Chapter 5. EVALUATION OF THE RETURN ON INVESTMENT

Objectives:

• To evaluate the yield of cash flows using various methods.

• To simulate mathematical and real content situations related to the cash flow management and analysis.

• To evaluate cash flows in a time sequence, to provide the comparative assessments, to present alternative suggestions.

Examined study results:

• Will apply the methods of cash flow evaluation.

• Will simulate mathematical and real content situations, will provide alternative suggestions.

• Will assess the return on investments.

Student achievement assessment criteria

• Correct use of concepts.

• Proper use of formulas.

• Correct intermediate and final answers.

• Correct answers to questions.

Repeat the concepts: cash value, discounting, discount factor, the present and future value of a cash-flow, periodic payments, the present and future value of an ordinary annuity and a complex annuity.

The business environment is an integral part of the investment process, the essence of which can be briefly presented in the following way: with the business ideas and funds people invest in investment projects in order to obtain a return the nominal value of which exceeds that of the invested money in the future. There is a natural question of how to determine where the prepared investment project will succeed or not? This section will explore the assessment methods for the return on investment of investment projects.

5.1 Method of the present value

Investment projects are related to cash flows, which include both income and expenses (investments). We will analyse these flows and will evaluate the return on investment with the help of mathematical methods. When studying a cash flow we will discount the funds of the entire flow to the selected moment in time (a focal date), and at this point we will compare income with outlay. On the basis of this income-outlay difference we will provide the conclusions on the return on an investment project. Thus, the main evaluation methods are related to the current value calculation and an equation at the selected point in time. In relation to this issue, the concept of cash value at the moment of the present time is very important.

Example. Which of the following alternatives should be a priority: A the return of 6000 after two years or the return of 10000 after five years: the alternative B the return of 7000 now and the return of 7000 after seven years. Assume the cash value is 11%.

Consider these alternatives by choosing a focal date as now." By analyzing the alternative A we obtain the value of 6000 after two years having recalculated it at the present moment (having discounted within the two-year interval) is equal to $6000 \cdot 1.11^2 = 4870$. Meanwhile, the nominal value of 10000 will have a value of $10000 \cdot 1.11^{-5} = 59351$ after five years having recalculated it at the present period. Then the value of the alternative A now is equal to the sum of these values 10805. We will analyze the alternative B. The current value of 7000 is 7000. Meanwhile, after seven years the present value of 7000 is $7000 \cdot 1.11^{-7} = 3372$. Then the total value of the alternative B is 10372.

The alternative A should be given a priority because its present value is bigger than that of B. On the other hand, in nominal terms, the alternative B is greater than A.

Example. The insurance company offers a contribution of 50000 now (Project A) or ten years at the end of each year by the regular payments of 8000 (Project B). Which project should be chosen by a client, if the interest rate is 12% which converted on an annual basis?

The present value of 8000 coincides with that of a conventional annuity, the fixed payments of which are $R = 8000, n = 10, i = 0.12$. Then $A_n = 8000 a_{10\rceil 0.12} = 45202$. We see that in this case a customer should choose the alternative B.

Example. Let us consider two investment projects: A the investment in a business for 5 years with an interest rate of 8%, B the investment in the stock market for 3 years with an interest rate of 9%. Determine which project and at what conditions is more profitable?

We see that the investment terms are different. Thus, we need to determine the interest rate at which the three-year investment in the Project B would be equivalent to that of the Project B. With this in mind we calculate:

$$(1+0,09)^3(1+r)^2 = (1+0,08)^5; \;\; r = \sqrt{\frac{1,08^5}{1,09^3}} - 1 \approx 0,065.$$

We see that if the interest rate exceeds 6.5% after three years, it is better to choose the Project B.

Note. In the case of the project analysis the future cash value is also important, they may be evaluated by employing the methods of cash value prediction.

Let us formalize the above-discussed situation. Suppose that the time sequence t_k is linked to the payment sequence C_{t_k}, respectively, $k = 1, 2, ...n$. Here t_1 is the first payment period (the point in time), t_n the last payment period. Payments C_{t_i} can be both income and outlay at a specified period in time. In the case of non-fixed moments in time, and payment are possibly made at any period in time, a continuous parameter should be used for the cash flow indication, by marking a flow element with the symbol of C_t. If the amount C_t is invested, we say that an investor encounters outlay or makes payments. In this case the flow element will be marked by C_t^-. If an income is received, the flow element is marked by C_t^+. If at the same moment in time investments are made and an income is received, the following symbol can be used

$$C_t = C_t^+ - C_t^-.$$

The last value is generally referred to as the value of a net cash flow at the moment in time t. While defining cash flows, we distinguish three values, i.e. incomes and reserves, which are considered to be positive, and the net flow, which can be both positive and negative.

Note. The time interval between the adjacent elements of a flow may not necessarily be equal. What is more, a cash flow defined this way can be considered a continuous function in respect of time, while the net flow is marked by the symbol of

$$C_t = C(t), \;\; t \in [0, T].$$

During the analysis of cash flows we assume that if time is discrete when:

1. All incomes are received at the end of the period.
2. All reserves (investments) are made at the beginning of the period.

Note. When time is continuous, there is no need to talk about the beginning and the end of the period. In this case it is the moment in time that will be talked about. During the analysis of investment projects, tables are quite easy to use to describe cash flows, and their example is presented below.

Example. Assume that an initial investment of the 5-year investment project is 100000. It is planned that in the future investments have to be made four years in a row and each year these investments would decrease by 10000. Incomes from this project are started to be received from the third year. The first estimated payment must be 100000 and by each following year it must increase by 50000. Make a cash flow table.

N_o the number of year;

C^- costs;

C^+ income;

C the amount of cash flow.

Nr	C^-	C^+	C
0	100000	0	-10000
1	90000	0	-90000
2	80000	0	-80000
3	70000	100000	30000
4	60000	150000	90000
5	0	200000	200000
$\sum$	400000	450000	50000

Definition. *The book value* of a cash flow will be called the difference between the total income and the total outlay of a flow.

In other words, the book value of a flow is considered the time sum of the net flow elements. It is indicated in the lower-right corner of a flow table. The book value of the above-presented flow (in the Table) is 50000.

Suppose that when investment projects are compared on the basis of the book value for their book value method is applied. Suppose that the book value of the first project is B_1, and that of the second B_2. Then, on the basis of the book value method, the first project is preferred in respect of the second one, if $B_1 > B_2$.

The book value method shows the income-outlay balance of an investment project; however, during the application of this method the cash value is not taken into account. We will discuss another method, which eliminates this drawback.

Definition. The present value of a cash flow, which will be marked by PV_{in}, will be called a sum of the present values of all the income at the focal date point. The present value of a cash flow marked by PV_{out} will be called a sum of the total income at the focal date point.

The a focal date point is usually (but not necessarily) the moment in time of the first payment.

Formalize this definition. Suppose that the investment period is T and $0 < t < T$. Then

$$PV_{in}(T) = \sum_{t \in [0,T]} \frac{C_t^+}{(1+r_t)^t}, \quad PV_{out}(t) = \sum_{t \in [0,T]} \frac{C_t^-}{(1+r_t)^t}.$$

Note. Please note that PV_{in} and PV_{out} are the sums of the discounted income and outlay, respectively. A general formula given above may depend on the time when the interest rate is r_t, i.e. it can be discounted with the interest rate of the flow element moment t. Unless noted separately, we will explore the situations where $r_t = r$, i.e. we will discount using the present interest rate (that of a focal date). In other words, the discounting process is performed with the interest rate, which was noticed at the moment of the conclusion of an investment project (now). If t is any point in time (the time interval from the beginning of an investment to the moment in time t), the exact method is applied during the discounting in the case of a compound interest.

If the income (reserve) intervals are constant, their total number is n and the actual interest rate during these intervals is i, then in this case the present values (for the fixed n) is the function of r, i.e.

$$PV_{in}(r,T) = \sum_{j=1}^{n} \frac{C_j^+}{(1+r)^j}, \quad PV_{out}(r,T) = \sum_{j=1}^{n} \frac{C_j^-}{(1+r)^j}.$$

Note. The functions of $PV(r,T)_{out}$ and $PV(r,T)_{in}$ are always non-negative.

Example. Determine the present value of a cash flow, when the income is fixed and every year equals to 1000, if payments are made for the period of five years at the end of the year when:

a) the interest rate is 0.1;

b) the interest rate is 0.2.

We have that

$$PV_n(r) = 1000 a_{5\rceil r}.$$

Then

$$PV_{in}(0.1) = 1000 \frac{1-(1,1)^{-5}}{0,1} = 3790,79;$$

and

$$PV_{in}(0,2) = 1000 \frac{1-(1,2)^{-5}}{0,2} = 2290,61.$$

We see that

$$PV_{in}(0.1) > P_{in}(0.2).$$

Thus, the function is decreasing.

Note. In the rest of this section the discount interest rate is marked with the letter r.

Definition. The present value of a net cash flow which will be marked by $NPV(r)$, will be called the sum of the total net income at the point of the focal date point, where the actual interest rate is$r$. Denote the element of the moment in time t of the net cash flow

$$C_t(r) = C_t^+(r) - C_t^-(r), \quad t \in [0,T].$$

Then the net present value is calculated as follows:

$$NPV(r) = \sum_{t\in[0,T]} \frac{C_t}{(1+r)^t} = PV_{in}(r) - PV_{out}(r).$$

Suppose that P is an initial investment. The other version of the formula displayed above is very often used in literature:

$$NPV(r) = \sum_{t\in(0,T]} \frac{C_t}{(1+r)^t} - P.$$

5.2 Evaluation of the investment projects

Let us assume that we examine an investment projects. Suppose that the return on an investment project is determined based on the net present value method, if $NPV > 0$. Otherwise, i.e. if $NVP < 0$, then an investment project is detrimental and should not be developed (based on net present value method). The net present value is the function of a variable r (interest rate) (NPV: NPV = (r)), assuming that an investment period is fixed. It is known that if $PV_{in} > PV_{out}$, then $NPV > 0$; $PV_{in} = PV_{out}$, then $NPV = 0$; $PV_{in} < PV_{out}$, then $NPV < 0$;

5.2.1 Net Present Value Method

While comparing investment projects, the projects with a non-negative net present value are approved. Otherwise, the projects are rejected. If we have an opportunity to choose between projects we prefer the one with a higher net present value. The interest rate r used in the formula of $NPV(r)$ is called the rate of return on an investment project.

Note that T the moment in time, starting from now (at a focal date), $C_t, t \in [0,T]$ the payment at the moment in time t; r_t the interest rate at the moment in time t.

Suppose that the following set of investment projects is given: $IP_i, ..., IP_n$ is given. Suppose that the net present values of these investment projects are $NPV_l(r), ..., NPV_n(r)$, respectively. Then the investment project IP_k is the best for investments, if

$$NPV_k(r) = \max_{j\in\{1,\ldots,n\}} NPV_j(r).$$

An investment project is considered the most appropriate when its net present value, at the same interest rate of return is the highest. This method is called the net present value method. We see that the present value of a cash flow is a function depending on time t; in addition, in the general case, the interest rate can also be considered a function of time. Thus, while calculating the present value of an investment project the attention has to be paid to the fact that each flow can be discounted not only at different times, but also with a different discount interest rate which is marked by $r_t = f(t)$. Note that we will examine the case when the rate of return $f(t) = r$ is constant. Otherwise, the rate of return is a function of time, so while calculating the net value of an investment project, the interest rate is used to model the function, or to agree at what functional dependence on time the rate of return is used in an investment project.

Example. Suppose that you can invest 20000 in a business, which will guarantee you the following return at the end of the year:

Year	C
0	-20000
2	10000
3	8000
5	6000

Assume that the interest rate is 7 percent; the interest is converted on an annual basis. Find the present value of reserves. By subtracting an initial investment from the sum of the present values of reserves we obtain:

$$NPV(0.02) = 10000(1.02)^{-2} + 8000(1.07)^{-3} + 6000(1.07)^{-5} - 20000$$

$$\approx 8734.39 + 6530.384 + 4277.916 - 2000 = -457.31.$$

We see that $NVP < 0$. We make a conclusion that the business is not profitable (according to this method.)

Example. Assume that an investment project is defined in the table below.

Metai	C^-	C^+	C
0	100000	0	-10000
1	90000	0	-90000
2	80000	0	-80000
3	70000	100000	30000
4	60000	150000	90000
5	0	200000	200000
totals	400000	450000	50000

We calculate the net present values for the different rates of return. We have that

$$NPV(0,04) = -100000 - \frac{90000}{1,04} - \frac{80000}{1,04^2} + \frac{30000}{1,04^3} +$$

$$\frac{90000}{1,04^4} + \frac{200000}{1,04^5} \approx 7484,$$

and

$$NPV(0,06) = -100000 - \frac{90000}{1,06} - \frac{80000}{1,06^2} + \frac{30000}{1,06^3} +$$

$$\frac{90000}{1,06^4} + \frac{200000}{1,06^5} \approx -10176.$$

Note that the function of $NPV(r)$ is continuous in respect of a variable $r > 0$. Therefore, there is a rate of return $r \in (0.04, 0.06)$ such as that $NPV(r) = 0$.

Tasks for the practice

1. Determine which proposal should be given a priority: If the Proposal A will be selected at the beginning 500000 will have to be invested, and then every three months 50000 should have been received for exactly nine years. In the case of the Proposal B 70000 will have to be invested annually for eight years, and in addition to this, the annual income of 210000 will be received for nine years. If the Proposal C will be selected, every year 110000 have to be invested nine years in a row and in addition the monthly income of 13000 will be received. Determine which option should be selected using the current value method, if it is known that the rate of return is 12% and the interest is converted on a quarterly basis?

2. When replacing the old equipment with a new one the amount of 5000000 needs to be invested now and the amount of 100000 every year for their maintenance. However, the replacement could result in monthly savings of 120000 for the period of ten years. Determine whether the equipment should be replaced, if the interest rate is 12

3. As a part of the investment project of 8 years a joint stock company invested 450000 in the first year, and then it has been investing 100000 every year for the period of five years. Determine the net present value of this project if it is known that starting from the end of the third year the income will amount to 400000, and then every year it will increase by 10%. The cash value of the period is 10%.

4. While building an office space a construction company invested 600000 for the first year and it will have to invest another 400000 for the renovation of these offices after six years. It expectd to lease the premises for twelve years and it is anticipated that after twelve years the value of these premises will be 1000000. The annual income of these premises is 160000. Find the net present value of this investment project during the period of twelve year, if the alternative investment includes the rate of 12%.

5.2.2 Internal rate of return (IRR) method.

Definition. The interest rate r_0 will be called an internal rate of return (profit rate), briefly $IRR = r_0$, where the following equality is correct:

$$NPV(r_0) = \sum_{t=0}^{n} \frac{C_t}{(1+r_0)^t} = 0.$$

In other words, the IRR is an interest rate at which the present value of reserves is equal to the present value of income. An internal rate of return is a marginal interest rate based on which the rate of return on a project is determined.

The project is considered profitable if its rate of return r is greater than or equal to the internal rate of return $r > r_0$. Otherwise, the project is considered unprofitable. Suppose that we analyze two projects. Let us say that the internal rates of return on different projects are r_1 and r_2, respectively. Suppose that the first project is more profitable than the second one, if $r_1 > r_2$.

Definition. The method for the assessment of the return on an investment project which is based on the size of the IRR is called the internal rate of return method. While searching for the internal rate of return the n-th degree equation needs to be solved. The latter equation can have no more than n solutions. We are interested in the question concerning the fact that this equation has only one solution.

Let us consider the following equation:

$$NPV(r) = \sum_{t=0}^{n} \frac{C_t}{(1+r)^t} = 0.$$

Multiplying both sides of the last equality by $(1+r)^n$ we get

$$C_0(1+r)^n + C_1(1+r)^{n-1} + C_{n-1}(1+r) + C_n = 0$$

The question is: when this polynomial has only one solution?

Answer: Assume that a cash flow is given. Then, there is only one IRR for this flow in the case a natural number $m \in (0,n)$ is such that $C_i < 0,\ i = 0, 1, \ldots, m$ and $C_i > 0, \quad i = m+1, \ldots, n$.

Otherwise, the number of solutions can be bigger. In other words, the solution will be the only one if at the beginning you start investing more than you receive, and after a certain moment in time the situation changes, i.e. incomes exceed investments.

Example. Set the interest rate r, with which an investment of 1000 now and an investment of 1520 after two years will give the income of 2500 after one year. This project can be formalized by the flow:

$$-1000, 2500, -1520.$$

We have

$$1000 + \frac{1520}{(1+r)^2} = \frac{2500}{(1+r)}.$$

Then

$$r^2 - 0.5r = 0.02 = 0.$$

Having solved this equation we obtain that

$$r_1 \approx 0,456, \quad r_2 \approx 0,044.$$

Note. In the event that the rate of return is not the only one it raises the problem of the investment project comparison. In other words, within the same interval of the rate of return variation one project may be more appropriate than the other, and vice versa. I.e. we cannot explicitly compare investment projects.

5.2.3 IRR evaluation

What you should know: 1. If the net present value is positive, when the rate of return is higher than the IRR, i.e.r_0; ($NPV(r) > 0$, then $r > r_0$.)

2. If the net present value is negative, the rate of return is lower than the IRR; ($NPV(r) < 0$, then $r < r_0$.)

3. If the net present value is zero, the rate of return equals the IRR, i.e. r_0. ($NPV(r) = 0$, then $r = r_0$;)

a) The Method of Averages

We make an assumption that the function $y = NPV(r)$ satisfies the conditions ensuring that there exists the only solution to the equation $NPV(r) = 0$. As we have mentioned above, the $NPN(r), r > 0$ is a continuous function of the variable r.

We will present a na?ve algorithm for the finding of this solution. In order to determine the IRR we choose the rate of return q_1 and calculate $NPV(q_i)$. The following values are possible: $NPV(q_i) > 0$ or $NPV(q_i) < 0$.

Assume that $NPV(q_1) > 0$. As a solution exists, we set the rate of return q_2, with which $NPV(q_2) < 0$. Usually this is easy to do if there is the only one solution, since in this case the considered function is either increasing or decreasing. In our case suppose that $q_1 < q_2$. Since the function $NPV(r)$ is defined and continuous for all the values of $r \geq 0$, there is such the number $r_0 \in (q_1, q_2)$ that $NPV(r_0) = 0$.

We choose the value $q_3 = q_1 + q_2$ and calculate the function value at this point. Suppose that $NPV(q_3) < 0$. Based on the same reasoning as mentioned above we can conclude that $r_0 \in (q_1, q_3)$. I.e. the search range of r_0 is narrowed by half. In the same way we obtain the subsequent value

$$q_4 = \frac{q_1 + q_3}{2}$$

and we calculate a function value at this point, suppose $NPV(q_4) > 0$. Thus, $r_0 \in (q_4, q_3)$. During this step the search interval is narrowed by half. We choose the point

$$q_5 = \frac{q_4 + q_3}{2}$$

and calculate the function value, etc. at this point. With each step the search interval narrows by half and this way we slowly approach the searched value of $IRR = r_0$. Continuing this algorithm we get a sequence $\{q_n, n \in \mathcal{N}\}$, the limit of which is equal to r_0.

Example. As a part of an investment project the amount of 25000 has been invested. It is predicted that seven years in a row (at the end of the year) the income of 7000 will be generated. Find the IRR of the project. The income forms a conventional annuity. Then we have that

$$PV_{in} = 7000 \cdot a_{7\rceil i}.$$

The reserves are equal to Then

$$PV_{out} = 25000.$$

Thus

$$NPV(r) = 7000a_{7\rceil r} - 25000 =$$

$$7000\frac{(1+r)^7 - 1}{r(1+r)^7} - 25000.$$

When performing a simple analysis of this function we find that for the large r values this function acquires negative values, while the values close to zero acquires positive values. Let us calculate the value of this function at the point $q_1 = 0.12$. We have that$NPV(0.12)6946.3$. To obtain a negative value we take the higher rate. Suppose that $q_2 = 0.24$. Having calculated everything we obtain that $NPV(0.24) - 2304$. Based on the middle point method we choose, $q_3 = 0.18$. Having calculated we have that $NPV(0.18)$ 1680. It follows that

$$r_0 \in (0, 18, 0, 24).$$

. By using the subsequent member $q_4 = 0.21$, we obtain that $NPV(0,21) - 444$. By repeating the algorithmic steps we obtain the following sequence:

$$q_5 = \frac{0,21 + 0,18}{2} = 0,195$$

and

$$NPV(0,195) \approx 581.$$

If

$$q_6 = \frac{0.21 + 0,195}{2} = 0.2025,$$

then we obtain that

$$NPV(0.2025) \approx 60.$$

We see $NPV(0.2025) \approx 60$. Thus $r_0 = 0.2...$ This method is very simple, but during its realization a lot of calculations need to be done. We will analyze another method of the IRR determination, which is much more efficient in terms of calculation.

b) Newtons Method When using this method the sequence is formed, a limit of which is the IRR. Let us discuss briefly the conditions which should satisfy the function for us to be able to use Newtons method. Assume that the continuous function $y = f(x)$, in the interval (a, b) satisfies this correlation $\text{sgn} f(a) = \text{sgn} f(b)$; in addition, the function in this interval is strictly even $x \in (a, b)$. Then there exists $r_0 \in (a, b)$ such, that $f(r_0) = 0$. .

Let us make a sequence

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}.$$

Then this sequence converges to r_0. Note that x_0 is optional, but it should not be too far from the possible IRR value.

Let us examine the example above in another way. Make a sequence based on Newtons method, which is converged to the IRR. A reader is suggested to make sure that the function $y = NPV(r)$ satisfies the above-mentioned conditions. We have that

$$r_n = r_{n-1} - \frac{NPV(r_{n-1}}{NPV'(r_{n-1})}.$$

Then

$$NPV(r) = 7000\frac{(1+r)^7 - 1}{r(1+r)^7} - 25000$$

and

$$NPV'(r) = 7000\frac{1 + r \cdot 8 - (1+r)^8}{r^2(1+r)^8}.$$

By choosing $r_0 = 0.15$ we obtain that $r_1 = 0.197,\ r_2 = 0.2038$.

Consequently, already the second member of the sequence is sufficiently close to the IRR. When applying Newtons method for the evaluation of investment projects, the derivatives of $NPV(r)$ need to be calculated. The function $NPV(r)$ is often defined by an annuity factor an]i. A derivative of this function, which has to be used in calculations, is presented below.

$$\frac{\mathrm{d}a_{n\rceil i}}{\mathrm{d}r} = \Big(\frac{(1+r)^n - 1}{r(1+r)^n}\Big)' = \frac{1 + (n+1)r - (1+r)^{n+1}}{r^2(1+r)^{n+1}}.$$

When calculating the rate of return a return index PI, which is defined by the following way, is often used:

$$PI = \frac{PV_{in}}{PV_{out}}.$$

It is clear
if $PI > 1$, then $r_0 > i$;
if $PI < 1$, then $r_0 < i$;
if $PI = 1$, then $r_0 = i$.

It follows from the last correlations that the rate of return is consistent with the discount interest rate, when the return index equals $PI = 1$.

It is easy to understand that the return index shows for how many percentage points an income and outlay of an investment project differ. For example, if the return index $PI = 1.15$, the income of this investment project is by 15% higher than the incurred outlay, and if $PI = 0.78$, the income of this investment project is by 22% percent lower than its outlay. This relative measure is quite often used in the economic terminology, as the net present value indicates the differential value of an investment project (indicates how different are values in total) at the focal date point. The advantage of a relative value is that this value is independent from the absolute values, and it enables the comparison of different investment projects in terms of their return, when the invested amounts differ significantly.

5.2.4. Modified internal rate of return

When considering a project rate of return in this section we will note that while characterizing the return of an investment project the income, which has been reinvested during the period of an investment project, should also be included. Definition. The modified internal rate of return will be called the interest rate p, based on which the present value of the total cost of an investment project (calculated with the internal rate of return of an investment project r_0) is equal to the present value of the reinvested income until the end of the period (invested with the expected market rate $i(t)$), discounted with the rate p. Suppose that the projects internal rate of return is r_0, while an alternative investment rate is i at the period T. Let us denote

$$A(T) = \sum_t \frac{C_t^-}{(1 + r_0)^t}$$

the present value of outlay and

$$S(T) = \sum_t C_t^+ (1 + i(t))^{T-t}.$$

Then the MIRR (modified internal rate of return) is calculated as follows:

$$A(T) = S(T)(1 + p)^{-T}$$

or

$$p = \sqrt[T]{\frac{S(T)}{A(T)}} - 1,$$

here r i, p are the annual interest rates, and time is calculated at the base of years. Suppose that we have two investment projects IP_1 and IP_2. Let p_1 and p_2 be the MIRR of these projects, respectively.

Suppose that the investment project IP_1 has an advantage in respect of the project IP_2, according to the method of the modified internal rate of return, if $p_1 > p_2$. The MIRR method becomes interesting when the market situation including the search for the most appropriate investment projects is modelled taken into regards the way the market interest rate changes.

5.2.5 Calculation of the account rate of return

This section will discuss a method used to determine the account rate of return during the period of a year when during this period the account balance is changing; in addition, it is known that the account balance can be subject to various interest rates.

Let t be the moment in time starting from now (the zero time point). The period is one year. A is the account balance at the beginning of the period, B the account balance at the end of the period, C_t the value of a cash flow, $0 \leq t \leq 1$,it is both positive and negative, and r_t is the interest rate at the moment in time t.

Let I be the interest of the period (one year). In this case a flow is considered negative, if the amount of money is withdrawn from the account, while multiplying this sum by the interest rate of a period; at the same time we obtain the lost income for the amount that was withdrawn from the account. Using the defined values we obtain that $B = A + C + I$, where $C = \sum_t C_t$.It is known that $I = B - A - C$, while on the other hand

$$I = rA + \sum_t C_t (5.1)$$

where r is a simple interest rate which we have to determine. Then the interest rate of the term $1 - t$ is $r_t = (1 - t)r$. Based on the last note and combining it with the equality (5.1) we obtain that

$$I = rA + \sum_t C_t$$

or

$$r = \frac{I}{A + \sum_t C_t(1 - t)}.$$

Example. Assume that an account balance of the company is 12000. After two months 1600 were used, and after 4 and 7 months the account was supplemented by the amounts of 1700 and 1300, respectively. 1200 were used after 9 months. At the end of the year 13000 could be found in the account. Find the IRR.

We have that

$$A = 12000, B = 13000, C_2^- = -1600,$$

$$C_4^+ = 1700, C_7^+ = 1300, C_9^- = -1200.$$

Then, based on the correlation $I = B - A - C$ we obtain that: $I = 13000 - (12000 - 1600 + 1700 + 1300 - 1200) = 800$. Thus

$$r = \frac{I}{A + \sum_t (1 - t)C_t} = \frac{800}{12000 - \frac{10}{12} \cdot 1600 + \frac{8}{12} \cdot 1700 + \frac{5}{12} \cdot 1300 - \frac{3}{12} \cdot 1200}.$$

The formula can also be used when account balances at the beginning and end of the year are known; in addition the received income and the performed investments during the period of a year are known. In this case, we make an assumption that in the middle of the year the income has been received and the outlay has incurred, i.e., $1t = 0.5$. Then we obtain the following formula for calculating the rate of return:

$$r = \frac{2I}{A + B - I}.$$

Example. It is known that at the beginning of a year 1500000 and at the end of the year 1680000 could be found in the account. The received interest amounted to 160000, and the investment outlay to 15000. Find the IRR.

We have that

$$A = 1500000, B = 1680000,$$

$$I = 160000 - 15000 = 145000.$$

Flow periods are unknown. Having assumed that a cash flow is distributed continuously we obtain that

$$r = \frac{2 \cdot 145000}{1500000 + 1680000 - 145000}.$$

Note. It is easy to understand that this method can also be applied to a period which is longer than one year, if a return rate is not high during this term.

5.2.6 Average rate method

We will analyze the investment process in the time interval T, which consists of individual parts of time intervals $T = t_1 + \cdots + t_k$ and within each time interval the interest rate is $r_1, ..., r_k$, respectively. Then the interest rate r, which in the time interval T accumulate the same future value as the rates in the time intervals $t_1, ..., t_k$ together, will be called the k average of an interest rate.

Let us analyze this issue in further details. Suppose that we have k in the time interval $t_1, ..., t_k$ with a simple interest rates of $r_1, ...r_k$. Then the future value is

$$\overline{r} = \frac{\sum\limits_{j=1}^{k} t_j r_j}{T}, \quad T = \sum_{j=1}^{k} t_j.$$

It follows from the latter equality that the average of a simple interest is

$$\overline{r} = \frac{\sum\limits_{j=1}^{k} t_j r_j}{T}, \quad T = \sum_{j=1}^{k} t_j.$$

Assume that the interest is compound. Then with the help of an analogous reasoning, only considering the fact that the effective interest rate r_k is converted m_k times per year we obtain that

$$(1 + \overline{e_j})^T P = (\prod_{j=1}^{k} (1 + i_j)^{m_j t_j}) P.$$

We obtain that the average efficient rate is

$$\bar{e}_f = \sqrt[T]{\prod_{j=1}^{k}(1+i_j)^{m_j t_j}} - 1 = \sqrt[T]{(1+i_1)^{m_1 t_1} \cdot \dots \cdot (1+i_k)^{m_k t_k}} - 1, \quad T = \sum_{j=1}^{k} t_j.$$

Example. Suppose that for the first two years, the interest rate was 15%, and for the next three years it reached 20%, while the interest was compound. Let us find a five-year average of the interest rate.

$$\bar{e}_f = \sqrt[5]{(1.15)^2 \cdot (1.2)^3} - 1 \approx 0.1797.$$

In the case of the equal time intervals, although during each period the interest rate is calculated from different values, the medium-term interest rates can be found in the following way: 1) In the case of a simple interest

$$\sum_{j=1}^{k}(1+t\bar{r})P_j = \sum_{j=1}^{k}(1+ti_j)P_j.$$

Then

$$\bar{r} = \frac{\sum\limits_{j=1}^{k} i_j P_j}{\sum\limits_{j=1}^{k} P_j}.$$

2) In the case of a compound interest assume that i_k the actual interest rate corresponding to the k-th period, but converted in the same frequency, and the fixed period of time t includes the number of conversion periods m_k, respectively. Suppose that the capital existing at the end of each period is replaced by another. The total interval of time is $T = tk$. Then the average actual rate of the period t is

$$\bar{i}_f = \sqrt[t]{\frac{\sum\limits_{j=1}^{k} P_j(1+i_j)^{tm_j}}{\sum\limits_{j=1}^{k} P_j}} - 1.$$

5.2.7 Investment assessment based on the periodic investment costs

Let P be the periodic investment, and i be the period interest rate. Then $P = Kxi$, where K is the capitalized costs. Having rewritten the latter correlation using the expression (5.1)???? we obtain that

$$P = (OV + \frac{R}{(1+i)^n - 1}) \cdot i = OV \cdot i + \frac{R}{s_{\overline{n}|i}}.$$

Note. If an investment project does not have periodic renovation outlay only for its value to be restored at the end of the period, the following correlation needs to be observed: $R = OVRV$, RV is the rest (salvage) value.

The entities that have decided to make an investment have a natural question of which alternative of the possible ones should be chosen for the investments in production facilities? We will address this issue below by comparing investment projects under their annual investment costs.

Suppose that we analyze two investment projects with the annual interest rate r. Suppose that the annual investment costs of the first project are $P_1 = K_1 \cdot r$, while those of the second project are $P_2 = K_2 \cdot r$. If $P_1 < P_2$, then we assume that the first project is better than the second one.

Example. A company is preparing to invest in the acquisition of a new production line. The company managers have received two proposals:

The first proposal:

the line L1 costs 40000, and its lifetime is 15 years, the final value is zero.

The line L2 costs 35000, its lifetime is 10 years and its salvage value is 4000.

Which alternative is more valuable if the cash value is 12%?

Compare these options at the interest rate of 12%.

We calculate the periodic investment costs of the line L1. We have $OV_1 = 40000; R = 40000$

$$c = \frac{1}{\frac{1}{15}} = 15 \quad p = 1,12^{15} - 1 = 4,4735665.$$

Then

$$K_1 = 40000 + \frac{40000}{4,4735657} = 48941,4.$$

We obtain that to the periodic investment costs of the line L_1 are $P_1 = 48941.4 \cdot (0.12) = 5872$.

We calculate the L2 periodic costs. We have that $OV_2 = 35000; R = 31000$.

$$c = \frac{1}{\frac{1}{10}} = 10, \quad p = 1.12^{10} - 1 = 2,1058482.$$

Then the periodic costs of L2 are as follows $P_22 = 49720.9 \cdot 0.12 = 5966.5$.

We see that the periodic costs of the device L1 are lower than that of the device L2, i.e. the first device can be purchased more efficiently at the interest rate (cash value) of 12%.

Tasks for the practice

1. The joint stock company was prompted to choose one of two alternatives. The first one was to invest 400000 now; the return of 100000 is expected after three years, the return of 500000 after four and the return of 600000 after fourteen years. The second alternative: the income of 12000 received at the end of each month for fourteen years. Furthermore, additional 3000 will have to be invested at the end of each year. Give your recommendation on which investment project should be chosen. Provide the recommendations on the basis of the internal rate of return method using the linear (middle point) method.

2. The investment company has chosen the project, which is defined in the following way: at beginning it needs to invest 400000; it is known that for 8 years, at the end of every six months the income of 80000 will be generated and at the end of the period an additional premium of 100000 will be received. By using Newtons method find the internal rate of return of the project. Is it worth investing in this project, if with the help of an alternative investment the interest of 8

3. The entrepreneur has invested 200000 and later for six years he has been investing additional 50000 every quarter. With the help of the linear method find the internal rate of

return (using the precision of 2 decimal points), if after seven years he has been receiving the income of 500000 every six months for the period of five years. 4. The Bank has invested 500000 to a wood panel plant. Then starting with the third year, every year it had been additionally investing 300000 for four more years. The factory started to operate and after three years the return on investment of 600000 has been received every six months. Find the modified internal rate of return, if it is known that the received income has been invested with an interest of 8

5. The entrepreneurs account balance is 200000. He participated in an investment project which was based on the following cash flow: after a month 30000 was allocated for the outlay, after 3 months 20000 were additionally spent, and after 5 months the income of 40000 was received; then after 9 and 11 months the account was supplemented by the amounts of 7000 and 18000, respectively. At the end of the year 27000 were accumulated in the account. Find the IRR of the project.

6. Suppose that the account balance of a company is the 20000. Two months later the amount of 2600 was used for, and after 4 and 7 months 1700 and 1300 were added to the account, respectively. After 9 months another 1200 was used. At the end of the year the amount of 13000 could be found in the account. Determine the account rate of return.

7. It is known that at the beginning of the year the amount of 500000 could be found in the account, at the end the amount of 780000. The received income amounted to 60000, and the investment outlay were equal to 35000. Determine the account rate of return.

8. The company is preparing to invest in the acquisition of new equipment. The company managers have received two proposals: the first proposal: the line L1 costs 100000, and its lifetime is 10 years, the final value is 5 9. The device with a value of 70000 and a guaranteed lifetime of six years after the end of this period has the salvage value of 10000. After the modernization of this device its service lifetime may be extended up to nine years, and then the salvage value of the device may reach 6000. It is known that the interest rate is 14%. Set the maximum reserves that will make it possible to modernize the device.

10. We analyze two investment proposals for the period of 12 years:

A: Suppose that for the first five years the interest rate was 7%, and 11% during the next seven years.

B: For the first four years the interest rate was 10%, for the next four years 6%, and for the remaining four years 12%. Evaluate the projects based on the average rate method if:

a) the interest is compound and converted on a quarterly basis;

b) the interest is simple.

Self-control exercises

1. The investment company has to choose between two alternatives: the alternative A and the alternative B. If the alternative A is selected 10000 will be obtained every year for the period of ten years. The alternative B guarantees that after three years the income of 20000 will be received, after six years the income of 60000 will be obtained and after ten years the income of 40000 will be received. Determine which alternative would be more useful to choose if it is known that the interest rate is 14%.

Ans: The alternative A should be selected.

$$NPV_A(0.14) = 52161, NPV_B(0.14) = 51624.$$

2. The investor has to choose between two alternatives: the alternative A and the alternative B. If the alternative A is selected, 70000 will have to be invested at the beginning, and then every three months 5000 will continue to be received for nine years. Meanwhile, the alternative B would require an initial investment of 65000, and then for the period of eight years the annual income of 260000 will continue to be received. Determine which alternative would be the most cost-effective if it is known that the interest rate is 12% which is converted on a quarterly basis?

Ans: : The alternative A should be selected. The present values of both the alternatives are 39160 and 35970, respectively.

3. The company is considering the following option: The replacement of the old equipment with a new one would require the outlay of 65000 now and another 40000 after five years. However, this could result in the savings of 80000 every six months for the period of ten years. Determine whether the equipment should be replaced, if the interest rate is 14

Ans: The equipment should be replaced because NPV(0.14)=508.

4. The investment of 15000 in manufacturing equipment at the current moment could result in the income of 2000 received for its sale after twelve years. Furthermore, the following additional incomes will be generated: during the year 1,2,3 the annual amount of 2000; during the year 4,5,6,7 the annual amount of 5000; during the year 9,10,11,12 the annual amount of 3000. What is the net present value of the project if the cash value is 18%?

Ans: NPV(0.18) = 1286

5. The company participates in a ten-year investment project: it has to invest the annual amount of 30000 for four years in a row. At the end of this investment project the company receives a real estate, the value of which is 30000. It is known that after four years the income of 60000 will be received, after five years the income of 40000 will be acquired, and for the remaining five years the annual income of 20000 will continue to be received. With the help of an alternative investment the interest of 14% would be paid. Determine the net present value and the internal rate of return of this project.

Ans: NPV(0.14) = 404, IRR= 0.16.

6. The investment project is based on the initial contribution of 4500000. The return of this investment is the annual payments of 1400000 continued to be made for eight years. Find the return rate (internal rate of return). **Ans:** IRR 26. 3

7. The company is investing in new equipment. At the beginning it has to invest 10000; however, the annual return of 2000 is guaranteed during the period of twelve years. Furthermore, at the end of the 12-th year the company will receive an additional income of 40000. Find the projects internal rate of return.

Ans: IRR = 20%

8. While building an office space the construction company invested 500000 for the first year and it will have to invest another 300000 for the renovation of these offices after six years. It is expected to lease the premises for twelve years and it is anticipated that after twelve years the value of these premises will be 1000000. The annual income of these premises is 130000. Find the net present value of this investment project during the period of twelve year, if the interest rate is 12%.

Ans: NPV(0.2)=-12150

9. At the moment the hotel room-conditioning system costs 300000. This system can be operated for seven years and then its salvage value will be 50000. The annual outlay of the system maintenance is 400000 and the necessary maintenance outlay 150000. However, there is also another alternative. You can change this system by a new one which costs 3400000 and serves for 10 years, and its salvage value is 350000. The annual maintenance outlay of a new system is 360000, and the annual repair outlay is 120000. Determine how much can be saved by replacing the old equipment with new one if the cash value is 16%. **Ans:** 2174355.

10. The residential house can be painted for the amount of 1350. In the case the paint of 1) type of is used, it serve three years and then the building needs to be repainted, while if the paint of 2) type is used the building needs to be repainted every five years. Set the maximum outlay, if the paint of 2) type will be replaced by the paint of 1) type at the interest rate of 12%.

Ans: 2026.

11. Find the effective interest rate, which is equivalent to the interest rate of 16.75

Ans: 18.236%.

Homework exercises

1. The investment company has to choose between two alternatives: the alternative A and the alternative B. If the alternative A is selected 13000 will be obtained every year for the period of ten years. The alternative B guarantees that after three years the income of 20000 will be received, after six years the income of 60000 will be obtained and after ten years the income of 40000 will be received. Determine which alternative would be more useful to choose if it is known that the interest rate is 8%.

2. The company is considering the following option: The replacement of the old equipment with a new one would require the outlay of 75000 now and another 30000 after five years. However, this could result in the savings of 20000 every six months for the period of ten years. Determine whether the equipment should be replaced, if the interest rate is 6

3. The investment project is based on the initial contribution of 500000. The return of this investment is the annual payments of 100000 continued to be made for eight years. Find the return rate (internal rate of return).

4. The company is investing in new equipment. At the beginning it has to invest 200000; however, the annual return of 50000 is guaranteed during the period of twelve years. Furthermore, at the end of the 12-th year the company will receive an additional income of 40000. Find the projects internal rate of return.

5. The investment company has chosen the project, which is defined as follows: initially it has to invest 1200000 and it is known that for the period of 12 years at the end of every six months the income of 70000 will continue to be generated and at the end the additional premium of 100000 will be received. What is the project net present value if the interest rate is 14%? Is this a profitable investment project?

6. The entrepreneur has invested 200000 and later for the period of six years he has been investing the additional amount of 50000 every quarter. With the help of the linear method find the internal rate of return (using the precision of 2 decimal points), if after seven years he has been receiving the income of 250000 every six months for the period of five years.

7. The Bank has invested the amount of 500000 in a plant of parts. When starting with the

fifth year, every two years it has been additionally investing 400000 for the period of four more years. The plant started to work after three years and its return on investment was 200000 every six months. Find the internal rate of return if it is known that after fifteen years the factory went bankrupt, and after the sale of its equipment the bank recovered the amount of 100000.

8. Having invested the amount of 250000 for every year and a half the annual income 300000 is continued to be generated. The investment project is intended for fifteen years. The interest is converted every quarter. Find the modified internal rate of return if the income is invested with an interest rate of 10%, which is converted every six months.

9. The account balance of an investment company is 500000. After a month the amount of 40000 and after 4 months the additional amount of 80000 was invested. Meanwhile, after 3, 5 and 9 months the company received the incomes of 90000, 40000 and 70000, respectively. After 11 months the amount of 30000 was used. At the end of the year the amount of 630000 could be found in the account. Find the approximate IRR (internal rate of return).

10. At the beginning of the year the amount of 600000 could be found in the account of an insurance company, at the end of the end the amount of 890000. The received interest amounted to 100000, and the investment outlay was equal to 15000. Find the MIRR if alternative it is possible to invest interest with 6% .

To be able to calculate:

The present values of cash flows, the accounting balance of a flow, the project rate of return, the parameters based on which investment projects are compared (present value, project internal rate of return, a modified rate of return), capitalized costs.