mathrm{m}(\boldsymbol{n}) \to \infty]{} 0. \tag{3.9}$$

Note that (3.7) is as well equivalent to

$$\boldsymbol{\pi}(\boldsymbol{n}) P\Big(\|X_{\boldsymbol{1}}\| > n_m^{1/p} \boldsymbol{\pi}(\boldsymbol{n}_{-m})^{1/2}\Big) \xrightarrow[\mathrm{m}(\boldsymbol{n}) \to \infty]{} 0, \tag{3.10}$$

for any $m \in \{2, \dots, d\}$.

3.1.3 Tightness

In this subsection we prove the tightness of summation process $\boldsymbol{\pi}(\boldsymbol{n})^{-1/2} \xi_{\boldsymbol{n}}$ in the space $\mathbb{H}_\alpha^o([0,1]^d)$ for the mean-zero i.i.d. collection of random variables $\{X_{\boldsymbol{j}}, \boldsymbol{j} \in \mathbb{N}^d\}$. We use tightness criteria, theorem 5, so we have to check two conditions. The first about asymptotic tightness of the net $\xi_{\boldsymbol{n}}$ at each point $\boldsymbol{t}$ readily follows from finite-dimensional convergence, which requires that $\mathbf{E}\,\|X_{\boldsymbol{1}}\|^2 < \infty$.

Recalling the estimate (2.34) from the section 2.1.3 and the relation (1.4) from the section 1.2.3 it follows that

$$\lim_{J \to \infty} \limsup_{\mathrm{m}(\boldsymbol{n}) \to \infty} P(\sup_{j \ge J} 2^{\alpha j} \max_{\boldsymbol{v} \in V_j} |\lambda_{j,\boldsymbol{v}}(\boldsymbol{\pi}(\boldsymbol{n})^{-1/2} \xi_{\boldsymbol{n}})| > \varepsilon) = 0$$

holds if

$$n_m^\alpha \boldsymbol{\pi}(\boldsymbol{n})^{-1/2} Z_{\boldsymbol{n}}^{(m)} \xrightarrow[\mathrm{m}(\boldsymbol{n}) \to \infty]{\Pr} 0 \tag{3.11}$$

and

$$\lim_{J \to \infty} \limsup_{\mathrm{m}(\boldsymbol{n}) \to \infty} P\Big(\max_{J \le j \le \log n_1} 2^{\alpha j} \boldsymbol{\pi}(\boldsymbol{n})^{-1/2} \max_{0 \le k < 2^j} \psi_{\boldsymbol{n}}(t_{k+1}, t_k) > \varepsilon'\Big) = 0 \tag{3.12}$$

hold for each $m = 1, \dots, d$. It turns out that condition

$$\boldsymbol{\pi}(\boldsymbol{n})P\Big(\|X_{\mathbf{1}}\| > n_m^{1/p}\boldsymbol{\pi}(\boldsymbol{n}_{-m})^{1/2}\Big) \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{} 0, \quad m = 1, \dots, d, \tag{3.13}$$

is sufficient for (3.11) and (3.12) to hold. Since (3.13) ensures that $\mathbf{E}\,\|X_{\mathbf{1}}\|^2 < \infty$, then condition (3.13) alone is sufficient for tightness of summation process $\pi(\boldsymbol{n})^{1/2}\xi_{\boldsymbol{n}}$.

Proof of (3.11)

We prove (3.11) for $m = 1$, since the proof is the same for other m. For this reason we drop superscript $^{(1)}$ from $Z_{\boldsymbol{n}}$. Note first that really

$$Z_{\boldsymbol{n}} = \max_{\mathbf{1}\le\boldsymbol{k}\le\boldsymbol{n}} \left\| \sum_{i_2=1}^{k_2} \dots \sum_{i_d=1}^{k_d} X_{(k_1,i_2,\dots,i_d)} \right\|.$$

Fix $\varepsilon > 0$ and associate to any $\delta \in (0,1)$ the truncated random variables $\widetilde{X}_{\boldsymbol{j}}$ and $X'_{\boldsymbol{j}}$ defined as

$$\begin{aligned}\widetilde{X}_{\boldsymbol{j}} &:= X_{\boldsymbol{j}}\mathbf{1}\left(\|X_{\boldsymbol{j}}\| \le \delta n_1^{1/p}(n_2\dots n_d)^{1/2}\right),\\ X'_{\boldsymbol{j}} &:= \widetilde{X}_{\boldsymbol{j}} - \mathbf{E}\,\widetilde{X}_{\boldsymbol{j}}, \quad \mathbf{1} \le \boldsymbol{j} \le \boldsymbol{n}.\end{aligned}$$

Substituting $X_{\boldsymbol{j}}$ by $\widetilde{X}_{\boldsymbol{j}}$, respectively $X'_{\boldsymbol{j}}$, in the definition of $Z_{\boldsymbol{n}}$ we obtain $\widetilde{Z}_{\boldsymbol{n}}$, respectively $Z'_{\boldsymbol{n}}$. Introducing the complementary events

$$E_{\boldsymbol{n}} := \Big\{\forall\boldsymbol{k} \le \boldsymbol{n},\ \|X_{\boldsymbol{k}}\| \le \delta n_1^{1/p}(n_2\dots n_d)^{1/2}\Big\}, \quad E^c_{\boldsymbol{n}} := \Omega \setminus E_{\boldsymbol{n}},$$

we have

$$P(Z_{\boldsymbol{n}} > \varepsilon n_1^{1/p}(n_2\dots n_d)^{1/2}) \le P(\{Z_{\boldsymbol{n}} > \varepsilon n_1^{1/p}(n_2\dots n_d)^{1/2}\} \cap E_{\boldsymbol{n}}) + P(E^c_{\boldsymbol{n}}).$$

Clearly $Z_{\boldsymbol{n}} = \widetilde{Z}_{\boldsymbol{n}}$ on the event $E_{\boldsymbol{n}}$. By identical distribution of the $X_{\boldsymbol{k}}$'s,

$$P(E^c_{\boldsymbol{n}}) \le n_1\dots n_d P(\|X_1\| > \delta n_1^{1/p}(n_2\dots n_d)^{1/2})$$

and this upper bound goes to zero when $\mathrm{m}(\boldsymbol{n})$ goes to infinity by condition (3.13). This leads to

$$\begin{aligned}&\limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} P(Z_{\boldsymbol{n}} > \varepsilon n_1^{1/p}(n_2\dots n_d)^{1/2})\\&\leq \limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} P(\widetilde{Z}_{\boldsymbol{n}} > \varepsilon n_1^{1/p}(n_2\dots n_d)^{1/2}).\end{aligned}\tag{3.14}$$

Because $n_1^{-1/p}(n_2\dots n_d)^{1/2}\|\mathbf{E}\,\widetilde{X_{\mathbf{1}}}\| \to 0$ as $\mathrm{m}(\boldsymbol{n}) \to \infty$ by lemma 23, the right-hand side of (3.14) does not exceed

$$\limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} P(n_1^{-1/p}(n_2\dots n_d)^{-1/2}Z'_{\boldsymbol{n}} > \varepsilon).$$

Using the extension of Doob inequality (1.12), we obtain with $q > p$

$$\begin{aligned}&P(n_1^{-1/p}(n_2\dots n_d)^{-1/2}Z'_{\boldsymbol{n}} > \varepsilon)\\&\leq n_1\, P\left(\max_{\mathbf{1}_{2:d}\leq \boldsymbol{k}_{2:d}\leq \boldsymbol{n}_{2:d}}\left\|\sum_{\boldsymbol{i}_{2:d}=\mathbf{1}_{2:d}}^{\boldsymbol{k}_{2:d}} X'_{(1,i_2,\dots,i_d)}\right\| > \varepsilon n_1^{1/p}(n_2\dots n_d)^{1/2}\right)\\&\leq \varepsilon^{-q} n_1^{1-q/p}(n_2\dots n_d)^{-q/2}\mathbf{E}\left\|\sum_{\boldsymbol{i}_{2:d}=\mathbf{1}_{2:d}}^{\boldsymbol{n}_{2:d}} X'_{(1,i_2,\dots,i_d)}\right\|^q.\end{aligned}$$

Applying Rosenthal inequality (1.10) together with the estimates (3.34), (3.35), provided in subsection 3.1.5 below, we obtain

$$\begin{aligned}&P(n_1^{-1/p}(n_2\dots n_d)^{-1/2}Z'_{\boldsymbol{n}} > \varepsilon)\\&\leq \varepsilon^{-q} n_1^{1-q/p}(n_2\dots n_d)^{-q/2}C_q\Big((n_2\dots n_d)^{q/2}(\mathbf{E}\,\|X'_{\mathbf{1}}\|^2)^{q/2}\\&+ n_2\dots n_d\mathbf{E}\,\|X'_{\mathbf{1}}\|^q\Big)\\&\leq C_q\varepsilon^{-q}\left(n_1^{1-q/p}(\mathbf{E}\,\|X_{\mathbf{1}}\|^2)^{q/2} + \frac{2^{q+1}c_{p,m}}{q-p}\delta^{q-p}\right).\end{aligned}$$

Combined with (3.14) this gives

$$\limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} P(n_1^{-1/p}(n_2\dots n_d)^{-1/2}Z_{\boldsymbol{n}} > \varepsilon) \leq c\delta^{q-p},$$

where the constant c depends on ε, p and q. Since $q > p$ and δ may be choosen arbitrarily small in $(0, 1)$, the convergence (3.11) follows.

Proof of (3.12)

Again we give the proof for $m = 1$ and subsequently drop the superscript. For notational simplification, let us agree to denote by ε' the successive splittings of ε, i.e. $\varepsilon' = c\varepsilon$ where the constant $c \in (0,1)$ may decrease from one formula to following one. Notations $\widetilde{\psi}_{\boldsymbol{n}}(t_{k+1}, t_k)$ and $\psi'_{\boldsymbol{n}}(t_{k+1}, t_k)$ mean that $X_{\boldsymbol{j}}$ are substituted by $\widetilde{X}_{\boldsymbol{j}}$ and $X'_{\boldsymbol{j}}$ respectively in the definition of $\psi_{\boldsymbol{n}}(t_{k+1}, t_k)$. Accordingly we introduce the notations $\widetilde{P}(J, \boldsymbol{n}; \varepsilon')$ and $P'(J, \boldsymbol{n}; \varepsilon')$ where

$$P(J, \boldsymbol{n}; \varepsilon) = P\Big(\max_{J \le j \le \log n_1} 2^{\alpha j} \boldsymbol{\pi}(\boldsymbol{n})^{-1/2} \max_{0 \le k < 2^j} \psi_{\boldsymbol{n}}(t_{k+1}, t_k) > \varepsilon' \Big) \tag{3.15}$$

Splitting Ω in complementary events

$$E_{\boldsymbol{n}} := \Big\{ \forall \boldsymbol{k} \le \boldsymbol{n},\ \|X_{\boldsymbol{k}}\| \le \delta n_1^{1/p} (n_2 \dots n_d)^{1/2} \Big\}, \quad E_{\boldsymbol{n}}^c := \Omega \setminus E_{\boldsymbol{n}},$$

like in previous subsection we obtain

$$P(J, \boldsymbol{n}; \varepsilon') \le \widetilde{P}(J, \boldsymbol{n}; \varepsilon) + n_1 \dots n_d P(\|X_{\mathbf{1}}\| \ge \delta n_1^{1/p} (n_2 \dots n_d)^{1/2}).$$

Then (3.12) is reduced by condition (3.13) to

$$\lim_{J \to \infty} \limsup_{\mathrm{m}(\boldsymbol{n}) \to \infty} \widetilde{P}(J, \boldsymbol{n}; \varepsilon') = 0. \tag{3.16}$$

The number of variables $\widetilde{X}_{\boldsymbol{k}}$ to be centered in the sum $\widetilde{\psi}_{\boldsymbol{n}}(t_{k+1}, t_k)$ is at most $n_1(t_{k+1} - t_k) n_2 \dots n_d \le n_1 2^{-J} n_2 \dots n_d$ and (3.32) yields

$$\begin{aligned} \max_{J \le j \le \log n_1} 2^{\alpha j} (n_1 \dots n_d)^{-1/2} \|\mathbf{E}\, \widetilde{X}_{\mathbf{1}}\| &\le n_1^{\alpha - 1/2} (2\delta^{1-p} c_{p,m}) n_1^{1/p-1} (n_2 \dots n_d)^{-1} \\ &= 2\delta^{1-p} c_{p,m} (n_1 \dots n_d)^{-1}. \end{aligned}$$

Therefore

$$\limsup_{\mathrm{m}(\boldsymbol{n}) \to \infty} \max_{J \le j \le \log n_1} 2^{\alpha j} (n_1 \dots n_d)^{-1/2} n_1 2^{-J} n_2 \dots n_d \|\mathbf{E}\, \widetilde{X}_{\mathbf{1}}\| \le \delta^{1-p} c_p 2^{-J+1}.$$

This upper bound going to zero when J goes to infinity, (3.16) is reduced to

$$\lim_{J \to \infty} \limsup_{\mathrm{m}(\boldsymbol{n}) \to \infty} P'(J, \boldsymbol{n}; \varepsilon') = 0. \tag{3.17}$$

We have with $q > p$

$$
\begin{aligned}
P'(J,\boldsymbol{n};\varepsilon') &\leq \sum_{j=J}^{\log n_1} P\Big(2^{\alpha j}(n_1\dots n_d)^{-1/2} \max_{0\leq k<2^j} \psi'_{\boldsymbol{n}}(t_{k+1},t_k) > \varepsilon'\Big)\\
&\leq \sum_{j=J}^{\log n_1} 2^{q\alpha j}(n_1\dots n_d)^{-q/2}\varepsilon'^{-q}2^j \mathbf{E}\,\psi'_{\boldsymbol{n}}(t_{k+1},t_k)^q. \qquad (3.18)
\end{aligned}
$$

Denote $u_k = [n_1 t_k]$ and observe that $u_{k+1} - u_k \leq n_1 2^{-j}$. By (1.12),

$$
\mathbf{E}\,\psi'_{\boldsymbol{n}}(t_{k+1},t_k)^q \leq \mathbf{E}\left\| \sum_{i_1=1+u_k}^{u_{k+1}} \sum_{\boldsymbol{i}_{2:d}=\boldsymbol{1}_{2:d}}^{\boldsymbol{n}_{2:d}} X'_{\boldsymbol{i}} \right\|^q.
$$

Estimating this last q-moment by Rosenthal inequality (1.10) with a number of summands $N \leq (n_1 2^{-j})n_2\dots n_d$, we obtain

$$
\begin{aligned}
\mathbf{E}\,\psi'_{\boldsymbol{n}}(t_{k+1},t_k)^q &\leq C_q\Big((n_1 2^{-j})^{q/2}(n_2\dots n_d)^{q/2}\mathbf{E}\,\|X'_{\boldsymbol{1}}\|^2 + n_1 2^{-j} n_2\dots n_d \mathbf{E}\,\|X'_{\boldsymbol{1}}\|^q\Big)\\
&\leq C_q \mathbf{E}\,\|X_{\boldsymbol{1}}\|^2 2^{-jq/2}(n_1\dots n_d)^{q/2}\\
&\quad + \frac{2^{q+1}C_q c_{p,m}}{q-p}\delta^{q-p}2^{-j}n_1^{q/p}(n_2\dots n_d)^{q/2}.
\end{aligned}
$$

Reporting this estimate into (3.18) we obtain

$$
P'(J,\boldsymbol{n};\varepsilon') \leq \Sigma_1(J,\boldsymbol{n};\varepsilon') + \Sigma_2(J,\boldsymbol{n};\varepsilon')
$$

with Σ_1 and Σ_2 explicited and bounded as follows. First

$$
\begin{aligned}
\Sigma_1(J,\boldsymbol{n};\varepsilon') &:= \frac{C_q}{\varepsilon'^q}\mathbf{E}\,\|X_{\boldsymbol{1}}\|^2 \sum_{J\leq j\leq \log n_1} 2^{(1+q(\alpha-1/2))j}\\
&\leq \frac{C_q}{\varepsilon'^q}\mathbf{E}\,\|X_{\boldsymbol{1}}\|^2 \sum_{j=J}^{\infty} 2^{-(q/p-1)j}\\
&= \frac{C_q}{\varepsilon'^q}\mathbf{E}\,\|X_{\boldsymbol{1}}\|^2 \frac{2^{-(q/p-1)J}}{1-2^{-(q/p-1)}}.
\end{aligned}
$$

Hence

$$
\lim_{J\to\infty} \limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} \Sigma_1(J,\boldsymbol{n};\varepsilon') = 0.
$$

Next

$$\Sigma_2(J,\boldsymbol{n};\varepsilon') := \frac{2^{q+1}C_q c_{p,m}}{(q-p)\varepsilon'^q}\delta^{q-p}n_1^{-q\alpha}\sum_{J\le j\le \log n_1} 2^{jq\alpha}$$
$$\le \frac{2^{q+1}C_q c_{p,m}}{(q-p)\varepsilon'^q}\delta^{q-p}n_1^{-q\alpha}\frac{n_1^{q\alpha}}{2^{q\alpha}-1}.$$

Noting that $m=\mathrm{m}(\boldsymbol{n})$ and $\limsup_{m\to\infty} c_{p,m}=c_p$, we obtain

$$\limsup_{\mathrm{m}(\boldsymbol{n})\to\infty}\Sigma_2(J,\boldsymbol{n};\varepsilon') \le \frac{2^{q+1}C_q c_p}{(q-p)(2^{q\alpha}-1)\varepsilon'^q}\delta^{q-p}.$$

Recalling (3.15) and summing up all the successive reductions leads to

$$\limsup_{J\to\infty}\limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} P(J,\boldsymbol{n};\varepsilon) \le \frac{2^{q+1}C_q c_p}{(q-p)(2^{q\alpha}-1)\varepsilon'^q}\delta^{q-p}.$$

Since $P(J,\boldsymbol{n};\varepsilon)$ does not depend on δ which may be choosen arbirarily small, the left-hand side is null and this gives (3.12).

3.1.4 Corollaries

We state now the invariance principle in the space $\mathbb{H}_\alpha^o([0,1]^d)$.

Theorem 20 *For $0<\alpha<1/2$, set $p=p(\alpha):=1/(1/2-\alpha)$. For $d\ge 2$, let $\{X_{\boldsymbol{i}};\ \boldsymbol{i}\in\mathbb{N}^d,\ \boldsymbol{i}\ge\boldsymbol{1}\}$ be an i.i.d. collection of square integrable centered random elements in the separable Hilbert space $\mathbb{H}$ and $\xi_{\boldsymbol{n}}$ be the summation process defined by*

$$\xi_{\boldsymbol{n}}(\boldsymbol{t}) = \sum_{\boldsymbol{i}\le\boldsymbol{n}}\pi(\boldsymbol{n})\left|\left[\frac{\boldsymbol{i}-\boldsymbol{1}}{\boldsymbol{n}},\frac{\boldsymbol{i}}{\boldsymbol{n}}\right)\cap[0,\boldsymbol{t}]\right|X_{\boldsymbol{i}}. \tag{3.19}$$

Let W be a $\mathbb{H}$-valued Brownian sheet with the same covariance operator as $X_{\boldsymbol{1}}$. Then the convergence

$$\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}\xi_{\boldsymbol{n}} \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{\mathbb{H}_\alpha^o([0,1]^d)} W \tag{3.20}$$

holds if and only if

$$\boldsymbol{\pi}(\boldsymbol{n})P\Big(\|X_{\boldsymbol{1}}\| > n_m^{1/p}\boldsymbol{\pi}(\boldsymbol{n}_{-m})^{1/2}\Big) \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{} 0, \tag{3.21}$$

for $m=1,\dots,d$.

Proof. It is really nothing left to prove, since necessity is proved in subsection 3.1.2, the convergence of finite-dimensional distributions follow since (3.21) ensures that $\mathbf{E}\,\|X_{\mathbf{1}}\|^2 < \infty$ and tightness is proved in subsection 3.1.3. $\square$

Though condition (3.21) looks rather technical it turns out that it is equivalent to the finiteness of the weak p-moment of $X_{\mathbf{1}}$, i.e.

$$\sup_{t>0} t^p P(\|X_{\mathbf{1}}\| > t) < \infty. \tag{3.22}$$

We prove this for $m = 1$ as the proof is the same for all m. Note that (3.21) is equivalent to

$$v_1^p (v_2 \cdots v_d)^2 P\Big(\|X_{\mathbf{1}}\| > v_1 v_2 \cdots v_d\Big) \xrightarrow[\mathrm{m}(\boldsymbol{v})\to\infty]{} 0 \tag{3.23}$$

and in return (3.23) is equivalent to the convergence

$$F(m) \xrightarrow[m\to\infty]{} 0, \tag{3.24}$$

where

$$F(m) := \sup_{\mathrm{m}(\boldsymbol{v})\geq m} v_1^p (v_2 \cdots v_d)^2 P\Big(\|X_{\mathbf{1}}\| > v_1 v_2 \cdots v_d\Big).$$

Now introducing the function $g(t) := P(\|X_{\mathbf{1}}\| > t)$ and the sets

$$H_{t,m} := \{\boldsymbol{v} \in \mathbb{R}^d;\ \boldsymbol{v} \geq m,\ v_1 v_2 \cdots v_d = t\},$$

we have

$$F(m) = \sup_{t\geq m^d} \sup_{\boldsymbol{v}\in H_{t,m}} v_1^{p-2} t^2 g(t) = \sup_{t\geq m^d} t^2 g(t) \sup_{\boldsymbol{v}\in H_{t,m}} v_1^{p-2}.$$

When $t \geq m^d$, $H_{t,m}$ is non empty and on this set, $v_1 = t(v_2 \cdots v_d)^{-1}$ is maximal for $v_2 = \cdots = v_d = m$, so

$$t^2 g(t) \sup_{\boldsymbol{v}\in H_{t,m}} v_1^{p-2} = t^p g(t) m^{-(d-1)(p-2)}.$$

Finally

$$F(m) = m^{-(d-1)(p-2)} \sup_{t\geq m^d} t^p g(t).$$

Recalling that $d > 1$ and $p > 2$, this reduces the convergence (3.24) to the finiteness of $\sup_{t\geq m_0^d} t^p g(t)$ for some $m_0 > 0$. As $t^p g(t)$ is bounded on any interval $[0, a]$ for $a < \infty$, this finiteness is equivalent to (3.22). Thus we proved the following theorem.

Theorem 21 *The convergence*

$$\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}\xi_{\boldsymbol{n}} \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{\mathbb{H}^o_\alpha([0,1]^d)} W$$

holds if and only if

$$\sup_{t>0} t^{p(\alpha)} P(\|X_{\mathbf{1}}\| > t) < \infty, \quad p(\alpha) = (1/2-\alpha)^{-1}.$$

As condition (3.22) is weaker than $\mathbf{E}\,\|X_1\|^p < \infty$, then theorem 20 improves when $\mathbb{H} = \mathbb{R}$, Erickson's [12] result for $\mathcal{Q}_d$:

$$(n_1 \cdots n_d)^{-1/2}\xi_{\boldsymbol{n}} \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{\mathrm{H}^o_\alpha([0,1]^d)} W,$$

if $0 < \alpha < 1/2$ and $\mathbf{E}\,|X_{\mathbf{1}}|^q < \infty$, where $q > dp(\alpha)$.

Considering the convergence of random fields $(\xi_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^d)$ along the fixed path $\boldsymbol{n} = (n, \ldots, n) \in \mathbb{N}^d$, $n \in \mathbb{N}$ we obtain the following result.

Theorem 22 *The convergence*

$$n^{-d/2}\xi_{(n,\ldots,n)} \xrightarrow[n\to\infty]{\mathbb{H}^o_\alpha([0,1]^d)} W \tag{3.25}$$

holds if and only if

$$\lim_{t\to\infty} t^{\frac{2d}{d-2\alpha}} P(\|X_{\mathbf{1}}\| > t) = 0. \tag{3.26}$$

Proof. Looking back at the proofs in previous subsections and having in mind the extra assumption that $n_1 = n_2 = \cdots = n_d = n$, it should be clear that the weak $\mathbb{H}^o_\alpha([0,1]^d)$ convergence of $n^{-d/2}\xi_{(n,\ldots,n)}$ to W is equivalent to the condition obtained by reporting this equality of the n_i's in (3.9), namely to

$$n^d P\Big(\|X_{\mathbf{1}}\| > n^{1/p+(d-1)/2}\Big) \xrightarrow[n\to\infty]{} 0. \tag{3.27}$$

It is easily checked that in (3.27) the integer n can be replaced by a positive real number s and then puting $t = s^{1/p+(d-1)/2}$, we obtain the equivalence of (3.27) with

$$\lim_{t\to\infty} t^{\frac{2pd}{2+p(d-1)}} P\Big(\|X_{\mathbf{1}}\| > t\Big) = 0. \tag{3.28}$$

Finally recalling that $p = p(\alpha) = 2/(1-2\alpha)$, we get

$$\frac{2pd}{2+p(d-1)} = \frac{2d}{d-2\alpha},$$

which reported in (3.28) gives (3.26) and completes the proof. □

Since $2d/(d-2\alpha) < 2d/(d-1)$ we see that $\mathbf{E}\,\|X_{\mathbf{1}}\|^{2d/(d-1)} < \infty$ yields (3.26). In particular $\mathbf{E}\,\|X_{\mathbf{1}}\|^4 < \infty$ gives the convergence (3.25) for any $d \geq 2$ and any $0 < \alpha < 1/2$. This contrasts with the corresponding result for Hölder convergence of the usual Donsker-Prokhorov polygonal line processes where necessarily $\mathbf{E}\,|X_1|^q < \infty$ for any $q < p(\alpha)$ as follows from (1).

Of course, Theorem 22 is only a striking special case and similar results can be obtained adapting the proof of Theorem 20 for summation processes with index going to infinity along some various paths or surfaces.

As passing from n to $n+1$ brings $O(n^{d-1})$ new summands in the summation process of Theorem 22, one may be tempted to look for similar weakening of the assumption in the Hölderian FCLT for $d = 1$, when restricting for subsequences. In fact even so, the situation is quite different: it is easy to see that for any increasing sequence of integers n_k such that $\sup_{k\geq 1} n_{k+1}/n_k < \infty$, the convergence to zero of $n_k^{p(\alpha)} P(|X_1| > n_k)$ when k tends to infinity implies (1). As $n_k^{p(\alpha)} P(|X_1| > n_k) = o(1)$ is a necessary condition for $(\xi_{n_k})_{k\geq 1}$ to satisfy the FCLT in $\mathrm{H}_\alpha^o([0,1]^d)$ when $d = 1$, there is no hope to obtain this FCLT for $(\xi_{n_k})_{k\geq 1}$ under some condition weaker than (1).

3.1.5 Truncated variables

In this subsection we complete the technical details about the estimates of moment of truncated variables used above. Such estimates are obtained under the assumption:

$$n_1 \cdots n_d P\Big(\|X_{\mathbf{1}}\| > n_1^{1/p} n_2^{1/2} \cdots n_d^{1/2}\Big) \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{} 0. \tag{3.29}$$

Let $\delta \in (0,1)$ be an arbitrary number. Define

$$\widetilde{X}_{\boldsymbol{j}} := X_{\boldsymbol{j}} \mathbf{1}\left(\|X_{\boldsymbol{j}}\| \leq \delta n_1^{1/p}(n_2 \ldots n_d)^{1/2}\}\right), \tag{3.30}$$

$$X'_{\boldsymbol{j}} := \widetilde{X}_{\boldsymbol{j}} - \mathbf{E}\,\widetilde{X}_{\boldsymbol{j}}, \quad \mathbf{1} \leq \boldsymbol{j} \leq \boldsymbol{n}. \tag{3.31}$$

Though we give the proof for (3.30) definition of truncated variable, the estimates hold for any permutation of indexes $1, \ldots, d$ in (3.30) combined with the same permutation in (3.29).

Denote for $m \geq 0$

$$c(m) := \sup_{u\geq m} \sup_{\boldsymbol{v}_{2:d}\geq m} u v_2 \ldots v_d P(\|X_{\mathbf{1}}\| > u^{1/p}(v_2 \ldots v_d)^{1/2})$$

$$c_p := \sup_{t\geq 0} t^{d/(d/2-\alpha)} P(\|X_{\mathbf{1}}\| > t).$$

Evidently condition (3.29) yields $c(m) \to 0$ as $m \to \infty$ and $c_p < \infty$. Set

$$c_{p,m} := \max\{c_p;\ c(m)\}.$$

Lemma 23 *With $m = \mathrm{m}(\boldsymbol{n})$ and any $q > p$*

$$\|\mathbf{E}\,\widetilde{X}_{\mathbf{1}}\| \le 2\delta^{1-p} c_{p,m} n_1^{1/p-1} (n_2 \dots n_d)^{-1/2}; \tag{3.32}$$

$$\mathbf{E}\,\|\widetilde{X}_{\mathbf{1}}\|^q \le \frac{2c_{p,m}}{q-p} \delta^{q-p} n_1^{q/p-1} (n_2 \dots n_d)^{q/2-1}; \tag{3.33}$$

$$\mathbf{E}\,\|X'_{\mathbf{1}}\|^2 \le \mathbf{E}\,\|X_{\mathbf{1}}\|^2; \tag{3.34}$$

$$\mathbf{E}\,\|X'_{\mathbf{1}}\|^q \le \frac{2^{q+1} c_{p,m}}{q-p} \delta^{q-p} n_1^{q/p-1} (n_2 \dots n_d)^{q/2-1}. \tag{3.35}$$

Proof. To check (3.32), we observe first that since $\mathbf{E}\,X_{\mathbf{1}} = 0$,

$$\begin{aligned}
\|\mathbf{E}\,\widetilde{X}_{\mathbf{1}}\| &= \|\mathbf{E}\,X_{\mathbf{1}} - \mathbf{E}\,X_{\mathbf{1}}\mathbf{1}\left(\|X_{\mathbf{1}}\| > \delta n_1^{1/p}(n_2 \dots n_d)^{1/2}\right)\| \\
&\le \int_{\delta n_1^{1/p}(n_2 \dots n_d)^{1/2}}^{\infty} P(\|X_{\mathbf{1}}\| > t)\,\mathrm{d}t \\
&\quad + \delta n_1^{1/p}(n_2 \dots n_d)^{1/2} P\Big(\|X_{\mathbf{1}}\| > \delta n_1^{1/p}(n_2 \dots n_d)\Big)^{1/2}).
\end{aligned}$$

Next we have

$$\begin{aligned}
&\int_{\delta n_1^{1/p}(n_2 \dots n_d)^{1/2}}^{\infty} P(\|X_{\mathbf{1}}\| > t)\,\mathrm{d}t \\
&= \delta n_1^{1/p-1}(n_3 \dots n_d)^{-1/2} \int_{n_2^{1/2}}^{\infty} v^2 n_1 n_3 \dots n_d \\
&P(\|X_{\mathbf{1}}\| > \delta v n_1^{1/p}(n_3 \dots n_d)^{1/2}) \frac{\mathrm{d}v}{v^2} \\
&\le \delta n_1^{1/p-1}(n_3 \dots n_d)^{-1/2} b(m,\delta) \int_{n_2^{1/2}}^{\infty} v^{-2}\,\mathrm{d}v \\
&\le \delta b(m,\delta) n_1^{1/p-1}(n_2 \dots n_d)^{-1/2},
\end{aligned}$$

where

$$b(m,\delta) := \sup_{u \ge m} \sup_{\boldsymbol{v}_{2:d} \ge m} u v_2 \dots v_d P(\|X_{\mathbf{1}}\| > \delta u^{1/p}(v_2 \dots v_d)^{1/2}).$$

We complete the proof of (3.32) noting that

$$\begin{aligned}
b(m;\delta) &= \delta^{-p}\sup_{u\ge\delta^p m}\sup_{\boldsymbol{v}_{2:d}\ge m} uv_2\dots v_d P(\|X_{\mathbf{1}}\|>u^{1/p}(v_2\dots v_d)^{1/2}) \\
&= \delta^{-p}\max\Big\{\sup_{m\ge u\ge\delta^p m}\sup_{\boldsymbol{v}_{2:d}\ge m} uv_2\dots v_d P(\|X_{\mathbf{1}}\|>u^{1/p}(v_2\dots v_d)^{1/2}); \\
&\qquad \sup_{u\ge m}\sup_{\boldsymbol{v}_{2:d}\ge m} uv_2\dots v_d P(\|X_{\mathbf{1}}\|>u^{1/p}(v_2\dots v_d)^{1/2})\Big\} \\
&\le \delta^{-p}c_{p,m}, \qquad (3.36)
\end{aligned}$$

since

$$\begin{aligned}
&\sup_{u\le m}\sup_{\boldsymbol{v}_{2:d}\ge m} uv_2\dots v_d P(\|X_{\mathbf{1}}\|>u^{1/p}(v_2\dots v_d)^{1/2}) \\
&\le \sup_{u\le m}\sup_{\boldsymbol{v}_{2:d}\ge m} uv_2\dots v_d c_p(u^{1/p}(v_2\dots v_d)^{1/2})^{-d/(d/2-\alpha)} \\
&= c_p\sup_{u\le m} u^{2\alpha(d-1)/(d-2\alpha)}\sup_{\boldsymbol{v}_{2:d}\ge m}(v_2\dots v_d)^{-2\alpha/(d-2\alpha)} = c_p.
\end{aligned}$$

Next we have

$$\begin{aligned}
\mathbf{E}\,\|\widetilde{X_{\mathbf{1}}}\|^q &\le \int_0^{\delta n_1^{1/p}(n_2\dots n_d)^{1/2}} t^{q-1}P(\|X_{\mathbf{1}}\|>t)\,\mathrm{d}t \\
&= \int_0^{\delta(n_2\dots n_d)^{1/2}} t^{q-1}P(\|X_{\mathbf{1}}\|>t)\,\mathrm{d}t \\
&\quad + \int_{\delta(n_2\dots n_d)^{1/2}}^{\delta n_1^{1/p}(n_2\dots n_d)^{1/2}} t^{q-1}P(\|X_{\mathbf{1}}\|>t)\,\mathrm{d}t.
\end{aligned}$$

By Chebyshev inequality $P(\|X_{\mathbf{1}}\|>t)\le t^{-2}$, hence the first integral does not exceed $(q-2)^{-1}\delta^{q-2}(n_2\dots n_d)^{q/2-1}$. As $\int_1^{n_1^{1/p}}\le n_1^{q/p-1}$, the second integral does not exceed

$$\begin{aligned}
&\delta^q(n_2\dots n_d)^{q/2-1}\int_1^{n_1^{1/p}} n_2\dots n_d u^p P(\|X_{\mathbf{1}}\|>\delta u(n_2\dots n_d)^{1/2})u^{q-p-1}\,\mathrm{d}u \\
&\le \delta^q(n_2\dots n_d)^{q/2-1}\sup_{\boldsymbol{v}_{2:d}\ge m}\sup_{1\le u\le n_1} uv_2\dots v_d P(\|X_{\mathbf{1}}\|>\delta u^{1/p}(v_2\dots v_d)^{1/2})n_1^{q/p-1} \\
&\le \frac{1}{q-p}\max\{b'(m,\delta);\ b(m;\delta)\}\delta^q n_1^{q/p-1}(n_2\dots n_d)^{q/2-1},
\end{aligned}$$

where

$$b'(m,\delta) := \sup_{\boldsymbol{v}_{2:d}\geq m} \sup_{1\leq u\leq m} uv_2\ldots v_d P(\|X_{\mathbf{1}}\| > \delta u^{1/p}(v_2\ldots v_d)^{1/2})$$
$$\leq \delta^{-2d/(d/2-\alpha)}c_p \leq \delta^{-p}c_p,$$

recalling that $0<\delta<1$ and $p=(1/2-\alpha)^{-1}$. Accounting (3.36) inequality (3.33) now follows.

To check (3.34), let us denote by $(e_k,\ k\in\mathbb{N})$ some orthonormal basis of the separable Hilbert space $\mathbb{H}$. Then we have

$$\|X'_{\mathbf{1}}\|^2 = \sum_{k=0}^{\infty}\left|\langle \widetilde{X}_{\mathbf{1}} - \mathbf{E}\,\widetilde{X}_{\mathbf{1}}, e_k\rangle\right|^2 = \sum_{k=0}^{\infty}\left|\langle \widetilde{X}_{\mathbf{1}}, e_k\rangle - \mathbf{E}\,\langle \widetilde{X}_{\mathbf{1}}, e_k\rangle\right|^2,$$

whence

$$\mathbf{E}\,\|X'_{\mathbf{1}}\|^2 = \sum_{k=0}^{\infty}\mathrm{Var}(\langle \widetilde{X}_{\mathbf{1}}, e_k\rangle) \leq \sum_{k=0}^{\infty}\mathbf{E}\,\left|\langle \widetilde{X}_{\mathbf{1}}, e_k\rangle\right|^2$$
$$= \mathbf{E}\sum_{k=0}^{\infty}\left|\langle \widetilde{X}_{\mathbf{1}}, e_k\rangle\right|^2 = \mathbf{E}\,\|\widetilde{X}_{\mathbf{1}}\|^2 \leq \mathbf{E}\,\|X_{\mathbf{1}}\|^2,$$

which gives (3.34).

Finally we note that (3.35) is obviously obtained from (3.33) since the convexity inequality $\|X'_{\mathbf{1}}\|^q \leq 2^{q-1}\|\widetilde{X}_{\mathbf{1}}\|^q + 2^{q-1}\|\mathbf{E}\,\widetilde{X}_{\mathbf{1}}\|^q$ together with $\mathbf{E}\,\|\widetilde{X}_1\| \leq (\mathbf{E}\,\|\widetilde{X}_1\|^q)^{1/q}$ gives $\mathbf{E}\,\|X'_1\|^q \leq 2^q\mathbf{E}\,\|\widetilde{X}_1\|^q$. □

3.2 Triangular array

3.2.1 Finite dimensional distributions

In this subsection we find the limits of finite-dimensional distributions of process $\Xi_{\boldsymbol{n}}$. We show that the convergence to Brownian sheet is only a special case. In general case the limiting process is Gaussian, if the limit exists, but that is not always the case. As in the i.i.d. case we show that it is more convenient to analyze the jump version $Z_{\boldsymbol{n}}$ of process $\Xi_{\boldsymbol{n}}$. Then we give some examples for which convergence to the Brownian sheet fails. Finally we give the conditions and assumptions, under which the finite-dimensional distributions converge to some Gaussian process.

Recall definitions $\boldsymbol{B}(\boldsymbol{k})$ and $b_l(k_l)$ from equations (2.41) and (2.42) in the section 2.2.1. Define then the jump process as

$$Z_{\boldsymbol{n}}(\boldsymbol{t}) = \sum_{\boldsymbol{1} \leq \boldsymbol{k} \leq \boldsymbol{k_n}} \mathbf{1}\left(\boldsymbol{B}(\boldsymbol{k}) \in [0, \boldsymbol{t}]\right) X_{\boldsymbol{n},\boldsymbol{k}}.$$

The following result holds.

Lemma 24 *Assume*

$$\max_{1 \leq l \leq d} \max_{1 \leq k_l \leq k_{\boldsymbol{n}}^l} \Delta b_l(k_l) \to 0, \ \ as \ \mathrm{m}(\boldsymbol{n}) \to \infty. \tag{3.37}$$

Then

$$\mathbf{E}\left|\Xi_{\boldsymbol{n}}(\boldsymbol{t}) - Z_{\boldsymbol{n}}(\boldsymbol{t})\right|^2 \to 0, \ \ as \ \mathrm{m}(\boldsymbol{n}) \to \infty \tag{3.38}$$

and subsequently

$$\left|\Xi_{\boldsymbol{n}}(\boldsymbol{t}) - Z_{\boldsymbol{n}}(\boldsymbol{t})\right| \xrightarrow{P} 0, \ \ as \ \mathrm{m}(\boldsymbol{n}) \to \infty.$$

Proof. For each $\boldsymbol{t}$ we have

$$\left|\Xi_{\boldsymbol{n}}(\boldsymbol{t}) - Z_{\boldsymbol{n}}(\boldsymbol{t})\right| = \sum_{\boldsymbol{1} \leq \boldsymbol{k} \leq \boldsymbol{k_n}} \alpha_{\boldsymbol{n},\boldsymbol{k}} X_{\boldsymbol{n},\boldsymbol{k}},$$

where

$$\alpha_{\boldsymbol{n},\boldsymbol{k}} = |Q_{\boldsymbol{n},\boldsymbol{k}}|^{-1} |Q_{\boldsymbol{n},\boldsymbol{k}}| - \mathbf{1}\{\boldsymbol{B}(\boldsymbol{k}) \in [0, \boldsymbol{t}]\}.$$

Now $|\alpha_{\boldsymbol{n},\boldsymbol{k}}| < 1$, and vanishes if $Q_{\boldsymbol{n},\boldsymbol{k}} \subset [0, \boldsymbol{t}]$, or $Q_{\boldsymbol{n},\boldsymbol{k}} \cap [0, \boldsymbol{t}] = \varnothing$. Actually $\alpha_{\boldsymbol{n},\boldsymbol{k}} \neq 0$ if and only if $\boldsymbol{k} \in I$, where I is defined as

$$I := \{\boldsymbol{i} \leq \boldsymbol{n};\, \forall k \in \{1, \ldots, d\},\ i_k \leq b_k(u_k(t_k) + 1) \text{ and } \exists l \in \{1, \ldots, d\},\ i_l = b_l(u_l(t_l) + 1)\}.$$

Thus

$$\mathbf{E}\left|\Xi_{\boldsymbol{n}}(\boldsymbol{t}) - \zeta_{\boldsymbol{n}}(\boldsymbol{t})\right|^2 = \sum_{\boldsymbol{k} \in I} \alpha_{\boldsymbol{n},\boldsymbol{k}} \sigma_{\boldsymbol{n},\boldsymbol{k}}^2 \leq \sum_{\boldsymbol{k} \in I} \sigma_{\boldsymbol{n},\boldsymbol{k}}^2 \leq \sum_{l=1}^{d} \Delta b_l(u_l(t_l) + 1).$$

Now due to (3.37) we have

$$\mathbf{E}\left|\Xi_{\boldsymbol{n}}(\boldsymbol{t}) - Z_{\boldsymbol{n}}(\boldsymbol{t})\right|^2 \to 0, \text{ as } \mathrm{m}(\boldsymbol{n}) \to \infty,$$

which coupled with the estimate $P(|Y| > r) \leq r^{-2} \mathbf{E}\,|Y|^2$, for any random variable Y gives us

$$|\Xi_{\boldsymbol{n}}(\boldsymbol{t}) - Z_{\boldsymbol{n}}(\boldsymbol{t})| \xrightarrow{P} 0, \text{ as } \mathrm{m}(\boldsymbol{n}) \to \infty. \qquad \square$$

The condition (3.37) ensures that the grid gets thinner and thinner as $\mathrm{m}(\boldsymbol{n})$ approach the infinity. It is more restrictive than the condition of asymptotic negligibility. Define now

$$\mu_{\boldsymbol{n}}(\boldsymbol{t}) = \sum_{\boldsymbol{1} \le \boldsymbol{k} \le \boldsymbol{k_n}} \mathbf{1}\{\boldsymbol{B}(\boldsymbol{k}) \in [0, \boldsymbol{t}]\} \sigma^2_{\boldsymbol{n},\boldsymbol{k}}. \tag{3.39}$$

It is clear that $\mu_{\boldsymbol{n}}(\boldsymbol{t}) = \mathbf{E}\, Z_{\boldsymbol{n}}(\boldsymbol{t})^2$. If the limiting finite-dimensional distributions of $Z_{\boldsymbol{n}}$ were those of Brownian sheet, then $\mu_{\boldsymbol{n}}(\boldsymbol{t})$ would converge to $\boldsymbol{\pi}(\boldsymbol{t})$ for each $\boldsymbol{t}$. Consider the following example of triangular array.

Example 1 *For $\boldsymbol{n} = (n, n)$ and $\boldsymbol{k_n} = (2n, 2n)$ take $X_{\boldsymbol{n},\boldsymbol{k}} = a_{\boldsymbol{n},\boldsymbol{k}} Y_k$, with $\{Y_{\boldsymbol{k}}, \boldsymbol{k} \le \boldsymbol{k_n}\}$ i.i.d. random variables with standard normal distribution, and*

$$a^2_{\boldsymbol{n},\boldsymbol{k}} = \begin{cases} \frac{1}{10n^2}, & \text{for } \boldsymbol{k} \le (n, n) \\ \frac{3}{10n^2}, & \text{otherwise.} \end{cases} \tag{3.40}$$

Thus defined triangular array satisfies the condition (3.37), but simple algebra shows that for such an array

$$\mu_{\boldsymbol{n}}(\boldsymbol{t}) \to \nu(\boldsymbol{t}) := \frac{1}{10}\left(\frac{5}{2}t_1 \wedge 1\right)\left(\frac{5}{2}t_2 \wedge 1\right) + \frac{(5t_1 - 2) \vee 0}{10}\left(\frac{5}{2}t_2 \vee 1\right) + \frac{(5t_2 - 2) \vee 0}{10}\left(\frac{5}{2}t_1 \vee 1\right) + \frac{\big((5t_1 - 2) \vee 0\big)\big((5t_2 - 2) \vee 0\big)}{30}.$$

Furthermore for the following example $\mu_{\boldsymbol{n}}(\boldsymbol{t})$ does not converge for any $\boldsymbol{t}$.

Example 2 *For $\boldsymbol{n} = (n, n)$ and $\boldsymbol{k_n} = (n, n)$ take $X_{\boldsymbol{n},\boldsymbol{k}} = b_{\boldsymbol{n},\boldsymbol{k}} Y_k$ with $\{Y_{\boldsymbol{k}}, \boldsymbol{k} \le \boldsymbol{k_n}\}$ i.i.d. random variables with standard normal distribution, and*

$$b^2_{\boldsymbol{n},\boldsymbol{k}} = \begin{cases} \pi(\boldsymbol{k_n})^{-1}, & \text{for } \boldsymbol{n} = (2l-1, 2l-1),\ l \in \mathbb{N} \\ a^2_{\boldsymbol{n},\boldsymbol{k}}, & \text{for } \boldsymbol{n} = (2l, 2l),\ l \in \mathbb{N} \end{cases}$$

where $a_{\boldsymbol{n},\boldsymbol{k}}$ are defined as in (3.40).

Nevertheless we can get some fruitful results by making following assumption.

Assumption 1 *There exists a function* $\mu : [0,1]^d \to \mathbb{R}$ *such that*

$$\forall \boldsymbol{t} \in [0,1]^d, \qquad \lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \mu_{\boldsymbol{n}}(\boldsymbol{t}) = \mu(\boldsymbol{t}). \tag{3.41}$$

With this assumption we can state the following result whose proof is postponed for a while.

Theorem 25 *Given assumption 1 there exists a Gaussian process* $\{G(\boldsymbol{t}), \boldsymbol{t} \in [0,1]^d\}$ *with covariance function* $\mathbf{E}\, G(\boldsymbol{t})G(\boldsymbol{s}) = \mu(\boldsymbol{t} \wedge \boldsymbol{s})$. *Furthermore if*

$$\max_{1\le l\le d} \max_{1\le k_l\le k_{\boldsymbol{n}}^l} \Delta b_l(k_l) \to 0, \ \text{as } \mathrm{m}(\boldsymbol{n}) \to \infty \tag{3.42}$$

and for every $\varepsilon > 0$

$$\lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \sum_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{k}_{\boldsymbol{n}}} \mathbf{E}\, X_{\boldsymbol{n},\boldsymbol{k}}^2 \mathbf{1}\{|X_{\boldsymbol{n},\boldsymbol{k}}| \ge \varepsilon\} = 0, \tag{3.43}$$

then for any collection of m *points* $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_m \in [0,1]^d$

$$(Z_{\boldsymbol{n}}(\boldsymbol{t}_1), \ldots, Z_{\boldsymbol{n}}(\boldsymbol{t}_m)) \xrightarrow{D} (G(\boldsymbol{t}_1), \ldots, G(\boldsymbol{t}_m)).$$

For the process $\Xi_{\boldsymbol{n}}$, the following theorem holds.

Theorem 26 *If there exists a function* $\mu : [0,1]^d \to \mathbb{R}$ *such that*

$$\forall \boldsymbol{t} \in [0,1]^d, \qquad \lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \mathbf{E}\, \Xi_{\boldsymbol{n}}^2(\boldsymbol{t}) = \mu(\boldsymbol{t}) \tag{3.44}$$

and

$$\max_{1\le l\le d} \max_{1\le k_l\le k_{\boldsymbol{n}}^l} \Delta b_l(k_l) \to 0, \ \text{as } \mathrm{m}(\boldsymbol{n}) \to \infty \tag{3.45}$$

and for every $\varepsilon > 0$

$$\lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \sum_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{k}_{\boldsymbol{n}}} \mathbf{E}\, X_{\boldsymbol{n},\boldsymbol{k}}^2 \mathbf{1}\{|X_{\boldsymbol{n},\boldsymbol{k}}| \ge \varepsilon\} = 0, \tag{3.46}$$

then given any collection of m *points* $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_m \in [0,1]^d$

$$(\Xi_{\boldsymbol{n}}(\boldsymbol{t}_1), \ldots, \Xi_{\boldsymbol{n}}(\boldsymbol{t}_m)) \xrightarrow{D} (G(\boldsymbol{t}_1), \ldots, G(\boldsymbol{t}_m)),$$

where G *is a Gaussian process satisfying* $\mathbf{E}\, G(\boldsymbol{t})G(\boldsymbol{s}) = \mu(\boldsymbol{t} \wedge \boldsymbol{s})$ *for* $\boldsymbol{t}, \boldsymbol{s} \in [0,1]^d$.

Proof. Since (3.45) is satisfied lemma 24 gives us

$$|\mathbf{E}\,\Xi_{\boldsymbol{n}}^2(\boldsymbol{t}) - \mathbf{E}\,Z_{\boldsymbol{n}}^2(\boldsymbol{t})| \xrightarrow{P} 0, \text{ as } \mathrm{m}(\boldsymbol{n}) \to \infty,$$

thus the limits $\mu(\boldsymbol{t})$ in (3.41) and (3.44) coincide. The theorem 25 then gives us the existence of the process G and the same theorem combined again with lemma 24 gives us the proof. □

For triangular arrays with certain variance structure, the limiting process is always a Brownian sheet. Take double indexed triangular array $\{X_{n,ij}, 1 \le i \le I_n, 1 \le j \le J_n\}$ with $\mathbf{E}\,X_{n,ij}^2 = a_i b_j$, where $a_i, b_j > 0$ and $\sum a_i = 1 = \sum b_j$. Recalling notations (2.41) we get

$$b_1(k) = \sum_{i=1}^{k} a_i, \quad b_2(l) = \sum_{j=1}^{l} b_j$$

so our grid rectangle is now

$$Q_{n,kl} = \Big[\sum_{i=1}^{k-1} a_i, \sum_{i=1}^{k} a_i, \Big) \times \Big[\sum_{j=1}^{l-1} b_j, \sum_{j=1}^{l} b_j, \Big).$$

We see that grid points on x-axis are defined only by a_i and on y-axis by b_j. Now the variance of jump process in this case will be

$$\begin{aligned}\mu_{\boldsymbol{n}}(\boldsymbol{t}) &= \sum_{i=1}^{I_n}\sum_{j=1}^{J_n} \mathbf{1}\left(\Big(\sum_{k=1}^{i} a_k, \sum_{l=1}^{j} b_l\Big) \in [0,t_1]\times[0,t_1]\right) a_i b_j \\ &= \sum_{i=1}^{I_n} \mathbf{1}\left(\sum_{k=1}^{i} a_k \in [0,t_1]\right) a_i \sum_{j=1}^{J_n} \mathbf{1}\left(\sum_{l=1}^{j} b_l \in [0,t_2]\right) b_j\end{aligned}$$

and if we assume the condition (3.45) which in this case translates to

$$\max_{1\le i\le I_n} a_i \to 0, \quad \max_{1\le j\le J_n} b_j \to 0,$$

we see that

$$\mu_{\boldsymbol{n}}(\boldsymbol{t}) \to t_1 t_2,$$

which is the variance of Brownian sheet. Thus assuming the Lindeberg condition (3.46), theorem 25 implies that limiting finite-dimensional distributions in this case are those of Brownian sheet.

Proof of theorem 25

Define

$$g(\boldsymbol{t},\boldsymbol{s}) = \lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \mu_{\boldsymbol{n}}(\boldsymbol{t}\wedge\boldsymbol{s}).$$

If we prove that $g(\boldsymbol{t},\boldsymbol{s})$ is positive definite, then the existence of zero mean Gaussian process $\{G(\boldsymbol{t}), \boldsymbol{t}\in[0,1]^d\}$ with covariance function $\mathbf{E}\, G(\boldsymbol{t})G(\boldsymbol{s}) = g(\boldsymbol{t},\boldsymbol{s})$ is ensured by theorem 7. Take $p\in\mathbb{N}$, $v_1,\dots,v_p\in\mathbb{R}$ and $\boldsymbol{t}_1,\dots,\boldsymbol{t}_p\in[0,1]^d$. Note that for any $\boldsymbol{t},\boldsymbol{s},\boldsymbol{r}\in[0,1]^d$ we have

$$\mathbf{1}\{\boldsymbol{r}\in[0,\boldsymbol{t}\wedge\boldsymbol{s}]\} = \mathbf{1}\{\boldsymbol{r}\in[0,\boldsymbol{t}]\cap[0,\boldsymbol{s}]\} = \mathbf{1}\{\boldsymbol{r}\in[0,\boldsymbol{t}]\}\mathbf{1}\{\boldsymbol{r}\in[0,\boldsymbol{s}]\}. \quad (3.47)$$

Then

$$\begin{aligned}\sum_{i=1}^p\sum_{j=1}^p v_i\mu_{\boldsymbol{n}}(\boldsymbol{t}_i\wedge\boldsymbol{t}_j)v_j &= \sum_{i=1}^p\sum_{j=1}^p v_iv_j\sum_{\boldsymbol{k}\le\boldsymbol{k_n}}\mathbf{1}\{B_{\boldsymbol{n}}(\boldsymbol{k})\in[0,\boldsymbol{t}_i\wedge\boldsymbol{t}_j]\}\sigma^2_{\boldsymbol{n},\boldsymbol{k}}\\ &= \sum_{\boldsymbol{k}\le\boldsymbol{k_n}}\sigma^2_{\boldsymbol{n},\boldsymbol{k}}\left(\sum_{i=1}^p v_i\mathbf{1}\{B_{\boldsymbol{n}}(\boldsymbol{k})\in[0,\boldsymbol{t}_i]\}\right)^2 \ge 0.\end{aligned}$$

Since this holds for each $\boldsymbol{n}$, taking the limit as $\mathrm{m}(\boldsymbol{n})\to\infty$ gives the positive definiteness of $g(\boldsymbol{t},\boldsymbol{s})$. So the first part of the theorem is proved.

Now fix $\boldsymbol{t}_1,\dots,\boldsymbol{t}_r\in[0,1]^d$ and $v_1,\dots,v_r$ real, and set

$$V_{\boldsymbol{n}} = \sum_{p=1}^r v_jZ_{\boldsymbol{n}}(\boldsymbol{t}_p) = \sum_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{k_n}}\alpha_{\boldsymbol{n},\boldsymbol{k}}X_{\boldsymbol{n},\boldsymbol{k}},$$

where

$$\alpha_{\boldsymbol{n},\boldsymbol{k}} = \sum_{p=1}^r v_p\mathbf{1}\left(\boldsymbol{B}(\boldsymbol{k})\in[0,\boldsymbol{t}_p]\right).$$

Now using (3.47) we get

$$\begin{aligned}b_{\boldsymbol{n}} := \mathbf{E}\, V_{\boldsymbol{n}}^2 &= \sum_{\boldsymbol{k}\le\boldsymbol{k_n}}\alpha^2_{\boldsymbol{n},\boldsymbol{k}}\sigma^2_{\boldsymbol{n},\boldsymbol{k}}\\ &= \sum_{\boldsymbol{k}\le\boldsymbol{k_n}}\sum_p\sum_q v_pv_q\mathbf{1}\left(\boldsymbol{B}(\boldsymbol{k})\in[0,\boldsymbol{t}_p]\right)\mathbf{1}\left(\boldsymbol{B}(\boldsymbol{k})\in[0,\boldsymbol{t}_q]\right)\sigma^2_{\boldsymbol{n},\boldsymbol{k}}\\ &= \sum_p\sum_q v_pv_q\mu_{\boldsymbol{n}}(\boldsymbol{t}_p\wedge\boldsymbol{t}_q).\end{aligned}$$

Letting $\mathrm{m}(\boldsymbol{n})$ tend to infinity and using assumption 1, we obtain

$$b_{\boldsymbol{n}} \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{} \sum_p \sum_q v_p v_q \mu(\boldsymbol{t}_p \wedge \boldsymbol{t}_q) = \mathbf{E}\left(\sum v_p G(\boldsymbol{t}_p)\right)^2 =: b.$$

If $b = 0$, then V_n converges to zero in distribution since $\mathbf{E}\, V_n^2$ tends to zero. In this special case we also have $\sum_p v_p G(\boldsymbol{t}_p) = 0$ almost surely, thus the convergence of finite dimensional distributions holds.

Assume now, that $b > 0$. For convenience put $Y_{\boldsymbol{n},\boldsymbol{k}} = \alpha_{\boldsymbol{n},\boldsymbol{k}} X_{\boldsymbol{n},\boldsymbol{k}}$ and $v = \sum_p \sum_q v_p v_q$. Conditions (3.42) and (3.43) ensures that triangular array $X_{\boldsymbol{n},\boldsymbol{j}}$ satisfies the conditions for central limit theorem: infinitesimal negligibility and Lindeberg condition. The same is true for triangular array $\{Y_{\boldsymbol{n},\boldsymbol{k}}\}$. We have

$$Y_{\boldsymbol{n},\boldsymbol{k}}^2 \leq v X_{\boldsymbol{n},\boldsymbol{k}}^2,$$

thus $Y_{\boldsymbol{n},\boldsymbol{k}}$ satisfies the condition of infinitesimal negligibility. For $\mathrm{m}(\boldsymbol{n})$ large enough to have $b_{\boldsymbol{n}} > b/2$, we get

$$\begin{aligned} &\frac{1}{\mathbf{E}\, V_{\boldsymbol{n}}^2} \sum_{\boldsymbol{1}\leq\boldsymbol{k}\leq\boldsymbol{k}_{\boldsymbol{n}}} \mathbf{E}\left(Y_{\boldsymbol{n},\boldsymbol{k}}^2 \mathbf{1}\{|Y_{\boldsymbol{n},\boldsymbol{k}}|^2 > \varepsilon^2 \mathbf{E}\, V_{\boldsymbol{n}}^2\}\right) \\ &\leq \frac{2v}{b} \sum_{\boldsymbol{1}\leq\boldsymbol{k}\leq\boldsymbol{k}_{\boldsymbol{n}}} \mathbf{E}\left(X_{\boldsymbol{n},\boldsymbol{k}}^2 \mathbf{1}\Big\{|X_{\boldsymbol{n},\boldsymbol{k}}|^2 > \frac{b\varepsilon^2}{2v}\Big\}\right). \end{aligned}$$

Thus Lindeberg condition for $V_{\boldsymbol{n}}$ is also satisfied and that gives us the convergence of finite dimensional distributions and the proof of the theorem.

3.2.2 Tightness

To prove tightness of process $\Xi_{\boldsymbol{n}}$ only certain moment conditions are required. There is no need for additional variance structure assumptions as proving the convergence of finite-dimensional distributions. This is quite clear, since due to results from section 2.2.2 and corollary 6, the process $\Xi_{\boldsymbol{n}}$ is tight if

$$\lim_{a\to\infty} P\left(\sup_{\boldsymbol{t}\in[0,1]^d} |\Xi_{\boldsymbol{n}}(\boldsymbol{t})| > a\right) = 0 \tag{3.48}$$

and for every $\varepsilon > 0$ and $m = 1, \dots, d$,

$$\lim_{\mathrm{m}(\boldsymbol{n})\to\infty} P\left(Z_{\boldsymbol{n}}^{(m)} > \varepsilon\right) = 0; \tag{3.49}$$

$$\lim_{J\to\infty} \lim_{\mathrm{m}(\boldsymbol{n})\to\infty} P\left(\sup_{j\geq J} 2^{-j\alpha} \max_{r\in D_j} \psi_{\boldsymbol{n}}(r, r^-)^{(m)} > \varepsilon\right) = 0; \tag{3.50}$$

$$\lim_{J\to\infty} \lim_{\mathrm{m}(\boldsymbol{n})\to\infty} P\left(\sup_{j\geq J} 2^{-j\alpha} \max_{r\in D_j} \psi_{\boldsymbol{n}}(r, r^+)^{(m)} > \varepsilon\right) = 0, \tag{3.51}$$

recalling respectively the definitions (2.51),(2.50) and(2.52).

Using Doob inequality (1.12) we have

$$\begin{aligned} P(\sup_{\boldsymbol{t}\in[0,1]^d} |\Xi_{\boldsymbol{n}}(\boldsymbol{t})| > a) &= P(\max_{\boldsymbol{k}\leq\boldsymbol{k_n}} |S_{\boldsymbol{n}}(\boldsymbol{k})| > a) \\ &\leq a^{-2}\mathbf{E}\, S_{\boldsymbol{n}}(\boldsymbol{k_n})^2 = a^{-2} \to 0, \text{ as } a \to \infty, \end{aligned}$$

thus (3.48) is satisfied leaving us with checking (3.49) to (3.51). As the expressions in the probability involve only sums we can use similar techniques as in proving tightness of process $\xi_{\boldsymbol{n}}$. We give now two sets of conditions.

Theorem 27 *For $0 < \alpha < 1/2$, set $p(\alpha) := 1/(1/2 - \alpha)$. If*

$$\max_{1\leq l\leq d} \max_{1\leq k_l\leq k_{\boldsymbol{n}}^l} \Delta b_l(k_l) \to 0, \text{ as } \mathrm{m}(\boldsymbol{n}) \to \infty. \tag{3.52}$$

and for some $q > p(\alpha)$

$$\lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \sum_{\mathbf{1}\leq\boldsymbol{k}\leq\boldsymbol{k_n}} \sigma_{\boldsymbol{n},\boldsymbol{k}}^{-2q\alpha} \mathbf{E}\, |X_{\boldsymbol{n},\boldsymbol{k}}|^q = 0, \tag{3.53}$$

then the net $\{\Xi_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}\}$ is asymptotically tight in the space $\mathrm{H}_\alpha^o([0,1]^d)$.

Introduce for every $\tau > 0$, the truncated random variables:

$$X_{\boldsymbol{n},\boldsymbol{k},\tau} := X_{\boldsymbol{n},\boldsymbol{k}} \mathbf{1}\{|X_{\boldsymbol{n},\boldsymbol{k}}| \leq \tau\sigma_{\boldsymbol{n},\boldsymbol{k}}^{2\alpha}\}.$$

Theorem 28 *Assume that*

$$\max_{1\leq l\leq d} \max_{1\leq k_l\leq k_{\boldsymbol{n}}^l} \Delta b_l(k_l) \to 0, \text{ as } \mathrm{m}(\boldsymbol{n}) \to \infty. \tag{3.54}$$

and that the following conditions hold.

(a). For every $\varepsilon > 0$,

$$\lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \sum_{\mathbf{1}\le\boldsymbol{k}\le\boldsymbol{k_n}} P(|X_{\boldsymbol{n},\boldsymbol{k}}| \ge \varepsilon\sigma_{\boldsymbol{n},\boldsymbol{k}}^{2\alpha}) = 0. \tag{3.55}$$

(b). For every $\varepsilon > 0$,

$$\lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \sum_{\mathbf{1}\le\boldsymbol{k}\le\boldsymbol{k_n}} \mathbf{E}\, X_{\boldsymbol{n},\boldsymbol{k}}^2 \mathbf{1}\{|X_{\boldsymbol{n},\boldsymbol{k}}| \ge \varepsilon\} = 0. \tag{3.56}$$

(c). For some $q > 1/(1/2-\alpha)$,

$$\lim_{\tau\to 0} \lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \sum_{\mathbf{1}\le\boldsymbol{k}\le\boldsymbol{k_n}} \sigma_{\boldsymbol{n},\boldsymbol{k}}^{-2q\alpha} \mathbf{E}\,|X_{\boldsymbol{n},\boldsymbol{k},\tau}|^q = 0. \tag{3.57}$$

Then the net $\{\Xi_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}\}$ is asymptoticaly tight in the space $\mathrm{H}_\alpha^o([0,1]^d)$.

Proof of the theorem 27

We only need to check that (3.52) and (3.53) ensure (3.49), (3.50), (3.51). We check only the case $m = 1$, since the proof is the same for other m, thus in following proofs we drop the superscript m.

Proof of (3.49). Using Markov and Doob (1.12) inequalities for $q > 1/(1/2-\alpha)$ we get

$$\begin{aligned} P\Big(Z_{\boldsymbol{n}} > \varepsilon\Big) &\le \sum_{k=1}^{k_{\boldsymbol{n}}^1} P\Big(\max_{\boldsymbol{k}_{2:d}\le\boldsymbol{k}_{\boldsymbol{n},2:d}} |\Delta_k^{(1)} S_{\boldsymbol{n}}(\boldsymbol{k})| > \varepsilon(\Delta b_1(k))^\alpha\Big) \\ &\le \sum_{k=1}^{k_{\boldsymbol{n}}^1} \varepsilon^{-q}(\Delta b_1(k))^{-q\alpha} \mathbf{E}\left(\max_{\boldsymbol{k}_{2:d}\le\boldsymbol{k}_{\boldsymbol{n},2:d}} |\Delta_k^{(1)} S_{\boldsymbol{n}}(\boldsymbol{k})|\right)^q \\ &\le \sum_{k=1}^{k_{\boldsymbol{n}}^1} \varepsilon^{-q}(\Delta b_1(k))^{-q\alpha} \mathbf{E}\,|\Delta_k^{(1)} S_{\boldsymbol{n}}(\boldsymbol{k_n})|^q. \end{aligned}$$

Rosenthal inequality (1.10) gives

$$P\Big(Z_{\boldsymbol{n}} > \varepsilon\Big) \le c\sum_{k=1}^{k_{\boldsymbol{n}}^1} \varepsilon^{-q}(\Delta b_1(k))^{-q\alpha}\Big((\Delta b_1(k))^{q/2} + \sum_{k_2=1}^{k_{\boldsymbol{n}}^2} \cdots \sum_{k_d=1}^{k_{\boldsymbol{n}}^d} \mathbf{E}\,|X_{\boldsymbol{n},\boldsymbol{k}}|^q\Big). \tag{3.58}$$

We have

$$\sum_{k=1}^{k_{\boldsymbol{n}}^1} (\Delta b_1(k))^{q(1/2-\alpha)} \le \left(\max_{1\le k\le k_{\boldsymbol{n}}^1} \Delta b_1(k) \right)^{q(1/2-\alpha)-1} \sum_{k=1}^{k_{\boldsymbol{n}}^1} \Delta b_1(k) \tag{3.59}$$
$$= \left(\max_{1\le k\le k_{\boldsymbol{n}}^1} \Delta b_1(k) \right)^{q(1/2-\alpha)-1} \to 0, \text{ as } \mathrm{m}(\boldsymbol{n}) \to \infty,$$

due to (3.52) and the fact that $q > (1/2-\alpha)^{-1}$. Also

$$\sum_{k=1}^{k_{\boldsymbol{n}}^1} (\Delta b_1(k))^{-q\alpha} \sum_{k_2=1}^{k_{\boldsymbol{n}}^2} \cdots \sum_{k_d=1}^{k_{\boldsymbol{n}}^d} \mathbf{E}\,|X_{\boldsymbol{n},\boldsymbol{k}}|^q = \sum_{\boldsymbol{k}\le \boldsymbol{k}_{\boldsymbol{n}}} (\Delta b_1(k_1))^{-q\alpha} \mathbf{E}\,|X_{\boldsymbol{n},\boldsymbol{k}}|^q$$
$$\le \sum_{\boldsymbol{k}\le \boldsymbol{k}_{\boldsymbol{n}}} \sigma_{\boldsymbol{n},\boldsymbol{k}}^{-2q\alpha} \mathbf{E}\,|X_{\boldsymbol{n},\boldsymbol{k}}|^q \to 0,$$

as $\mathrm{m}(\boldsymbol{n}) \to \infty$, due to (3.53), since $(\Delta b_1(k_1))^{-q\alpha} \le \sigma_{\boldsymbol{n},\boldsymbol{k}}^{-2q\alpha}$ for all $\mathbf{1} \le \boldsymbol{k} \le \boldsymbol{k}_{\boldsymbol{n}}$. Reporting these estimates to (3.58) we see that (3.52) and (3.53) imply (3.49).

Proof of (3.50) *and* (3.51). We check only (3.50) since (3.51) is treated similarly. Define

$$\Pi(J, \boldsymbol{n}, \varepsilon) = P\left(\sup_{j\ge J} 2^{-j\alpha} \max_{r\in D_j} \psi_{\boldsymbol{n}}(r, r^-)^{(m)} > \varepsilon \right)$$

then

$$\Pi(J, \boldsymbol{n}, \varepsilon) \le \sum_{j\ge J} P(2^{\alpha j} \max_{r\in D_j} \psi_{\boldsymbol{n}}(r, r^-) > \varepsilon) \le \sum_{j\ge J} \sum_{r\in D_j} \varepsilon^{-q} 2^{\alpha j q} \mathbf{E}\,|\psi_{\boldsymbol{n}}(r, r^-)|^q.$$

Doob (1.12) and then Rosenthal (1.9) inequalities give us

$$\mathbf{E}\,\psi_{\boldsymbol{n}}(r, r^-)^q \le \mathbf{E} \left| \sum_{\boldsymbol{k}_{2:d} \le \boldsymbol{k}_{\boldsymbol{n},2:d}} \left(\sum_{k_1=u_1(r^-)+2}^{u_1(r)} X_{\boldsymbol{n},\boldsymbol{k}} \right) \right|^q$$
$$\le c \left(\left(\sum_{k_1=u_1(r^-)+2}^{u_1(r)} \sum_{\boldsymbol{k}_{2:d} \le \boldsymbol{k}_{\boldsymbol{n},2:d}} \sigma_{\boldsymbol{n},\boldsymbol{k}}^2 \right)^{q/2} + \sum_{k_1=u_1(r^-)+2}^{u_1(r)} \sum_{\boldsymbol{k}_{2:d} \le \boldsymbol{k}_{\boldsymbol{n},2:d}} \mathbf{E}\,|X_{\boldsymbol{n},\boldsymbol{k}}|^q \right).$$

Due to definition of $u_1(r)$

$$\sum_{k_1=u_1(r^-)+2}^{u_1(r)} \sum_{\mathbf{1}\leq \boldsymbol{k}_{2:d}\leq \boldsymbol{k}_{\boldsymbol{n},2:d}} \sigma^2_{\boldsymbol{n},\boldsymbol{k}} = \sum_{k_1=u_1(r^-)+2}^{u_1(r)} \Delta b_1(k_1) \leq r - r^- = 2^{-j},$$

thus

$$\Pi(J,\boldsymbol{n},\varepsilon) \leq \frac{c}{\varepsilon^q}\sum_{j\geq J} 2^{(q\alpha+1-q/2)j} + \frac{c}{\varepsilon^q}\sum_{j\geq J}\sum_{r\in D_j} 2^{q\alpha j}\sum_{k_1=u_1(r^-)+2}^{u_1(r)} \sum_{\boldsymbol{k}_{2:d}\leq(\boldsymbol{k}_{\boldsymbol{n}})_{2:d}} \mathbf{E}\left|X_{\boldsymbol{n},\boldsymbol{k}}\right|^q. \tag{3.60}$$

Denote by $I(J,\boldsymbol{n},q)$ the second sum without the constant $c\varepsilon^{-q}$. By changing the order of summation we get

$$I(J,\boldsymbol{n},q) = \sum_{\mathbf{1}\leq\boldsymbol{k}\leq\boldsymbol{k}_{\boldsymbol{n}}} \mathbf{E}\left|X_{\boldsymbol{n},\boldsymbol{k}}\right|^q \sum_{j\geq J} 2^{\alpha qj}\sum_{r\in D_j}\mathbf{1}\{u_1(r^-)+1<k_1\leq u_1(r)\}. \tag{3.61}$$

The proof further proceeds as in [28]. Consider for fixed k_1 the condition

$$u_1(r^-)+1<k_1<u_1(r). \tag{3.62}$$

Suppose that there exists $r\in D_j$ satisfying (3.62) and take another $r'\in D_j$. Since u_1 is non decreasing, if $r'<r^-$ we have $u_1(r')<u_1(r^-)+1<k$, and thus r' cannot satisfy (3.62). If $r'>r$, then $r'^->r$, whence $k\leq u_1(r)\leq u_1(r'^-)<u_1(r'^-)+1$ and again if follows that r' cannot satisfy (3.62). Thus there will exists at most only one r satisfying (3.62). If such r exists we have

$$r^-\leq \sum_{i=1}^{u_1(r^-)+1}\Delta b_1(i) < \sum_{i=1}^{k_1}\Delta b_1(i) \leq \sum_{i=1}^{u_1(r)}\Delta b_1(i)\leq r.$$

Thus $\Delta b_1(k_1)\leq 2^{-j}$. So

$$\forall k_1 = 1,\dots,k^1_{\boldsymbol{n}},\ \sum_{r\in D_j}\mathbf{1}\{u_1(r^-)+1<k_1\leq u_1(r)\}\leq \mathbf{1}\{\Delta b_1(k_1)\leq 2^{-j}\}$$

so

$$\sum_{j\geq J}2^{\alpha qj}\sum_{r\in D_j}\mathbf{1}\{u_1(r^-)+1<k_1\leq u_1(r)\}\leq \frac{2^{q\alpha}}{2^{q\alpha}-1}(\Delta b_1(k_1))^{-\alpha q} \tag{3.63}$$

(we can sum only those j, for which $\Delta b_1(k_1) \le 2^{-j}$, because for larger j, r and r^- will be closer together and will fall in the same $R_{\boldsymbol{n},\boldsymbol{k}}$).

Reporting estimate (3.63) to (3.61) we get

$$I(J, \boldsymbol{n}, q) \le C \sum_{\boldsymbol{k} \le \boldsymbol{k_n}} (\Delta b_1(k_1))^{-q\alpha} \mathbf{E} \, |X_{\boldsymbol{n},\boldsymbol{k}}|^q \le \sum_{\boldsymbol{k} \le \boldsymbol{k_n}} \sigma_{\boldsymbol{n},\boldsymbol{k}}^{-2q\alpha} \mathbf{E} \, |X_{\boldsymbol{n},\boldsymbol{k}}|^q$$

and substituting this into inequality (3.60) we get

$$\Pi(J, \boldsymbol{n}; \varepsilon) \le C_1 \varepsilon^{-q} 2^{-Jq\alpha+1-q/2} + C_2 \sum_{\boldsymbol{k} \le \boldsymbol{k_n}} \sigma_{\boldsymbol{n},\boldsymbol{k}}^{-2q\alpha} \mathbf{E} \, |X_{\boldsymbol{n},\boldsymbol{k}}|^q.$$

Thus

$$\lim_{\mathrm{m}(\boldsymbol{n}) \to \infty} \Pi(J, \boldsymbol{n}; \varepsilon) = 0$$

follows from (3.53), which gives us the proof of the theorem.

Proof of theorem 28

As in proof of the theorem 27 we check (3.49), (3.50), (3.51) and give a proof only for case $m = 1$.

Proof of (3.49) Define:

$$S_{\boldsymbol{n},\tau}(\boldsymbol{k}) = \sum_{\boldsymbol{1} \le \boldsymbol{j} \le \boldsymbol{k}} X_{\boldsymbol{n},\boldsymbol{j},\tau}, \quad S_{\boldsymbol{n},\tau}(\boldsymbol{k})' = \sum_{\boldsymbol{1} \le \boldsymbol{j} \le \boldsymbol{k}} (X_{\boldsymbol{n},\boldsymbol{j},\tau} - \mathbf{E} \, X_{\boldsymbol{n},\boldsymbol{j},\tau})$$

and

$$A_{\boldsymbol{n}} = \left\{ \max_{\boldsymbol{1} \le \boldsymbol{k} \le \boldsymbol{k_n}} |X_{\boldsymbol{k}}| \le \tau \sigma_{\boldsymbol{n},\boldsymbol{k}}^{2\alpha} \right\}.$$

Then we can estimate the probability in (3.49) by

$$P(Z_{\boldsymbol{n}} > \varepsilon) =: P(\boldsymbol{n}, \varepsilon) \le P_1(\boldsymbol{n}, \varepsilon, \tau) + P(A_{\boldsymbol{n}}^c)$$

where

$$P_1(\boldsymbol{n}, \varepsilon, \tau) = P\left(\max_{\boldsymbol{1} \le \boldsymbol{k} \le \boldsymbol{k_n}} \frac{|\Delta_{k_1}^{(1)} S_{\boldsymbol{n},\tau}(\boldsymbol{k})|}{(\Delta b_1(k_1))^{\alpha}} > \varepsilon \right). \tag{3.64}$$

Due to (3.55) the probability $P(A_{\boldsymbol{n}}^c)$ tends to zero so we need only to study the asymptotics of $P_1(\boldsymbol{n}, \varepsilon, \tau)$.

Using the splitting

$$\Delta_{k_1}^{(1)} S_{\boldsymbol{n},\tau}(\boldsymbol{k}) = \Delta_{k_1}^{(1)} S'_{\boldsymbol{n},\tau}(\boldsymbol{k}) + \mathbf{E}\,\Delta_{k_1}^{(1)} S_{\boldsymbol{n},\tau}(\boldsymbol{k}),$$

let us begin with some estimate of the expectation term, since $X_{\boldsymbol{n},\boldsymbol{k},\tau}$ are not centered.

We have

$$\mathbf{E}\,|X_{\boldsymbol{n},\boldsymbol{k},\tau}| \le \mathbf{E}^{1/2} X_{\boldsymbol{n},\boldsymbol{k}}^2 P^{1/2}(|X_{\boldsymbol{n},\boldsymbol{k}}| > \tau\sigma_{\boldsymbol{n},\boldsymbol{k}}^{2\alpha}).$$

By applying Cauchy inequality we get

$$\begin{aligned}
\max_{\mathbf{1}\le\boldsymbol{k}\le\boldsymbol{k}_{\boldsymbol{n}}} \frac{|\mathbf{E}\,\Delta_{k_1}^{(1)} S_{\boldsymbol{n},\tau}(\boldsymbol{k})|}{(\Delta b_1(k_1))^\alpha} &\le \max_{1\le k_1\le k_{\boldsymbol{n}}^1} \frac{\sum_{\boldsymbol{k}_{2:d}=1}^{\boldsymbol{k}_{\boldsymbol{n},2:d}} \mathbf{E}\,|X_{\boldsymbol{n},\boldsymbol{j},\tau}|}{(\Delta b_1(k_1))^\alpha} \\
&\le \max_{1\le k_1\le k_{\boldsymbol{n}}^1} \frac{(\Delta b_1(k_1))^{1/2}\Big(\sum_{\boldsymbol{k}_{2:d}=1}^{\boldsymbol{k}_{\boldsymbol{n},2:d}} P(|X_{\boldsymbol{n},\boldsymbol{k}}| > \tau\sigma_{\boldsymbol{n},\boldsymbol{k}}^{2\alpha})\Big)^{1/2}}{(\Delta b_1(k_1))^\alpha} \\
&\le \max_{1\le k_1\le k_{\boldsymbol{n}}^1} (\Delta b_1(k_1))^{1/2-\alpha}\Bigg(\sum_{\mathbf{1}\le\boldsymbol{k}\le\boldsymbol{k}_{\boldsymbol{n}}} P(|X_{\boldsymbol{n},\boldsymbol{k}}| > \tau\sigma_{\boldsymbol{n},\boldsymbol{k}}^{2\alpha})\Bigg)^{1/2}.
\end{aligned}$$

Due to (3.54) and (3.55) the last expression is bounded by $\varepsilon/2$ for $\boldsymbol{n} \ge \boldsymbol{n}_0$, where $\boldsymbol{n}_0$ depends on ε and τ. Thus for $\boldsymbol{n} \ge \boldsymbol{n}_0$ we have $P_1(\boldsymbol{n},\varepsilon,\tau) \le P_1'(\boldsymbol{n},\varepsilon,\tau)$, where

$$P_1'(\boldsymbol{n},\varepsilon,\tau) = P\Bigg(\max_{\mathbf{1}\le\boldsymbol{k}\le\boldsymbol{k}_{\boldsymbol{n}}} \frac{|\Delta_{k_1}^{(1)} S'_{\boldsymbol{n},\tau}(\boldsymbol{k})|}{(\Delta b_1(k_1))^\alpha} > \varepsilon/2\Bigg). \tag{3.65}$$

Since

$$\mathrm{Var} X_{\boldsymbol{n},\boldsymbol{k},\tau} \le \mathbf{E}\,X_{\boldsymbol{n},\boldsymbol{k},\tau}^2 \le \mathbf{E}\,X_{\boldsymbol{n},\boldsymbol{k}}^2 = \sigma_{\boldsymbol{n},\boldsymbol{k}}^2,$$

using Markov, Doob and Rosenthal inequalities for $q > 1/(1/2-\alpha)$ we get

$$\begin{aligned} P_1'(\boldsymbol{n},\varepsilon,\tau) &\le \sum_{k=1}^{k_{\boldsymbol{n}}^1} (\varepsilon/2)^{-q}(\Delta b_1(k))^{-q\alpha}\mathbf{E}\,|\Delta_k^{(1)} S_{\boldsymbol{n},\tau}'(\boldsymbol{k}_{\boldsymbol{n}})|^q \\ &\le c\sum_{k=1}^{k_{\boldsymbol{n}}^1} (\varepsilon/2)^{-q}(\Delta b_1(k))^{-q\alpha}\Bigg((\Delta b_1(k))^{q/2} + \sum_{\boldsymbol{k}_{2:d}=1}^{\boldsymbol{k}_{\boldsymbol{n},2:d}} \mathbf{E}\,|X_{\boldsymbol{n},\boldsymbol{k},\tau}|^q\Bigg) \\ &\le c(\varepsilon/2)^{-q}\Bigg(\sum_{k=1}^{k_{\boldsymbol{n}}^1} (\Delta b_1(k))^{q(1/2-\alpha)} + \sum_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{k}_{\boldsymbol{n}}} \sigma_{\boldsymbol{n},\boldsymbol{k}}^{-2q\alpha}\mathbf{E}\,|X_{\boldsymbol{n},\boldsymbol{k},\tau}|^q\Bigg). \end{aligned}$$

Note that this estimate holds for each $\tau > 0$. Combining all the estimates we get

$$\forall\tau > 0, \quad \limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} P(\boldsymbol{n},\varepsilon) \le c \limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} \sum_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{k}_{\boldsymbol{n}}} \sigma_{\boldsymbol{n},\boldsymbol{k}}^{-2q\alpha}\mathbf{E}\,|X_{\boldsymbol{n},\boldsymbol{k},\tau}|^q.$$

with the constant c depending only on q. By letting $\tau \to 0$ due to (3.57), (3.49) follows.

Proof of (3.50) *and* (3.51) We again prove only (3.50) since (3.51) is treated similarly. Introduce definitions $\psi_{\boldsymbol{n},\tau}(r,r^-)$ and $\psi_{\boldsymbol{n},\tau}'(r,r^-)$ by exchanging variables $X_{\boldsymbol{n},\boldsymbol{k}}$ with variables $X_{\boldsymbol{n},\boldsymbol{k},\tau}$ and $X_{\boldsymbol{n},\boldsymbol{k},\tau}' := X_{\boldsymbol{n},\boldsymbol{k},\tau} - \mathbf{E}\,X_{\boldsymbol{n},\boldsymbol{k},\tau}$ respectively. Define

$$P(J,\boldsymbol{n},\varepsilon) = P\Bigg(\sup_{j\ge J} 2^{\alpha j}\max_{r\in D_j}\psi_{\boldsymbol{n}}(r,r^-) > \varepsilon\Bigg).$$

Similar to the proof of (3.49) we need only to deal with asymptotics of $P_1(J,\boldsymbol{n},\varepsilon,\tau)$, where

$$P_1(J,\boldsymbol{n},\varepsilon,\tau) = P\Bigg(\sup_{j\ge J} 2^{\alpha j}\max_{r\in D_j}\psi_{\boldsymbol{n},\tau}(r,r^-) > \varepsilon\Bigg).$$

Again we need to estimate the expectation term. We have

$$\begin{aligned}
&\sup_{j\geq J} 2^{\alpha j} \max_{r\in D_j} \max_{\mathbf{1}_{2:d}\leq \boldsymbol{k}_{2:d}\leq \boldsymbol{k}_{\boldsymbol{n},2:d}} \left| \sum_{i=u_1(r^-)+2}^{u_1(r)} \Delta_i^{(1)} \mathbf{E}\, S_{\boldsymbol{n},\tau}((i,\boldsymbol{k}_{2:d}))\right| \\
&\quad \leq \sup_{j\geq J} 2^{\alpha j} \max_{r\in D_j} \left(\sum_{i=u_1(r^-)+2}^{u_1(r)} \Delta b_1(i)\right)^{1/2} \left(\sum_{i=u_1(r^-)+2}^{u_1(r)} \sum_{\boldsymbol{k}_{2:d}=1}^{\boldsymbol{k}_{\boldsymbol{n},2:d}} P(|X_{\boldsymbol{n},\boldsymbol{k}}| > \sigma_{\boldsymbol{n},\boldsymbol{k}}^{2\alpha})\right)^{1/2} \\
&\quad \leq 2^{J(\alpha-1/2)} \left(\sum_{\mathbf{1}\leq \boldsymbol{k}\leq \boldsymbol{k}_{\boldsymbol{n}}} P(|X_{\boldsymbol{n},\boldsymbol{k}}| > \sigma_{\boldsymbol{n},\boldsymbol{k}}^{2\alpha})\right)^{1/2}.
\end{aligned}$$

The last expression is bounded by $\varepsilon/2$ for $\boldsymbol{n} \geq \boldsymbol{n}_0$, due to (3.55) where $\boldsymbol{n}_0$ depends on ε and τ, but not on J. Thus $P_1(J,\boldsymbol{n},\varepsilon,\tau) \leq P_1'(J,\boldsymbol{n},\varepsilon,\tau)$, where

$$P_1'(J,\boldsymbol{n},\varepsilon,\tau) := P\left(\sup_{j\geq J} 2^{\alpha j} \max_{r\in D_j} \psi_{\boldsymbol{n},\tau}'(r,r^-) > \varepsilon/2\right). \tag{3.66}$$

Applying the same arguments as in proving (3.60) we get

$$\begin{aligned}
P_1'(J,\boldsymbol{n},\varepsilon,\tau) &\leq \frac{c}{\varepsilon^q} \sum_{j\geq J} 2^{(q\alpha+1-q/2)j} \\
&\qquad + \frac{c}{\varepsilon^q} \sum_{j\geq J} \sum_{r\in D_j} 2^{q\alpha j} \sum_{k_1=u_1(r^-)+2}^{u_1(r)} \sum_{\mathbf{1}\leq \boldsymbol{k}_{2:d}\leq \boldsymbol{k}_{\boldsymbol{n},2:d}} \mathbf{E}\,|X_{\boldsymbol{n},\boldsymbol{k},\tau}|^q.
\end{aligned}$$

Now using estimate (3.63) we get

$$P_1'(J,\boldsymbol{n},\varepsilon,\tau) \leq C_1 2^{(q\alpha+1-q/2)J} + C_2 \sum_{\mathbf{1}\leq \boldsymbol{k}\leq \boldsymbol{k}_{\boldsymbol{n}}} \sigma_{\boldsymbol{n},\boldsymbol{k}}^{-2q\alpha} \mathbf{E}\,|X_{\boldsymbol{n},\boldsymbol{k},\tau}|^q,$$

where constants C_1 and C_2 depend on q, α and ε. Note that this inequality holds for each $\tau > 0$. Combining all the estimates we get

$$\forall \tau > 0, \quad \lim_{J\to\infty} \limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} P(J,\boldsymbol{n},\varepsilon) \leq C_2 \limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} \sum_{\mathbf{1}\leq \boldsymbol{k}\leq \boldsymbol{k}_{\boldsymbol{n}}} \sigma_{\boldsymbol{n},\boldsymbol{k}}^{-2q\alpha} \mathbf{E}\,|X_{\boldsymbol{n},\boldsymbol{k},\tau}|^q.$$

By letting $\tau \to 0$ due to (3.57), (3.50) follows.

3.2.3 Corollaries

Recalling the results from previous subsections we have the following functional central limit theorem for the summation process $\Xi_{\boldsymbol{n}}$.

Theorem 29 *Suppose there exists a function* $\mu : [0,1]^d \to \mathbb{R}$ *such that*

$$\forall \boldsymbol{t} \in [0,1]^d, \qquad \lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \mathbf{E}\,\Xi_{\boldsymbol{n}}^2(\boldsymbol{t}) = \mu(\boldsymbol{t}). \tag{3.67}$$

For $0 < \alpha < 1/2$, *set* $p(\alpha) := 1/(1/2-\alpha)$. *If*

$$\max_{1\le l\le d} \max_{1\le k_l\le k_{\boldsymbol{n}}^l} \Delta b_l(k_l) \to 0, \ \textit{as}\ \mathrm{m}(\boldsymbol{n}) \to \infty \tag{3.68}$$

and for some $q > p(\alpha)$

$$\lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \sum_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{k_n}} \sigma_{\boldsymbol{n},\boldsymbol{k}}^{-2q\alpha} \mathbf{E}\,|X_{\boldsymbol{n},\boldsymbol{k}}|^q = 0, \tag{3.69}$$

then

$$\Xi_{\boldsymbol{n}} \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{\mathrm{H}_\alpha^o([0,1]^d)} G, \tag{3.70}$$

where G *is a centered Gaussian process satisfying* $\mathbf{E}\,G(\boldsymbol{t})G(\boldsymbol{s}) = \mu(\boldsymbol{t}\wedge\boldsymbol{s})$ *for* $\boldsymbol{t},\boldsymbol{s} \in [0,1]^d$.

Proof. We have

$$\sum_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{k_n}} \mathbf{E}\,X_{\boldsymbol{n},\boldsymbol{k}}^2 \mathbf{1}\{|X_{\boldsymbol{n},\boldsymbol{k}}| \ge \varepsilon\} \le \frac{1}{\varepsilon^{q-2}} \sum_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{k_n}} \mathbf{E}\,|X_{\boldsymbol{n},\boldsymbol{k}}|^q.$$

Since $\sigma_{\boldsymbol{n},\boldsymbol{k}}^2 \le 1$, condition (3.69) ensures that $\sum_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{k_n}} \mathbf{E}\,|X_{\boldsymbol{n},\boldsymbol{k}}|^q$ converges to zero, thus conditions of theorem 26 are satisfied and we have convergence of finite-dimensional distributions. Furthermore the conditions for theorem 27 are satisfied also, so the process $\Xi_{\boldsymbol{n}}$ is asymptotically tight in the space $\mathrm{H}_\alpha^o([0,1]^d)$. The theorem then follows. □

Our motivation for introducing special construction for the collections of random variables with non-uniform variance was to get one limiting process in functional central limit theorem for all possible variance structures of the collection. It is clear from theorem 29 that this goal was not achieved. Nevertheless we think that this is an improvement compared to using non-modified construction. The convergence of process $\xi_{\boldsymbol{n}}$ in case of non-uniform variance was investigated by Goldie and Greenwood [13], [14]. Their focus was on non-independent variables and although the domain of summation process was wider, for $[0,1]^d$ their process coincides with $\xi_{\boldsymbol{n}}$. They proved the convergence to Brownian sheet in case $\boldsymbol{n} = (n,\dots,n)$ in the space of continuous functions, but naturally their result requires that $\mathbf{E}\,\xi_{\boldsymbol{n}}(\boldsymbol{t}) \to \boldsymbol{\pi}(\boldsymbol{t})$ for all $\boldsymbol{t} \in [0,1]^d$, which is the special case of our requirement (3.67). Furthermore to achieve the convergence they place quite strict conditions on variances of random variables by requiring that

$$\lim_{c\to\infty} \sup_{\mathbf{1}\le\boldsymbol{k}\le\boldsymbol{k_n}} \mathbf{E}\, n^d X_{\boldsymbol{n},\boldsymbol{k}}^2 \mathbf{1}\left(|n^{d/2} X_{\boldsymbol{n},\boldsymbol{k}}| > c\right) = 0.$$

This limit follows from Lindeberg condition if we take $\mathbf{E}\, X_{\boldsymbol{n},\boldsymbol{k}}^2 = n^{-d}$. Thus the variances in Goldie-Greenwood case have the additional restriction, which is unnecessary when using our proposed construction. Furthermore for the special structure of variances the convergence $\mathbf{E}\,\Xi_{\boldsymbol{n}}(\boldsymbol{t}) \to \boldsymbol{\pi}(\boldsymbol{t})$ is always satisfied.

Corollary 30 *Let $\sigma_{\boldsymbol{n},\boldsymbol{k}}^2 = \pi(\boldsymbol{a}_{\boldsymbol{n},\boldsymbol{k}})$, where $\{\boldsymbol{a}_{\boldsymbol{n},\boldsymbol{k}} = (a_{\boldsymbol{n},k_1}^1, \dots, a_{\boldsymbol{n},k_d}^d)\}$ is a triangular array of real vectors satisfying the following conditions for each $i = 1, \dots, d$ and for all $\boldsymbol{k} \le \boldsymbol{k_n}$.*

i) $\sum_{k=1}^{k_{\boldsymbol{n}}^i} a_{\boldsymbol{n},k}^i = 1$ *with* $a_{\boldsymbol{n},k_i}^i > 0$.

ii)

$$\max_{1\le k\le k_{\boldsymbol{n}}^i} a_{\boldsymbol{n},k}^i \to 0, \ \textit{as} \ \mathrm{m}(\boldsymbol{n}) \to \infty.$$

Then condition (3.69) is sufficient for weak convergence of summation process $\Xi_{\boldsymbol{n}}$ in the space $\mathrm{H}_\alpha^o([0,1]^d)$ and the limiting process is then Brownian sheet W.

Proof. The result follows from theorem 29 if we check that

$$\mathbf{E}\,\Xi_{\boldsymbol{n}}^2(\boldsymbol{t}) \to \boldsymbol{\pi}(\boldsymbol{t}), \tag{3.71}$$

since evidently the condition (3.37) is satisfied. We have

$$b_i(k_i) = \sum_{k=1}^{k_i} a_{\boldsymbol{n},k}^i,$$

thus

$$\mathbf{1}\{\boldsymbol{B}(\boldsymbol{k}) \in [0,\boldsymbol{t}]\} = \prod_{i=1}^d \mathbf{1}\{b_i(k_i) \in [0,t_i]\},$$

so for jump process $Z_{\boldsymbol{n}}$ we have

$$\mathbf{E}\, Z_{\boldsymbol{n}}^2(\boldsymbol{t}) = \sum_{\boldsymbol{k}\le\boldsymbol{k_n}} \mathbf{1}\{\boldsymbol{B}(\boldsymbol{t}) \in [0,\boldsymbol{t}]\}\sigma_{\boldsymbol{n},\boldsymbol{k}}^2 = \prod_{i=1}^d \sum_{k_i=1}^{k_{\boldsymbol{n}}^i} \mathbf{1}\{b_i(k_i) \in [0,t_i]\} a_{\boldsymbol{n},k_i}^i.$$

But

$$\sum_{k_i=1}^{k^i_{\boldsymbol{n}}} \mathbf{1}\{b_i(k_i) \in [0, t_i]\} a^i_{\boldsymbol{n},k_i} = \sum_{k_i=1}^{u_i(t_i)} a^i_{\boldsymbol{n},k_i} \to t_i,$$

thus (3.71) holds due to lemma 24 and the result follows. □

The weak convergence of summation process for random variables with such variance structure was investigated by Bickel and Wichura [5] for case $d = 2$. They investigated weak convergence in the space of càdlàg functions. Naturally the convergence to Brownian sheet was proved.

Since in i.i.d. case we have $\mathbf{E}\, X^2_{\boldsymbol{n},\boldsymbol{k}} = \boldsymbol{\pi}(\boldsymbol{k_n})$ this corollary then shows that theorem 29 is a generalization of invariance principle 20 in case of real valued random variables. The moment condition (3.69) in i.i.d. case then becomes

$$\lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \sum_{\boldsymbol{i}\le \boldsymbol{n}} \boldsymbol{\pi}(\boldsymbol{n})^{q\alpha} \mathbf{E}\, |X_{\boldsymbol{i}}/\pi(\boldsymbol{n})^{1/2}|^q = \lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \boldsymbol{\pi}(\boldsymbol{n})^{q(\alpha-1/2)} \mathbf{E}\, |X_{\mathbf{1}}|^q = 0$$

for some $q > 1/2 - \alpha$ and $0 < \alpha < 1/2$ which holds whenever $\mathbf{E}\, |X_{\mathbf{1}}|^q < \infty$. Compared to requirement $\sup_{t>0} t^{1/2-\alpha} P(|X_{\mathbf{1}}| > t)$ we see that our moment condition is not optimal, but not very far from optimality. We can further weaken it by introducing truncated variables

$$X_{\boldsymbol{n},\boldsymbol{k},\tau} := X_{\boldsymbol{n},\boldsymbol{k}} \mathbf{1}\{|X_{\boldsymbol{n},\boldsymbol{k}}| \le \tau\sigma^{2\alpha}_{\boldsymbol{n},\boldsymbol{k}}\}.$$

Then following theorem holds.

Theorem 31 *Suppose there exists a function* $\mu : [0, 1]^d \to \mathbb{R}$ *such that*

$$\forall \boldsymbol{t} \in [0, 1]^d, \qquad \lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \mathbf{E}\, \Xi^2_{\boldsymbol{n}}(\boldsymbol{t}) = \mu(\boldsymbol{t}). \tag{3.72}$$

If

$$\max_{1\le l\le d} \max_{1\le k_l \le k^l_{\boldsymbol{n}}} \Delta b_l(k_l) \to 0, \ \ \textit{as } \mathrm{m}(\boldsymbol{n}) \to \infty \tag{3.73}$$

and following conditions hold:

(a). For every $\varepsilon > 0$,

$$\lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \sum_{\mathbf{1}\le \boldsymbol{k}\le \boldsymbol{k_n}} P(|X_{\boldsymbol{n},\boldsymbol{k}}| \ge \varepsilon\sigma^{2\alpha}_{\boldsymbol{n},\boldsymbol{k}}) = 0; \tag{3.74}$$

(b). For every $\varepsilon > 0$,

$$\lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \sum_{\mathbf{1}\le\boldsymbol{k}\le\boldsymbol{k}_{\boldsymbol{n}}} \mathbf{E}\, X_{\boldsymbol{n},\boldsymbol{k}}^2 \mathbf{1}\{|X_{\boldsymbol{n},\boldsymbol{k}}| \ge \varepsilon\} = 0; \tag{3.75}$$

(c). For some $q > 1/(1/2-\alpha)$,

$$\lim_{\tau\to 0} \lim_{\mathrm{m}(\boldsymbol{n})\to\infty} \sum_{\mathbf{1}\le\boldsymbol{k}\le\boldsymbol{k}_{\boldsymbol{n}}} \sigma_{\boldsymbol{n},\boldsymbol{k}}^{-2q\alpha} \mathbf{E}\, |X_{\boldsymbol{n},\boldsymbol{k},\tau}|^q = 0; \tag{3.76}$$

then

$$\Xi_{\boldsymbol{n}} \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{\mathrm{H}_\alpha^o([0,1]^d)} G,$$

where G is a centered Gaussian process satisfying $\mathbf{E}\, G(\boldsymbol{t})G(\boldsymbol{s}) = \mu(\boldsymbol{t}\wedge\boldsymbol{s})$ for $\boldsymbol{t},\boldsymbol{s} \in [0,1]^d$.

Proof. The proof is identical to that of theorem 29. Just notice that the theorem conditions ensure the conditions of the theorems 28 and 19. □

Chapter 4

Applications

To apply our theoretical results we have to look at the examples where we can naturally assign multi-dimensional index to random observations. One of such examples is so called panel or longitudinal data, where a sample of individuals is observed over some period of time. In this case each observation has two indexes, one denoting the number of individual and another the time period at which the individual was observed. For such type of data all classical statistical problems can be discussed in a view of adjustments necessary for accomodation of the additional index. We restrict ourselves to regression and change point problems with the goal of developing the test for detecting the occurence of the change of the regression coefficient in a given sample.

We briefly recount the general setting. The classical panel data regression model which we investigate can be presented as

$$y_{it} = \alpha_i + \mathbf{x}'_{it}\boldsymbol{\beta} + u_{it}, \tag{4.1}$$

where $i = 1, \ldots, N$, $t = 1, \ldots, T$, y_{it} is an observation of dependent variable for individual i at time period t, $\mathbf{x}'_{it} = [x_{1it}, \ldots, x_{Kit}]$ is the $K \times 1$ vector of observations on the independent variables and u_{it} are zero mean disturbances. The classical panel regression problem is to estimate $\boldsymbol{\beta}$ in a view of various assumptions on intercepts α_i, $\mathbf{x}_{it}$ and u_{it}, see for example Baltagi [2], Hsiao [15]. After estimating $\boldsymbol{\beta}$ the usual statistical procedure is to test the goodness-of-fit and the validity of the model assumptions. One of the possible violations of the validity is that relationship (4.1) holds only for certain subsample of data, i.e. the true model is

$$y_{it} = \begin{cases} \alpha_i + \mathbf{x}'_{it}\boldsymbol{\beta}_0 + u_{it}, & \text{for } (i,t) \in I, \\ \alpha_i + \mathbf{x}'_{it}\boldsymbol{\beta}_1 + u_{it}, & \text{for } (i,t) \in I^c, \end{cases} \tag{4.2}$$

with $I \subset \{1,\dots,N\} \times \{1,\dots,T\}$. Such violation is called change point problem. It can also appear for larger class of models, usually in parametric problems, for more general treatment see Csörgo and Horvath [7]. First test for detecting the change point in the regression setting was developed by Brown, Durbin and Evans [6], for testing the model

$$y_t = \mathbf{x}_t'\boldsymbol{\beta} + u_t, \tag{4.3}$$

against the alternative

$$y_t = \begin{cases} \mathbf{x}_t'\boldsymbol{\beta}_0 + u_t, & t = 1,\dots,t_0, \\ \mathbf{x}_t'\boldsymbol{\beta}_1 + u_t, & t = t_0+1,\dots,T, \end{cases} \tag{4.4}$$

where u_t are independent standard normal variables. They examined the cumulative sums of so called recursive regression residuals:

$$w_r = \frac{y_r - \mathbf{x}_r'\boldsymbol{\beta}_{r-1}}{\sqrt{1 + \mathbf{x}_r'(\sum_{k=1}^{r-1}\mathbf{x}_k\mathbf{x}_k')^{-1}\mathbf{x}_r}},$$

where $\boldsymbol{\beta}_r$ is the least squares estimate of the model (4.3) calculated using first r observations. Suitably normalized jump sum process based on these residuals converges to Brownian motion. For the alternative model (4.4) they show the w_r no longer have zero mean, thus cumulative sum converges to infinity. The normality restriction was lifted by Sen [36], who proved similar result for the case of i.i.d. regression errors with finite variance. Ploberger and Kraämer[22] proved similar result for usual regression residuals. The limiting process in this case is the Brownian bridge.

All these three results use the same test statistic, the maximum of the cumulative sum. Since this is also a maximum norm of the jump sum process, and maximum norm is the continous functional, due to FCLT and continuous mapping theorem, the statistic converges to maximum of limiting process (Brownian motion or Brownian bridge) under null hypothesis of no change.

Other types of alternative models are also considered. For epidemic alternative:

$$y_t = \begin{cases} \mathbf{x}_t'\boldsymbol{\beta}_0 + u_t, & t = 1,\dots,t_0, t_1+1,\dots,T \\ \mathbf{x}_t'\boldsymbol{\beta}_1 + u_t, & t = t_0+1,\dots,t_1, \end{cases} \tag{4.5}$$

Račkauskas [25] proves that it is better to use the statistic

$$\max_{1<l<n} \frac{1}{l^\alpha} \max_{0<k<n-l} |S(k+l) - S(k) - \frac{l}{n} S_n|,$$

where $S(k)$ is the cumulative sum of the regression residuals. This statistic allows testing against shorter epidemics, than the usual maximum test.

In their paper Kao, Trapani and Urga [16] write "Despite the potential usefulness in economics, the econometric theory of the testing and estimation of structural changes in panels is still underdeveloped". Current results focus on testing the change point in presence of unit roots.

In light of these results we first develop the test against epidemic rectangles using techniques from Csörgő and Horvàth [7] and then apply these tests for panel regression to generalize the results of Ploberger and Krämer.

4.1 Tests for epidemic alternatives

4.1.1 Epidemic rectangles

The question arises of how to generalize epidemic alternatives for multi-indexed case. In case of panel data where we have interpretation of indexes as individuals and times several simple scenarios are immediately apparent.

- In some time interval the change occurs for all individuals.
- At the start of observation, the change occurs for certain individuals.
- At the end of observation, the change occurs for certain individuals.

For the moment assume that we are only testing the change of mean. Let $\{X_{ij}, 1 \leq i \leq n, 1 \leq j \leq m\}$ be a sample of panel data. The null hypothesis of no change then is

(H_0) : X_{ij} *have all the same mean* μ_0.

The scenarios we want to test against fall into general setting:

(H_A) : *There are integers* $1 < a^* \leq b^* < n$, $1 < c^* \leq d^* < m$ and a constant $\mu_1 \neq \mu_0$ such that

$$\mathbf{E}\, X_{ij} = \mu_0 + \mu_1 \mathbf{1}\left((i,j) \in [a^*, b^*] \times [c^*, d^*] \cap \mathbb{N}^2\right).$$

Classical log-likelihood statistic from Csörgő and Horvàth [7] for testing change of a mean in a certain set I (if it is known in advance) is

$$R = \frac{1}{\sqrt{n}} \left(\sum_{i \in I} X_i - \frac{|I|}{n} \sum_{i=1}^{n} X_n \right), \tag{4.6}$$

where $|I|$ is the cardinality of set $I \subset \{1, \dots, n\}$. This statistic is suitable for testing epidemics of size n^γ, where $\gamma > 1/2$. In order to test shorter epidemics you have to weight this statistic with some function of $|I|$.

For two-dimensional setting assume now, that integers a^*, b^*, c^*, d^* are known. Define the set

$$D^* = \left[\frac{a^*}{n}, \frac{b^*}{n} \right] \times \left[\frac{c^*}{m}, \frac{d^*}{m} \right]$$

and introduce the analog of statistic (4.6)

$$R = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{ij} \mathbf{1} \left(\left(\frac{i}{n}, \frac{j}{m} \right) \in D^* \right) - \frac{k^* l^*}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} X_{ij}$$

where $k^* = b^* - a^*$, $l^* = d^* - c^*$. Under hypothesis H_0, if X_{ij} are i.i.d. with finite variance σ^2

$$(nm)^{-1/2} R \to N(0, \sigma^2 |D^*|(1 - |D^*|))$$

when $n \wedge m \to \infty$. Under alternative hypothesis if $\{X_{ij}, (i/n, j/m) \in D^*\}$ and $\{X_{ij}, (i/n, j/m) \in [0,1]^2 \backslash D^*\}$ are separately i.i.d. but with different means, we have

$$(nm)^{-1/2} R = (nm)^{1/2} \frac{k^* l^*}{nm} \left(1 - \frac{k^* l^*}{nm} \right) (\mu_1 - \mu_0) + O_P(1).$$

and we see that the statistic will converge to infinity as long as $k^* \geq C_1 n^\gamma$ and $l^* \geq C_2 m^\delta$ with $\gamma, \delta > 1/2$ and some positive constants C_1, C_2.

In order to test shorter epidemics we have to weight the statistic R. One of the possible weights is $\operatorname{diam}(D)^\alpha$, since clearly $\operatorname{diam}(D) \to 0$, as $k^*/n \to 0$ and $l^*/m \to 0$ and vice versa.

Let us rewrite the $Q = R / \operatorname{diam}(D)^\alpha$ in terms of the summation process. Denote $s_i = i/n$, $t_j = j/m$. Then

$$
\begin{aligned}
Q &= \operatorname{diam}(D)^{-\alpha} \left(\sum_{i=a^*+1}^{b^*} \sum_{j=c^*+1}^{d^*} X_{ij} - \frac{k^* l^*}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} X_{ij} \right) \\
&= \frac{\Delta^1_{k^*} \Delta^2_{l^*} S_{b^*,d^*} - (s_{b^*} - s_{a^*})(t_{d^*} - t_{c^*}) S_{n,m}}{\max\{s_{b^*} - s_{a^*}, t_{d^*} - t_{c^*}\}^\alpha}
\end{aligned}
$$

When a^*, b^*, c^*, d^* are unknown it is reasonable to replace Q with maximum over all possible their combinations:

$$
DUI(n, m, \alpha) = \max_{\substack{1 \le a < b \le n \\ 1 \le c < d \le m}} \frac{|\Delta^1_{b-a} \Delta^2_{d-c} S_{b,d} - (s_b - s_a)(t_d - t_c) S_{n,m}|}{\max\{s_b - s_a, t_d - t_c\}^\alpha} \tag{4.7}
$$

For $\boldsymbol{n} = (n, m) \in \mathbb{N}^2$, consider the functionals $g_{n,m}$ defined on $\mathrm{H}^o_\alpha([0,1]^2)$ by

$$
g_{n,m}(x) := \max_{\boldsymbol{1} \le \boldsymbol{i} < \boldsymbol{j} \le \boldsymbol{n}} I(x, \boldsymbol{i}/\boldsymbol{n}, \boldsymbol{j}/\boldsymbol{n}), \tag{4.8}
$$

where

$$
I(x, \boldsymbol{s}, \boldsymbol{t}) = \frac{|\Delta^1_{t_1 - s_1} \Delta^2_{t_2 - s_2} x(\boldsymbol{t}) - (t_1 - s_1)(t_2 - s_2) x(\boldsymbol{1})|}{|\boldsymbol{t} - \boldsymbol{s}|^\alpha}.
$$

It is clear that

$$
DUI(n, m, \alpha) = g_{n,m}(\xi_{n,m}).
$$

The following theorem holds.

Theorem 32 *Functionals* $\{g_{n,m}, (n, m) \in \mathbb{N}^2\}$ *and* g *are continuous in the Hölder space* $\mathrm{H}^o_\alpha([0,1]^2)$. *Furthermore* $\{g_{n,m}, (n, m) \in \mathbb{N}^2\}$ *are equicontinuous and for each* $x \in \mathrm{H}^o_\alpha([0,1]^2)$

$$
\lim_{n \wedge m \to \infty} g_{n,m}(x) = g(x) \tag{4.9}
$$

where

$$
g(x) := \sup_{\boldsymbol{0} < \boldsymbol{s} < \boldsymbol{t} < \boldsymbol{1}} I(x, \boldsymbol{s}, \boldsymbol{t}). \tag{4.10}
$$

Proof. To show continuity of functionals $g_{n,m}$ and g and equicontinuity of family $\{g_{n,m}, (n, m) \in \mathbb{N}^2\}$ we use lemma 9. Clearly the functional $q = I(\cdot, \boldsymbol{s}, \boldsymbol{t})$ satisfies conditions (a) and (b) of lemma 9. Let us check condition (c). For all $\boldsymbol{t}, \boldsymbol{s} \in [0,1]^2$

$$\frac{(t_1 - s_1)(t_2 - s_2)}{|\boldsymbol{t} - \boldsymbol{s}|^\alpha} \leq 1.$$

Thus if $t_1 - s_1 \leq t_2 - s_2$, then

$$\begin{aligned} I(x, \boldsymbol{s}, \boldsymbol{t}) &\leq \frac{|x(t_1,t_2) - x(t_1,s_2)|}{|t_2 - s_2|^\alpha} + \frac{|x(s_1,t_2) - x(s_1,s_2)|}{|t_2 - s_2|^\alpha} + x(1,1) \\ &\leq 2\|x\|_\alpha. \end{aligned} \tag{4.11}$$

Similarly if $t_1 - s_1 > t_2 - s_2$

$$\begin{aligned} I(x, \boldsymbol{s}, \boldsymbol{t}) &\leq \frac{|x(t_1,t_2) - x(s_1,t_2)|}{|t_1 - s_1|^\alpha} + \frac{|x(t_1,s_2) - x(s_1,s_2)|}{|t_1 - s_1|^\alpha} + x(1,1) \\ &\leq 2\|x\|_\alpha. \end{aligned} \tag{4.12}$$

So functional $I(\cdot, \boldsymbol{s}, \boldsymbol{t})$ satisfies condition (c) with $C = 2$. Thus the continuity and equicontinuity follows immediately from (1.7).

For (4.9) it is sufficient to show that the function $(\boldsymbol{s}, \boldsymbol{t}) \to I(x, \boldsymbol{s}, \boldsymbol{t})$ can be extended by continuity to the compact set $T = \{(\boldsymbol{s}, \boldsymbol{t}) \in [0,1]^4; \boldsymbol{0} \leq \boldsymbol{s} \leq \boldsymbol{t} \leq \boldsymbol{1}\}$. From (4.11) and (4.12) we get $0 \leq I(x, \boldsymbol{s}, \boldsymbol{t}) \leq 2w_\alpha(x, |\boldsymbol{t} - \boldsymbol{s}|) + |\boldsymbol{t} - \boldsymbol{s}|^{1-\alpha} x(1,1)$, which allows continuous extension when $\boldsymbol{t} = \boldsymbol{s}$ putting $I(x, \boldsymbol{s}, \boldsymbol{s}) = 0$. □

Functionals $g_{n,m}$ and g satisfy the conditions of lemma 8 thus FCLT for X_{ij} gives the limiting distribution of statistic $DUI(n, m, \alpha)$. Due to results in previous sections, the FCLT in the space $\mathrm{H}^o_\alpha([0,1]^2)$ holds for the summation processes based on i.i.d. random variables. Thus we have to strengthen the null hypothesis:

(H_0') :X_{ij} *are independent identically distributed with mean denoted by* μ_0.

For better clarity for any real function x with two dimensional argument introduce definition

$$\Delta_{[\boldsymbol{s},\boldsymbol{t}]} x := x(\boldsymbol{t}) - x(s_1, t_2) - x(t_1, s_2) + x(\boldsymbol{s}).$$

This sometimes is called the increment of x around the rectangle $[\boldsymbol{s}, \boldsymbol{t}]$. Consider the following random variable

$$DUI(\alpha) = \sup_{\boldsymbol{0} \leq \boldsymbol{s} < \boldsymbol{t} \leq \boldsymbol{1}} \frac{|\Delta_{[\boldsymbol{s},\boldsymbol{t}]} W - (t_1 - s_1)(t_2 - s_2) W(\boldsymbol{1})|}{|\boldsymbol{t} - \boldsymbol{s}|^\alpha}. \tag{4.13}$$

Then following theorem holds.

Theorem 33 *Under H_0' assume that $0 < \alpha < 1/2$ and*

$$\sup_{t>0} t^p P(|X_{\mathbf{1}}| > t) > \infty$$

for $p = 1/(1/2 - \alpha)$. Then

$$\sigma^{-1}(nm)^{-1/2} DUI(n, m, \alpha) \xrightarrow{D} DUI(\alpha), \text{ as } n \wedge m \to \infty,$$

where $\sigma^2 = \mathbf{E}\, X_{\mathbf{1}}^2$.

Proof. Note first, that under H_0' the value of statistic $DUI(n, m, \alpha)$ does not change if $X_{\boldsymbol{i}}$ are exchanged with $X_{\boldsymbol{i}} - \mu_0$. Assume then that $\mu_0 = 0$. Theorem 21 together with theorem 32 and lemma 8 gives us the result. □

The consistency of the test is given by following theorem.

Theorem 34 *Assume under (H_A) that the X_{ij} are independent and $\sigma_0^2 = \sup_{\boldsymbol{n}} \operatorname{var}(X_{\boldsymbol{n}})$ is finite. If*

$$\lim_{\mathrm{m}(\boldsymbol{n})\to\infty} (nm)^{1/2} \frac{h_{n,m}}{d_{n,m}^{\alpha}} |\mu_1 - \mu_0| \to \infty, \tag{4.14}$$

where

$$h_{n,m} = \frac{k^* l^*}{nm}\left(1 - \frac{k^* l^*}{nm}\right) \text{ and } d_{n,m} = \max\left\{\frac{k^*}{n}, \frac{l^*}{m}\right\}, \tag{4.15}$$

then

$$(nm)^{-1/2} DUI(n, m, \alpha) \to \infty. \tag{4.16}$$

For the case $d = 1$ our result replicates the result of Račkauskas and Suquet [30]. In this case the test will be able to detect epidemics of order $n^{\frac{1-2\alpha}{2-2\alpha}}$. Now for two dimensional case assume that $k^* = n^\gamma$, $l^* = m^\delta$ and that $\mu_1 - \mu_0$ does not depend on (n, m). Then the condition (4.14) becomes

$$\frac{n^{\gamma-1/2} m^{\delta-1/2}}{[n^{\gamma-1} \vee m^{\delta-1}]^{\alpha}} \to \infty.$$

If $n^{\gamma-1} > m^{\delta-1}$ (4.14) reduces to

$$n^{\gamma(1-\alpha)+\alpha-1/2} m^{\delta-1/2} \to \infty,$$

thus we can detect very short epidemics of k^*, but we can never get better rate for epidemic length of l^*. Notice that for $\delta > 1/2$ condition $n^{\gamma-1} > m^{\delta-1}$ will be satisfied if $m > n^2$. So we see that to detect very short epidemics for one index we must have more data for the other index. In case $n = m$, $\gamma = \delta$ we have

$$n^{\gamma(1-\alpha)+\alpha-1/2+\gamma-1/2} = n^{\gamma(2-\alpha)+\alpha-1} \to \infty,$$

and the best rate is $\gamma > 1/3$. We get that in two-dimensional case the rates of epidemic are influenced not only by α, but also by the relationship between n and m.

The choice of α is important in the convergence of null hypothesis. In one dimensional case for the convergence we have the condition $\lim_{t\to\infty} t^p P(|X_1| > t) = 0$, where $p = 1/(1/2-\alpha)$. Since $\frac{1-2\alpha}{2-2\alpha} \to 0$, when $\alpha \to 1/2$, we get better rates with higher moment conditions.

For case $m = n^2$ the moment condition for convergence is $\lim_{t\to\infty} t^{\frac{3}{3/2-2\alpha}} P(|X_{\mathbf{1}}| > t) = 0$. Thus $\mathbf{E}\, X_{\mathbf{1}}^6$ is sufficient.

For case $m = n$ from theorem 22 it follows that 4^{th} moment is sufficient for convergence for any α, but the rate cannot be lower than $1/3$. In one dimensional case detecting an epidemic of length $n^{1/3}$ comes with the choice $\alpha = 1/4$, which means that we need 4^{th} moment.

Proof of the theorem 34

Define set $I_{n,m} = [a^*, b^*] \times [c^*, d^*] \cap \mathbb{N}^2$ and random variables

$$X_{ij} := \begin{cases} X_{ij} - \mu_0, (i,j) \in I^c_{n,m} \\ X_{ij} - \mu_1, (i,j) \in I_{n,m} \end{cases}$$

We have

$$\begin{aligned} \Delta^1_{k^*}\Delta^2_{l^*} S_{b^*,d^*} - (s_{b^*} - s_{a^*})(t_{d^*} - t_{c^*}) S_{n,m} &= S(I_{n,m}) - \frac{k^* l^*}{nm}(S(I_{n,m}) + S(I^c_{n,m})) \\ &= k^* l^* \left(1 - \frac{k^* l^*}{nm}\right)(\mu_1 - \mu_0) + R_{n,m}, \end{aligned} \tag{4.17}$$

where

$$R_{n,m} := -\frac{k^* l^*}{nm} \sum_{i \in I^c_{n,m}} X'_i + \left(1 - \frac{k^* l^*}{nm}\right) \sum_{i \in I_{n,m}} X'_i.$$

Now

$$\operatorname{var}((nm)^{-1/2}R_{n,m}) \le \frac{1}{nm}\left(\frac{k^*l^*}{nm}\right)^2 (nm - k^*l^*)\sigma_0^2 + \frac{1}{nm}\left(1-\frac{k^*l^*}{nm}\right)^2 k^*l^*\sigma_0^2 = \sigma_0^2 h_{n,m}.$$

This estimate together with (4.17) leads to the lower bound

$$(nm)^{-1/2}DUI(n,m,\alpha) \ge (nm)^{1/2}\frac{h_{n,m}}{d_{n,m}^{\alpha}}|\mu_1-\mu_0| + O_P\left(\frac{h_{n,m}^{1/2}}{d_{n,m}^{\alpha}}\right).$$

Now $h_{n,m} \le d_{n,m}^2$, thus $\lim_{d_{n,m}\to 0} h_{n,m}^{1/2}/d_{n,m}^{\alpha} = 0$, so the theorem follows due to condition (4.14).

4.1.2 Some special cases

In previous section we constructed statistic for detetecting the change in subrectangle of unit square. Our motivation for such statistic came from three simple scenarios:

S1. At the start of observation, the change occurs for certain individuals

S2. At the end of observation, the change occurs for certain individuals.

S3. In some time interval the change occurs for all individuals

Using results from the previous section we can adapt the general statistic $DUI(n,m,\alpha)$ for each of these scenarios. Recall that the alternative hypothesis was of the change in an epidemic rectangle

$$D^* = \left[\frac{a^*}{n}, \frac{b^*}{n}\right] \times \left[\frac{c^*}{m}, \frac{d^*}{m}\right].$$

Then the respective epidemic rectangles for the scenarios are

$$D_1^* = \left[0, \frac{b^*}{n}\right] \times \left[0, \frac{d^*}{m}\right],$$
$$D_2^* = \left[\frac{a^*}{n}, 1\right] \times \left[\frac{c^*}{m}, 1\right],$$
$$D_3^* = [0,1] \times \left[\frac{c^*}{m}, \frac{d^*}{m}\right].$$

Denote $s_i = i/n$, $t_j = j/m$. Then the statistic adapted for specific first scenario is

$$DUI_1(n,m,\alpha) = \max_{\substack{1\le b\le n\\ 1\le d\le m}} \frac{|S_{b,d} - s_b t_d S_{n,m}|}{\max\{s_b, t_d\}^\alpha},$$

since $S_{0,d} = S_{0,0} = S_{b,0} = 0$, and $s_a = t_c = 0$. For the second scenario we have

$$DUI_2(n,m,\alpha) = \max_{\substack{1\le b\le n\\ 1\le d\le m}} \frac{|\Delta^1_{n-a}\Delta^2_{m-c}S_{n,m} - (1-s_a)(1-t_c)S_{n,m}|}{\max\{(1-s_a),(1-t_c)\}^\alpha}.$$

For the third scenario the statistic is defined as

$$DUI_3(n,m,\alpha) = \max_{1\le c<d\le m} \frac{|S_{n,d} - S_{n,c} - (t_d - t_c)S_{n,m}|}{(t_d - t_c)^\alpha},$$

where we changed the denominator, since $S_{0,c} = S_{0,d} = 0$ and in the nominator only the difference of the second argument matters. All the three statistics are the functionals of summation process $\xi_{n,m}$ similar to functional $g_{\boldsymbol{n}}$ defined by (4.10). With minimal adaptation similar proposition to the theorem 32 holds. Define following functionals of Brownian sheet

$$\begin{aligned}
DUI_1(\alpha) &= \sup_{\mathbf{0}<\boldsymbol{t}<\mathbf{1}} \frac{|W(\boldsymbol{t}) - t_1t_2W(1,1)|}{|\boldsymbol{t}|^\alpha}\\
DUI_2(\alpha) &= \sup_{\mathbf{0}<\boldsymbol{t}<\mathbf{1}} \frac{|\Delta_{[\boldsymbol{t},\mathbf{1}]}W - (1-t_1)(1-t_2)W(\mathbf{1})|}{|\mathbf{1}-\boldsymbol{t}|^\alpha},\\
DUI_3(\alpha) &= \sup_{0<s<t<1} \frac{|W(1,t) - W(1,s) - (t-s)W(1,1)|}{|t-s|^\alpha}.
\end{aligned}$$

Then following theorem holds.

Theorem 35 *For i.i.d. sample of double-indexed data and under null hypothesis of no change for scenarios S1, S2 and S3 assume that* $0 < \alpha < 1/2$ *and*

$$\sup_{t>0} t^p P(|X_{\mathbf{1}}| > t) > \infty,$$

for $p = 1/(1/2-\alpha)$*. Then for* $i = 1,\ 2,\ 3.$

$$\sigma^{-1}(nm)^{-1/2}DUI_i(n,m,\alpha) \xrightarrow{D} DUI_i(\alpha),\ \ \text{as } n\wedge m\to\infty,$$

where $\sigma^2 = \mathbf{E}\, X_{\mathbf{1}}^2$.

Under the alternative hypothesis of the change of the mean in the rectangles D_i^*, $i = 1, 2, 3$, the lengths of epidemics are

$$\begin{aligned} &k^* = b^*, \quad l^* = d^*, \text{ for the rectangle } D_1^*, \\ &k^* = 1 - a^*, \quad l^* = 1 - c^*, \text{ for the rectangle } D_2^*, \\ &k^* = n, \quad l^* = d^* - c^*, \text{ for the rectangle } D_3^*. \end{aligned}$$

For the rectangles D_1^* and D_2^* the consistency is then the direct corollary of the theorem (34).

Corollary 36 *Given the independent family* X_{ij} *with* $\sigma_0^2 = \sup_{\boldsymbol{n}} \operatorname{var}(X_{\boldsymbol{n}})$ *finite under alternative hypothesis of the change of the mean in rectangles* D_1^* *and* D_2^* *we have*

$$(nm)^{-1/2} DUI_i(n, m, \alpha) \to \infty \tag{4.18}$$

for $i = 1, 2$ *if*

$$\lim_{\mathrm{m}(\boldsymbol{n}) \to \infty} (nm)^{1/2} \frac{h_{n,m}}{d_{n,m}^{\alpha}} |\mu_1 - \mu_0| \to \infty, \tag{4.19}$$

where

$$h_{n,m} = \frac{k^* l^*}{nm} \left(1 - \frac{k^* l^*}{nm} \right) \text{ and } d_{n,m} = \max \left\{ \frac{k^*}{n}, \frac{l^*}{m} \right\}. \tag{4.20}$$

For the rectangle D_3^* the conditions for the consistency are slightly different, since the denominator in the test statistic is different.

Corollary 37 *Given the independent family* X_{ij} *with* $\sigma_0^2 = \sup_{\boldsymbol{n}} \operatorname{var}(X_{\boldsymbol{n}})$ *finite under alternative hypothesis of the change of the mean in rectangle* D_3^* *we have*

$$(nm)^{-1/2} DUI_3(n, m, \alpha) \to \infty, \tag{4.21}$$

if

$$\lim_{\mathrm{m}(\boldsymbol{n}) \to \infty} (nm)^{1/2} \left(\frac{l^*}{m} \right)^{1-\alpha} \left(1 - \frac{l^*}{m} \right) |\mu_1 - \mu_0| \to \infty. \tag{4.22}$$

This corollary enables us to improve the detection of short epidemics for the scenario S3 . If we let $l^* = m^\delta$ for some $\delta > 0$, condition (4.22) becomes

$$n^{1/2} m^{\delta(1-\alpha)+\alpha-1/2}(1 - m^{\delta-1}) \to \infty,$$

for $\delta > \frac{1/2-\alpha}{1-\alpha}$. Since $0 < \alpha < 1/2$ we can make δ arbitrarily small. Thus for certain rectangles it is possible to get similar results as in one dimensional case.

4.2 Functional central limit theorems for panel data regressions

4.2.1 Models and the assumptions

Suppose we have a sample of panel data $\{(y_{ij}, \mathbf{x}'_{ij}), i = 1, \dots, n; j = 1, \dots, m\}$ where $\mathbf{x}'_{ij} = (x_{ij1}, \dots, x_{ijK})$. We investigate FCLT for following panel regression models

$$y_{ij} = \mathbf{x}'_{ij}\boldsymbol{\beta} + u_{ij}, \tag{4.23}$$
$$y_{ij} = \mathbf{x}'_{ij}\boldsymbol{\beta} + \mu_i + u_{ij}, \tag{4.24}$$

where u_{ij} are the disturbances, mean-zero random variables with finite variance independent of y_{ij} and $\mathbf{x}_{ij}$.

The goal of panel regression is to estimate coefficient vector $\boldsymbol{\beta}$. The model (4.23) is the classical linear regression model for observations with two dimensional indexes. The coefficients $\boldsymbol{\beta}$ are then usually estimated using least squares:

$$\widehat{\boldsymbol{\beta}} = \left(\sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{x}_{ij}\mathbf{x}'_{ij}\right)^{-1} \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{x}_{ij} y_{ij}. \tag{4.25}$$

In classical panel data literature (Baltagi [2], Hsiao [15]) this estimate is called pooled or ordinary least squares estimate, and it is assumed that $x_{ij1} = 1$ for all i, j, i.e. there is only one constant term. For the model (4.24) the constant term is allowed to vary through i and is considered as a nuisance parameter. The coefficient vector $\boldsymbol{\beta}$ in this case is estimated by solving least squares problem for the model

$$y_{ij} - \overline{y}_{i.} = (\mathbf{x}_{ij} - \overline{\mathbf{x}}_{i.})\boldsymbol{\beta} + u_{ij} - \overline{u}_{i.}, \tag{4.26}$$

where

$$\overline{y}_{i\cdot} = \frac{1}{m}\sum_{j=1}^{m} y_{ij}, \quad \overline{\mathbf{x}}_{i\cdot} = \frac{1}{m}\sum_{j=1}^{m} \mathbf{x}_{ij}, \quad \overline{u}_{i\cdot} = \frac{1}{m}\sum_{j=1}^{m} u_{ij}.$$

The estimate of $\boldsymbol{\beta}$ is then

$$\widehat{\boldsymbol{\beta}}^{FE} = \left(\sum_{i=1}^{n}\sum_{j=1}^{m}(\mathbf{x}_{ij} - \overline{\mathbf{x}}_{i.})(\mathbf{x}_{ij} - \overline{\mathbf{x}}_{i.})'\right)^{-1} \sum_{i=1}^{n}\sum_{j=1}^{m}(\mathbf{x}_{ij} - \overline{\mathbf{x}}_{i\cdot})(y_{ij} - y_{i\cdot}). \quad (4.27)$$

We are interested in proving FCLT for regression residuals. For model (4.23) they are defined as

$$\widehat{u}_{ij} = y_{ij} - \mathbf{x}_{ij}'\widehat{\boldsymbol{\beta}}. = u_{ij} - \mathbf{x}_{ij}'(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}).$$

By substituting the expression for y_{ij} we immediately get

$$\widehat{u}_{ij} = u_{ij} - \mathbf{x}_{ij}'(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}). \quad (4.28)$$

Now

$$\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta} = \left(\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{x}_{ij}\mathbf{x}_{ij}'\right)^{-1} \sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{x}_{ij}u_{ij},$$

and we see that FCLT for regression residuals depends on distributional properties of regression disturbances. For this section let us make the following assumption.

Assumption F *Let random variables u_{ij} have zero mean, variance σ^2 and be independent of $\mathbf{x}_{ij}$. Assume that the summation process based on these random variables defined as*

$$\xi_{n,m}(t,s) = nm\sum_{i=1}^{n}\sum_{j=1}^{m}\left|\left[\frac{i-1}{n},\frac{i}{n}\right) \times \left[\frac{j-1}{m},\frac{j}{m}\right) \cap [0,t]\times[0,s]\right| u_{ij}, \quad (4.29)$$

satisfies the functional central limit theorem:

$$\frac{1}{\sigma\sqrt{nm}}\xi_{n,m}(t,s) \xrightarrow{D} W(t,s), \text{ as } n\wedge m \to \infty,$$

in the space $\mathrm{H}_{\alpha}^{o}([0,1]^2)$ with $0 < \alpha < 1/2$.

From (4.28) it is also evident that we have to make assumptions on $\mathbf{x}_{ij}$ in the panel regression model (4.23).

Assumption A *Let* $x_{ij1} = 1$ *for all* $\mathbf{1} \le (i,j) \le (n,m)$*. Assume that*

$$\lim_{n\wedge m\to\infty} \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{x}_{ij}\mathbf{x}_{ij}' = R, \tag{4.30}$$

for some nonsingular $(K \times K)$ *matrix* R*. Furthermore assume that the model is reparameterized such that*

$$R = \begin{bmatrix} 1 & 0 \\ 0 & R^* \end{bmatrix}, \tag{4.31}$$

which in turn implies that

$$\boldsymbol{c} \equiv \lim_{n\wedge m\to\infty} \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{x}_{ij} = [1, 0, \dots, 0]'. \tag{4.32}$$

Assumption F implies that u_{ij} satisfy central limit theorem, which together with assumption A implies that

$$\frac{1}{\sqrt{nm}}\left(\sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{x}_{ij}\mathbf{x}_{ij}'\right)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{x}_{ij}u_{ij} \xrightarrow{D} N(0,\sigma^2 R), \text{ as } n\wedge m \to \infty. \tag{4.33}$$

Assumption A also implies that

$$\frac{1}{nm}\sum_{i=1}^{[nt]}\sum_{j=1}^{[ms]} \mathbf{x}_{ij} \to ts\boldsymbol{c}, \text{ as } n\wedge m \to \infty,$$

for each fixed t and s. Using results from section 2.1 we get that summation process based on $\mathbf{x}_{ij}$ also has the same limit

$$\begin{aligned} \boldsymbol{X}_{n,m}(t,s) &= \frac{1}{nm}\sum_{j=1}^{m} nm\left|\left[\frac{i-1}{n},\frac{i}{n}\right)\times\left[\frac{j-1}{m},\frac{j}{m}\right)\cap[0,t]\times[0,s]\right|\mathbf{x}_{ij} \\ &\to ts\boldsymbol{c} \end{aligned} \tag{4.34}$$

for each fixed t and s. But assumption A also ensures that $\{\boldsymbol{X}_{n,m}, (n,m) \in \mathbb{N}^2\}$ is equicontinuous in Hölder space $\mathrm{H}_\alpha^o([0,1]^2,\mathbb{R}^K)$. Thus we get that

$$\boldsymbol{X}_{n,m}(t,s) \to ts\boldsymbol{c}, \text{ as } n\wedge m \to \infty. \tag{4.35}$$

in $\mathrm{H}^o_\alpha([0,1]^2,\mathbb{R}^K)$. We can write

$$\sqrt{nm}(\widehat{\boldsymbol\beta}-\boldsymbol\beta)=\begin{bmatrix}1 & 0\\ 0 & R^*\end{bmatrix}\begin{bmatrix}\dfrac{1}{\sqrt{nm}}\displaystyle\sum_{i=1}^n\sum_{j=1}^m u_{ij}\\ \dfrac{1}{\sqrt{nm}}\displaystyle\sum_{i=1}^n\sum_{j=1}^m \mathbf{x}^*_{ij}u_{ij},\end{bmatrix}+o_P(1)$$

where $\mathbf{x}^*_{ij}=[x_{ij2},\dots,x_{ijK}]'$. This representation together with (4.34) gives us the convergence

$$\frac{1}{\sqrt{nm}}\sum_{j=1}^m nm\left|\left[\frac{i-1}{n},\frac{i}{n}\right)\times\left[\frac{j-1}{m},\frac{j}{m}\right)\cap[0,t]\times[0,s]\right|\mathbf{x}'_{ij}(\widehat{\boldsymbol\beta}-\boldsymbol\beta)$$
$$\xrightarrow{D} tsW(1,1),$$

as $n\wedge m\to\infty$ in the space $\mathrm{H}^o_\alpha([0,1]^2)$. We have proved the following theorem.

Theorem 38 *For the panel regression model*

$$y_{ij}=\mathbf{x}'_{ij}\boldsymbol\beta+u_{ij}, \tag{4.36}$$

define the summation process

$$\widehat{W}^{(n,m)}(t,s)=nm\sum_{i=1}^n\sum_{j=1}^m\left|\left[\frac{i-1}{n},\frac{i}{n}\right)\times\left[\frac{j-1}{m},\frac{j}{m}\right)\cap[0,t]\times[0,s]\right|\widehat{u}_{ij}.$$

Given the assumptions F and A we have

$$\frac{1}{\sigma\sqrt{nm}}\widehat{W}^{(n,m)}(t,s)\xrightarrow{D} W(t,s)-tsW(1,1),\ \text{as } n\wedge m\to\infty,$$

in the space $\mathrm{H}^o_\alpha([0,1]^2)$, *with* $0<\alpha<1/2$.

Let us turn now to the model (4.24). Since the fixed effect estimate $\widehat{\boldsymbol\beta}^{FE}$ comes from the adjusted regression (4.26) it is natural to define residuals as

$$\widehat{u}^{FE}=\widetilde{y}_{ij}-\widetilde{\mathbf{x}}_{ij}\widehat{\boldsymbol\beta}^{FE},$$

where $\widetilde{y}_{ij}=y_{ij}-\overline{y}_{i\cdot}$, and $\widetilde{\mathbf{x}}_{ij}$ is defined analogously. Substituting the model (4.24) we get that

$$\widehat{u}^{FE}_{ij}=u_{ij}-\overline{u}_{i\cdot}-\widetilde{\mathbf{x}}'_{ij}(\widehat{\boldsymbol\beta}^{FE}-\boldsymbol\beta).$$

For model (4.24) we make slightly different assumptions about $\mathbf{x}_{ij}$.

Assumption B *Assume that*

$$\lim_{n\wedge m\to\infty}\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\widetilde{\mathbf{x}}_{ij}\widetilde{\mathbf{x}}_{ij}'=\tilde{R} \tag{4.37}$$

for some nonsingular $(K\times K)$ *matrix* $\tilde{R}$, *and that*

$$\lim_{n\wedge m\to\infty}\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{x}_{ij}=\boldsymbol{c} \tag{4.38}$$

for some $\boldsymbol{c}\in\mathbb{R}^K$.

Now

$$\begin{aligned}\widehat{\boldsymbol{\beta}}^{FE}-\boldsymbol{\beta}&=\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\widetilde{\mathbf{x}}_{ij}\widetilde{\mathbf{x}}_{ij}'\right)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}\widetilde{\mathbf{x}}_{ij}(u_{ij}-\overline{u}_{i\cdot})\\&=\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\widetilde{\mathbf{x}}_{ij}\widetilde{\mathbf{x}}_{ij}'\right)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}\widetilde{\mathbf{x}}_{ij}u_{ij},\end{aligned}$$

since

$$\sum_{j=1}^{m}\widetilde{\mathbf{x}}_{ij}=0.$$

Thus assumption (F) gives us

$$\sqrt{nm}(\widehat{\boldsymbol{\beta}}^{FE}-\boldsymbol{\beta})\xrightarrow{D}N(0,\sigma^2\widetilde{R}).$$

From condition (4.38) we get

$$\frac{1}{nm}\sum_{i=1}^{[nt]}\sum_{j=1}^{[ms]}\widetilde{\mathbf{x}}_{ij}\to 0,$$

for each fixed t and s. Then similar to (4.35) for summation process $\widetilde{\boldsymbol{X}}_{n,m}(t,s)$ based on $\widetilde{\mathbf{x}}_{ij}$ we get that

$$\widetilde{\boldsymbol{X}}_{n,m}(t,s)\to 0,\ \text{ as } n\wedge m\to\infty, \tag{4.39}$$

in $\mathrm{H}^o_\alpha([0,1]^2, \mathbb{R}^K)$. Now relationships

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{m} nm \left| \left[\frac{i-1}{n}, \frac{i}{n}\right) \times \left[\frac{j-1}{m}, \frac{j}{m}\right) \cap [0,t] \times [0,s] \right| \overline{u}_{i\cdot} \\
&= \sum_{i=1}^{n}\sum_{l=1}^{m} n \left| \left[\frac{i-1}{n}, \frac{i}{n}\right) \cap [0,t] \right| u_{il} \sum_{j=1}^{m} \left| \left[\frac{j-1}{m}, \frac{j}{m}\right) \cap [0,s] \right| \\
&= s \sum_{i=1}^{n}\sum_{j=1}^{m} nm \left| \left[\frac{i-1}{n}, \frac{i}{n}\right) \times \left[\frac{j-1}{m}, \frac{j}{m}\right) \cap [0,t] \times [0,1] \right| u_{ij}
\end{aligned}$$

complete the proof of the following theorem.

Theorem 39 *For the panel regression model*

$$y_{ij} = \mathbf{x}'_{ij}\boldsymbol{\beta} + \mu_i + u_{ij}, \tag{4.40}$$

define the summation process

$$\widehat{W}^{FE}_{nm}(t,s) = nm \sum_{i=1}^{n}\sum_{j=1}^{m} \left| \left[\frac{i-1}{n}, \frac{i}{n}\right) \times \left[\frac{j-1}{m}, \frac{j}{m}\right) \cap [0,t] \times [0,s] \right| \widehat{u}^{FE}_{ij}.$$

Given the assumptions F and B, we have

$$\widehat{W}^{FE}_{nm}(t,s) \xrightarrow{D} W(t,s) - sW(t,1), \ \ \text{as } n \wedge m \to \infty,$$

in the space $\mathrm{H}^o_\alpha([0,1]^2)$, *with* $0 < \alpha < 1/2$.

4.2.2 Local alternatives

It is possible to get meaningful results if we alter the original regression models. Assume that coefficient $\boldsymbol{\beta}$ in the pooled regression actually varies accross i and j:

$$\boldsymbol{\beta}_{ij} = \boldsymbol{\beta} + \frac{1}{\sqrt{nm}} \boldsymbol{g}\left(\frac{i}{n}, \frac{j}{m}\right) \tag{4.41}$$

where $\boldsymbol{g}$ is K-vector valued function continuous on $[0,1]^2$.

As $n \wedge m \to \infty$ this alternative to the regresion model

$$y^{loc}_{ij} = \mathbf{x}'_{ij}\boldsymbol{\beta}_{ij} + u_{ij},$$

converges to model (4.23). Define least squares estimate for this regression as

$$\widehat{\boldsymbol{\beta}}^{loc} = \left(\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{x}_{ij}\mathbf{x}_{ij}'\right)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{x}_{ij}y_{ij}^{loc}.$$

The following theorem then holds

Theorem 40 *Define summation process $\widehat{W}^{(n,m)}$ based on residuals*

$$u_{ij}^{loc} = y_{ij}^{loc} - \mathbf{x}_{ij}'\widehat{\boldsymbol{\beta}}^{loc}.$$

Given the assumptions F and A we have

$$\widehat{W}^{(n,m)}(t,s) \xrightarrow{D} W(t,s) - tsW(1,1) + \int_0^t\int_0^s \boldsymbol{c}'\boldsymbol{g}(u,v)\,\mathrm{d}u\,\mathrm{d}v - ts\boldsymbol{c}'\int_0^1\int_0^1 \boldsymbol{g}(u,v)\,\mathrm{d}u\,\mathrm{d}v.$$

Proof. Note that

$$\widehat{\boldsymbol{\beta}}^{loc} = \widehat{\boldsymbol{\beta}} + \frac{1}{\sqrt{nm}}\boldsymbol{d}_{n,m},$$

where

$$\boldsymbol{d}_{n,m} = \left(\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{x}_{ij}\mathbf{x}_{ij}'\right)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{x}_{ij}\mathbf{x}_{ij}'\boldsymbol{g}\left(\frac{i}{n},\frac{j}{m}\right).$$

Thus we can decompose the residuals u_{ij}^{loc} into following sums:

$$\begin{aligned} u_{ij}^{loc} &= u_{ij} + \mathbf{x}_{ij}'\boldsymbol{\beta}_{ij} - \mathbf{x}_{ij}'\widehat{\boldsymbol{\beta}}^{loc} \\ &= u_{ij} - \mathbf{x}_{ij}'(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) + \frac{1}{\sqrt{nm}}\mathbf{x}_{ij}'\boldsymbol{g}\left(\frac{i}{n},\frac{j}{m}\right) - \frac{1}{\sqrt{nm}}\mathbf{x}_{ij}'\boldsymbol{d}_{n,m} \\ &= \widehat{u}_{ij} + \frac{1}{\sqrt{nm}}\mathbf{x}_{ij}'\boldsymbol{g}\left(\frac{i}{n},\frac{j}{m}\right) - \frac{1}{\sqrt{nm}}\mathbf{x}_{ij}'\boldsymbol{d}_{n,m}. \end{aligned}$$

Using assumption A and properties of $\boldsymbol{g}$ for each fixed t and s we get

$$\frac{1}{nm}\sum_{i=1}^{[nt]}\sum_{j=1}^{[ms]}\mathbf{x}_{ij}'\boldsymbol{g}\left(\frac{i}{n},\frac{j}{m}\right) \to \int_0^t\int_0^s \boldsymbol{c}'\boldsymbol{g}(u,v)\,\mathrm{d}u\,\mathrm{d}v, \tag{4.42}$$

as $n \wedge m \to \infty$. Using the same arguments as in (4.34) we get that summation process based on $\mathbf{x}_{ij}'\boldsymbol{g}\left(\frac{i}{n},\frac{j}{m}\right)$ has the same limit in Hölder space $\mathrm{H}_\alpha^o([0,1]^2)$.

Similarly we get

$$\boldsymbol{d}_{n,m} \to \int_0^1 \int_0^1 \boldsymbol{g}(u,v)\,\mathrm{d}u\,\mathrm{d}v, \text{ as } n \wedge m \to \infty. \tag{4.43}$$

Since $\boldsymbol{d}_{n,m}$ does not depend on i and j, the convergence (4.34) and theorem 38 complete the proof. $\square$

Consider the same local alternatives (4.41) for the fixed-effects panel regression. Then the alternative model is

$$y_{ij}^{loc} = \mu_i + \mathbf{x}_{ij}'\beta_{ij} + u_{ij}. \tag{4.44}$$

Define analogously fixed effect estimate

$$\boldsymbol{\beta}_{loc}^{FE} = \left(\sum_{i=1}^{n}\sum_{j=1}^{m} \widetilde{\mathbf{x}}_{ij}\widetilde{\mathbf{x}}_{ij}'\right)^{-1} \sum_{i=1}^{n}\sum_{j=1}^{m} \widetilde{\mathbf{x}}_{ij}\widetilde{y}_{ij}^{loc}. \tag{4.45}$$

Then the following theorem holds.

Theorem 41 *Define summation process $\widehat{W}^{(n,m)}$ based on residuals*

$$\widetilde{u}_{ij}^{loc} = \widetilde{y}_{ij}^{loc} - \widetilde{\mathbf{x}}_{ij}'\boldsymbol{\beta}_{loc}^{FE}.$$

Given the assumptions F and B we have

$$\widehat{W}^{(n,m)}(t,s) \xrightarrow{D} W(t,s) - sW(t,1) + \int_0^t \int_0^s \boldsymbol{c}'\boldsymbol{g}(u,v)\,\mathrm{d}u\,\mathrm{d}v - s\int_0^t \int_0^1 \boldsymbol{c}'\boldsymbol{g}(u,v)\,\mathrm{d}u\,\mathrm{d}v,$$

as $n \wedge m \to \infty$, in the space $\mathrm{H}_\alpha^o([0,1]^2)$, with $0 < \alpha < 1/2$.

Proof. Introduce definitions

$$\boldsymbol{g}_{ij} = \boldsymbol{g}\left(\frac{i}{n}, \frac{j}{m}\right), \ \bar{\boldsymbol{f}}_{i\cdot} = \frac{1}{m}\sum_{i=1}^{m} \mathbf{x}_{ij}'\boldsymbol{g}_{ij}.$$

Then for our alternative model (4.44) we get

$$\widetilde{y}_{ij}^{loc} = \widetilde{\mathbf{x}}_{ij}'\boldsymbol{\beta} + \widetilde{u}_{ij} + \frac{1}{\sqrt{nm}}\mathbf{x}_{ij}'\boldsymbol{g}_{ij} - \frac{1}{\sqrt{nm}}\bar{\boldsymbol{f}}_{i\cdot}.$$

Then from the definition (4.45) it follows that

$$\boldsymbol{\beta}_{loc}^{FE} = \widehat{\boldsymbol{\beta}}^{FE} + \frac{1}{\sqrt{nm}} \boldsymbol{d}_{n,m},$$

where

$$\boldsymbol{d}_{n,m} = \left(\sum_{i=1}^{n} \sum_{j=1}^{m} \widetilde{\mathbf{x}}_{ij} \widetilde{\mathbf{x}}_{ij}' \right)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{m} \widetilde{\mathbf{x}}_{ij} (\mathbf{x}_{ij}' \boldsymbol{g}_{ij} - \bar{\boldsymbol{f}}_{i\cdot}).$$

Substituting these expressions to the definition of the residuals we obtain

$$\begin{aligned}
\widehat{u}_{ij}^{loc} &= \widetilde{\mathbf{x}}_{ij}' \boldsymbol{\beta} + \widetilde{u}_{ij} + \frac{1}{\sqrt{nm}} (\mathbf{x}_{ij}' \boldsymbol{g}_{ij} - \bar{\boldsymbol{f}}_{i\cdot}) - \widetilde{\mathbf{x}}_{ij}' \boldsymbol{\beta}_{loc}^{FE} \\
&= \widetilde{u}_{ij} - \widetilde{\mathbf{x}}_{ij}' (\widehat{\boldsymbol{\beta}}^{FE} - \boldsymbol{\beta}) + \frac{1}{\sqrt{nm}} (\mathbf{x}_{ij}' \boldsymbol{g}_{ij} - \bar{\boldsymbol{f}}_{i\cdot}) - \frac{1}{\sqrt{nm}} \widetilde{\mathbf{x}}_{ij}' \boldsymbol{d}_{n,m} \\
&= \widehat{u}_{ij}^{FE} + \frac{1}{\sqrt{nm}} (\mathbf{x}_{ij}' \boldsymbol{g}_{ij} - \bar{\boldsymbol{f}}_{i\cdot}) - \frac{1}{\sqrt{nm}} \widetilde{\mathbf{x}}_{ij}' \boldsymbol{d}_{n,m}.
\end{aligned}$$

Now given assumption B, similar to (4.42) we get that

$$\frac{1}{nm} \sum_{i=1}^{[nt]} \sum_{j=1}^{[ms]} \mathbf{x}_{ij}' \boldsymbol{g} \left(\frac{i}{n}, \frac{j}{m} \right) \to \int_0^t \int_0^s \boldsymbol{c}' \boldsymbol{g}(u, v) \, \mathrm{d}u \, \mathrm{d}v,$$

as $n \wedge m \to \infty$ for each fixed t and s. Similarly

$$\frac{1}{nm} \sum_{i=1}^{[nt]} \sum_{j=1}^{[ms]} \bar{\boldsymbol{f}}_{i\cdot} = \frac{1}{nm} \sum_{i=1}^{[nt]} \sum_{j=1}^{m} \mathbf{x}_{ij}' \boldsymbol{g} \left(\frac{i}{n}, \frac{j}{m} \right) \to s \int_0^t \int_0^1 \boldsymbol{c}' \boldsymbol{g}(u, v) \, \mathrm{d}u \, \mathrm{d}v,$$

as $n \wedge m \to \infty$ for each fixed t and s. From

$$\sum_{j=1}^{m} \widetilde{\mathbf{x}}_{ij} = 0$$

it follows that

$$\frac{1}{nm} \sum_{i=1}^{m} \sum_{j=1}^{m} \widetilde{\mathbf{x}}_{ij} \bar{\boldsymbol{f}}_{i\cdot} = 0 \text{ and } \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \widetilde{\mathbf{x}}_{ij} \bar{\mathbf{x}}_{i\cdot}' = 0.$$

Since $\boldsymbol{g}$ is continuous on $[0,1]^2$, it is bounded:

$$\sup_{t,s\in[0,1]^2} |\boldsymbol{g}(t,s)| \leq C.$$

Then

$$\begin{aligned}
\boldsymbol{d}_{n,m} &= \left(\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\widetilde{\mathbf{x}}_{ij}\widetilde{\mathbf{x}}_{ij}'\right)^{-1}\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\widetilde{\mathbf{x}}_{ij}(\mathbf{x}_{ij}g_{ij}-\bar{\boldsymbol{f}}_{i\cdot}) \\
&= \left(\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\widetilde{\mathbf{x}}_{ij}\widetilde{\mathbf{x}}_{ij}'\right)^{-1}\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\widetilde{\mathbf{x}}_{ij}(\widetilde{\mathbf{x}}_{ij}'g_{ij}+\bar{\mathbf{x}}_{i\cdot}'g_{ij}) \leq C
\end{aligned}$$

and similar to the proof of the theorem 40 the convergence (4.39) and theorem 38 complete the proof. □

4.3 Change point statistics for panel regressions

4.3.1 Tests and their behaviour under null hypothesis

Combining results from previous sections we can now suggest statistics for detecting the change point in panel regression models and give their limiting distributions. Under assumption F and respective assumptions A and B the residuals for our regression models

$$y_{ij} = \mathbf{x}_{ij}'\boldsymbol{\beta} + u_{ij}, \tag{4.46}$$

$$y_{ij} = \mathbf{x}_{ij}'\boldsymbol{\beta} + \mu_i + u_{ij}, \tag{4.47}$$

satisfy functional central limit theorem. Then it is natural to "plug" respective partial sums of these regression residuals to statistic $DUI(n,m,\alpha)$ and its special variants $DUI_i(n,m,\alpha)$, $i=1,2,3$. For the model (4.46) define the partial sums

$$\widehat{S}_{kl} = \sum_{i=1}^{k}\sum_{j=1}^{l}\widehat{u}_{ij}$$

and the statistic

$$\widehat{DUI}(n,m,\alpha) = \max_{\substack{1\le a<b\le n\\ 1\le c<d\le m}} \frac{|\Delta^1_{b-a}\Delta^2_{d-c}\widehat{S}_{b,d} - (s_b - s_a)(t_d - t_c)\widehat{S}_{n,m}|}{\max\{s_b - s_a, t_d - t_c\}^\alpha} \tag{4.48}$$

with $s_k = k/n$ and $t_l = l/m$. We have then the following corollary

Corollary 42 *Under null hypothesis of no change in the regression coefficient $\boldsymbol{\beta}$ of the model 4.46 given the assumptions F and A*

$$\sigma^{-1}(nm)^{-1/2}\widehat{DUI}(n,m,\alpha) \xrightarrow{D} DUI(\alpha), \text{ as } n\wedge m \to \infty,$$

for the local alternatives model

$$\boldsymbol{\beta}_{ij} = \boldsymbol{\beta} + \frac{1}{\sqrt{nm}}\boldsymbol{g}\left(\frac{i}{n}, \frac{j}{m}\right)$$

with $\boldsymbol{g}$ continuous on $[0,1]^2$, we have similar result with the limiting statistic

$$DUI^{loc}(\alpha) = \sup_{\mathbf{0}\le \boldsymbol{s}<\boldsymbol{t}\le \mathbf{1}} \frac{|\Delta_{[\boldsymbol{s},\boldsymbol{t}]}W - \boldsymbol{\pi}(\boldsymbol{t}-\boldsymbol{s})W(\mathbf{1}) + \int_{[\boldsymbol{s},\boldsymbol{t}]}\boldsymbol{c}'\boldsymbol{g} - \boldsymbol{\pi}(\boldsymbol{t}-\boldsymbol{s})\int_{[0,1]^2}\boldsymbol{c}'\boldsymbol{g}|}{|\boldsymbol{t}-\boldsymbol{s}|^\alpha}. \tag{4.49}$$

Note that the limiting statistic remains the same as in the theorem 33. This corollary can be considered as the generalization of the results of Ploberger and Krämer [22] for the regression of double-indexed random variables.

For the regression model (4.47) define partial sums

$$\widehat{S}^{FE}_{kl} = \sum_{i=1}^{k}\sum_{j=1}^{l}\widehat{u}^{FE}_{ij}$$

and the statistic

$$\widehat{DUI}^{FE}(n,m,\alpha) = \max_{\substack{1\le a<b\le n\\ 1\le c<d\le m}} \frac{|\Delta^1_{b-a}\Delta^2_{d-c}\widehat{S}_{b,d} - (s_b - s_a)(t_d - t_c)\widehat{S}_{n,m}|}{\max\{s_b - s_a, t_d - t_c\}^\alpha}. \tag{4.50}$$

The limiting distribution then is

$$DUI^{FE}(\alpha) = \sup_{\mathbf{0}\le \boldsymbol{s}<\boldsymbol{t}\le \mathbf{1}} \frac{|\Delta_{[\boldsymbol{s},\boldsymbol{t}]}W - (t_2 - s_2)[W(t_1,1) - W(s_1,1)]|}{|\boldsymbol{t}-\boldsymbol{s}|^\alpha}.$$

and the corollary

Corollary 43 *Under null hypothesis of no change in the regression coefficient* $\boldsymbol{\beta}$ *of the model* (4.24) *given the assumptions* F *and* B

$$\sigma^{-1}(nm)^{-1/2}\widehat{DUI}^{FE}(n,m,\alpha) \xrightarrow{D} DUI^{FE}(\alpha), \;\; as\; n \wedge m \to \infty,$$

for the local alternatives model

$$\boldsymbol{\beta}_{ij} = \boldsymbol{\beta} + \frac{1}{\sqrt{nm}}\boldsymbol{g}\left(\frac{i}{n},\frac{j}{m}\right)$$

with $\boldsymbol{g}$ *continuous on* $[0,1]^2$*, we have similar result with the limiting statistic*

$$DUI_{loc}^{FE}(\alpha) = \tag{4.51}$$

$$\sup_{\mathbf{0}\le \boldsymbol{s}<\boldsymbol{t}\le \mathbf{1}} \frac{|\Delta_{[\boldsymbol{s},\boldsymbol{t}]}W - (t_2-s_2)\Delta^1_{t_1-s_1}W(t_1,1) + \int_{[\boldsymbol{s},\boldsymbol{t}]}\boldsymbol{c}'\boldsymbol{g} - (t_2-s_2)\int_{s_1}^{t_1}\int_0^1 \boldsymbol{c}'\boldsymbol{g}|}{|\boldsymbol{t}-\boldsymbol{s}|^{\alpha}}. \tag{4.52}$$

4.3.2 Consistency of the epidemic alternatives

Consider that there is a change of the regression coefficient $\boldsymbol{\beta}$ in rectangle

$$D^* = \left[\frac{a^*}{n},\frac{b^*}{n}\right] \times \left[\frac{c^*}{m},\frac{d^*}{m}\right],$$

or that the true panel regression models are

$$y_{ij} = \mathbf{x}_{ij}'\boldsymbol{\beta}_0 + \mathbf{x}_{ij}'\boldsymbol{d}_{ij} + u_{ij} \tag{4.53}$$

$$y_{ij} = \mathbf{x}_{ij}'\boldsymbol{\beta}_0 + \mathbf{x}_{ij}'\boldsymbol{d}_{ij} + \mu_i + u_{ij}, \tag{4.54}$$

where

$$\boldsymbol{d}_{ij} = (\boldsymbol{\beta}_1 - \boldsymbol{\beta}_0)\mathbf{1}\left(\left[\frac{i}{n},\frac{j}{m}\right] \cap D^*\right).$$

Denote by

$$\boldsymbol{e}_{n,m} = \left(\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{x}_{ij}\mathbf{x}_{ij}'\right)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{x}_{ij}\mathbf{x}_{ij}'\boldsymbol{d}_{ij}$$

and introduce quantity

$$\Delta(n,m,D^*) = \left(1 - \frac{k^*l^*}{nm}\right)\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{x}_{ij}'\boldsymbol{d}_{ij}$$
$$- \sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{x}_{ij}'\boldsymbol{e}_{n,m}\mathbf{1}\left(\left[\frac{i}{n},\frac{j}{m}\right]\cap D^*\right) + \frac{k^*l^*}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{x}_{ij}'\boldsymbol{e}_{n,m},$$

where $k^* = b^* - a^*$, $l^* = d^* - c^*$ are the lengths of epidemics.

Theorem 44 *Under alternative hypothesis of the change of the regression coefficient* $\boldsymbol{\beta}$ *of the model* (4.46)

$$(nm)^{-1/2}\widehat{DUI}(n,m,\alpha) \to \infty, \ \ \textit{as}\ n\wedge m \to \infty,$$

provided

$$(nm)^{-1/2}\max\left\{\frac{k^*}{n},\frac{l^*}{m}\right\}^{-\alpha}|\Delta(nm,D^*)| \to \infty, \ \ \textit{as}\ n\wedge m \to \infty.$$

Proof. The least squares estimate for the model (4.53) satisfies

$$\widehat{\boldsymbol{\beta}} = \left(\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{x}_{ij}\mathbf{x}_{ij}'\right)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{x}_{ij}'y_{ij}$$
$$= \boldsymbol{\beta}_0 + \left(\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{x}_{ij}\mathbf{x}_{ij}'\right)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{x}_{ij}u_{ij} + \boldsymbol{e}_{n,m}.$$

The regression residuals then satisfy

$$\widehat{u}_{ij}^{alt} = \widehat{u}_{ij} + \mathbf{x}_{ij}'\boldsymbol{d}_{ij} - \mathbf{x}_{ij}'\boldsymbol{e}_{n,m},$$

where $\widehat{u}_{ij}$ are the regression residuals of the model (4.46). Under alternative hypothesis then

$$\widehat{DUI}(n,m,\alpha) \geq \max\left\{\frac{k^*}{n},\frac{l^*}{m}\right\}^{-\alpha}\left|\Delta_{k^*}^1\Delta_{l^*}^2\widehat{S}_{b^*d}^{alt} - \frac{k^*l^*}{nm}\widehat{S}_{n,m}^{alt}\right|$$
$$\geq \max\left\{\frac{k^*}{n},\frac{l^*}{m}\right\}^{-\alpha}(|\Delta(nm,D^*)| - |T(n,m,D^*)|),$$

where

$$T(n,m,D^*) = \Delta^1_{k^*}\Delta^2_{l^*}\widehat{S}_{b^*d} - \frac{k^*l^*}{nm}\widehat{S}_{n,m}$$

since

$$(nm)^{-1/2}\max\left\{\frac{k^*}{n},\frac{l^*}{m}\right\}^{-\alpha} T(n,m,D^*) = O_P(1),$$

due to corollary 42, the proof is complete. $\square$

Denote by

$$\mathbf{x}^*_{ij} = \mathbf{x}_{ij} - \frac{1}{m}\sum_{j=c^*}^{d^*}\mathbf{x}_{ij},$$

and define

$$\Delta^{FE}(n,m,D^*) = \left(1-\frac{k^*l^*}{nm}\right)\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{x}'^*_{ij}\boldsymbol{d}_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{x}'^*_{ij}\boldsymbol{e}^{FE}_{n,m}\mathbf{1}\left(\left[\frac{i}{n},\frac{j}{m}\right]\cap D^*\right) + \frac{k^*l^*}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{x}'^*_{ij}\boldsymbol{e}^{FE}_{n,m},$$

where

$$\boldsymbol{e}^{FE}_{n,m} = \left(\sum_{i=1}^{n}\sum_{j=1}^{m}\widetilde{\mathbf{x}}_{ij}\widetilde{\mathbf{x}}'_{ij}\right)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}\widetilde{\mathbf{x}}_{ij}\mathbf{x}'^*_{ij}\boldsymbol{d}_{ij}.$$

Theorem 45 *Under alternative hypothesis of the change of the regression coefficient $\boldsymbol{\beta}$ of the model (4.47)*

$$(nm)^{-1/2}\widehat{DUI}^{FE}(n,m,\alpha) \to \infty, \text{ as } n\wedge m\to\infty,$$

provided

$$(nm)^{-1/2}\max\left\{\frac{k^*}{n},\frac{l^*}{m}\right\}^{-\alpha}|\Delta^{FE}(n,m,D^*)| \to \infty, \text{ as } n\wedge m\to\infty.$$

Proof. Due to definition of $\boldsymbol{d}_{ij}$ we have

$$\widetilde{y}_{ij} = \widetilde{\mathbf{x}}'_{ij}\boldsymbol{\beta}_0 + \mathbf{x}'^*_{ij}\boldsymbol{d}_{ij} + \widetilde{u}_{ij},$$

since $\widehat{\boldsymbol{\beta}}^{FE}$ is the least squares estimate of

$$\widetilde{y}_{ij} = \widetilde{\mathbf{x}}_{ij}'\boldsymbol{\beta} + \widetilde{u}_{ij}.$$

The proof is identical to the proof of the theorem 44 with notation changed. □

4.3.3 Practical considerations and discussion

Note that all of the results in this section relied on the assumption, that regression disturbances satisfy FCLT in Hölder space $\mathrm{H}_\alpha^o([0,1]^2)$. From the chapter 3 we know that FCLT holds if disturbances are i.i.d. and satisfy the moment condition

$$\sup_{t>0} t^{1/(1/2-\alpha)} P(|u_{11}| > t) < \infty.$$

For the practical applications the i.i.d. condition sometimes can be too restrictive. On the other hand to lift this restriction we just need to prove the FCLT for wider class of double-indexed random variables. Statistics for testing the change of the regression coefficients remain the same.

Throughout this chapter we focused on constructing statistics for testing against epidemic alternatives. Then the FCLT in Hölder space is needed, since the statistics are the functionals which are continuous only in Hölder space. If in the statistic $DUI(n, m, \alpha)$ we drop the denominator, then the statistics are continuous functionals in the spaces $C([0,1]^2)$ and $D([0,1]^2)$. Then all the results from previous section apply for such statistics given the assumption of the FCLT in $C([0,1]^d)$ or $D([0,1]^2)$. The FCLT in $D([0,1]^2)$ is proved for wider class of random variables, for strictly stationary multi-indexed random variables satisfying mixing condition by Deo [9] and for strictly stationary multi-indexed martingale differences by Basu and Dorea [4] to name a few. Thus we can apply these types of statistics for wider class of disturbances. In particular if we drop the denominator in the statistic $DUI_1(n, m, \alpha)$, our results are then a generalization of Ploberger and Krämer [22].

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