Chapter 3. AMORTIZATION OF LOAN. SINKING FUNDS

Objectives of the Topic:

• Being able to formalise and solve practical and mathematical problems, in which the subjects of loan amortisation and management of cumulative funds are analysed.

• Assessing financial flows in time, providing reasoned evaluations when comparing various loan repayment methods.

Assessed results of the studies:

• Will understand the methods of analysing loan amortisation and cumulative funds.

• Will apply knowledge of annuity when modelling mathematical and real life cases.

• Will provide mathematically supported recommendations.

Student achievement Assessment Criteria:

• Accurate use of terms.

• Appropriate application of formulas.

• Accurate interim and end answers.

• Accurate answers to questions.

3.1 Amortization (simple annuities)

Review the following terms: Periodic payments:

a) simple (ordinary, paid-up, deferred paid-up, deferred ordinary, ordinary life-long, paid up life-long, b) complex (ordinary, paid-up, deferred paid-up, deferred ordinary, ordinary life-long, paid-up life-long).

All repayments of interest-bearing debts by a series of payments, usually in size, made at equal intervals of time is called an *amortization*. Mortgages and many consumer loans are repaid by this method.

We consider a classical problem. Suppose that a bank loans B. This amount plus interest is to be repaid by equal payments of R each at he end of each n period. Further, let us assume that the bank charges interest at the nominal rate of r percent compounded in m times in year (actual $i = r/m$). Essentially, for B the bank is an annuity of n payments of R each. Using formula of a present value of an ordinary annuity we obtain that the monthly payment R is

$$R = \frac{B}{a_{n\rceil i}}.$$

The bank can consider each payment as consisting of two parts: (1) interest on outstanding loan, and (2) repayment of part of the loan.

The amount of the loan is the present value of the annuity. A portion of each payment is applied against the principal, and the remainder is applied against the interest. When a loan is repaid by an annuity, it is said to be *amortized*. In another words, a loan is amortized when part of each payment is used to pay interest and the remaining part is used to reduce the outstanding principal. Since each payment reduces the outstanding principal, the interest portion of a payment decreases as times goes on. Let us analyze the loan described above.

Suppose that the principal is B. At the end of the first month you pay R. The interest on the outstanding principal is $I_1 = iB$. The balance of the payment $P_1 = R - I_1$ is then applied to reduce the principal. Hence the principal outstanding now $B_1 = B - R_1$. Further, at the end of the second month, the interest is $I_2 = iB_1$. Thus the amount of the loan repaid is $P_2 = R - I_2$, and the outstanding principle is $B_2 = B_1 - P_2$. The interest due at the end of the third month is $I_3 = iB_2$ and so the amount of the loan repaid is $P_3 = R - I_3$. Hence the outstanding principle is $B_3 = B_2 - p_3$ and etc.. The interest due at the end of the nth and final month is $I_n = iB_{n-1}$ and the amount of the loan repaid is $B_n = R - I_n$. Hence the outstanding balance (principal) is $B_n = B_{n-1} - P_n$.

Actually, the debt should now be paid off, and the balance of B_n is due to rounding, if it is not 0. Often, banks will change the amount of the last payment to offset this. An analysis of how each payment in the loan is handled can be given in a table called an amortization schedule. Consider one example. Suppose that a bank loans 1500. Equal payments of R at the end of each three month. The nominal rates of 12 percent compounded monthly. Thus we have

$R = \frac{1500}{a_{3\rceil 0.01}} \approx \frac{1500}{2.940985} \approx 510.03.$

The allocation of each payment to first cover the interval due and then to reduce the principal may be shown in an amortization schedule. The simplified amortization schedule of the loan considered above:

No	P	I	R	B
1	1500	15	510.03	495.03
2	1004.97	10.05	510.03	499.98
3	504.99	5.05	510.03	504.98
Totals		30.10	1530.09	1499.99

Table 1

Here
N0- end of period;
B- principal balance;
I- interest paid;
R- amount paid;
P- Principal repaid at end of period.

The total interest paid is 30.10, which is often called the $f$inance charge. As mentioned before, the total of the entries in the last column would equal the original principal were it not for rounding errors.

When one is amortizing a loan, at the beginning of any period the principal outstanding is present value of the remaining payments. Using this fact together with our previous development, we obtain the formulas listed below that describe the amortization of an interest bearing loan of B dollars, at a rate i per period, by n equal payments of R each and such that a payment is made at the end of each period. Notice below that the formula for the periodic payment R involves $a_{n\rceil i}$.

1. Periodic payment:

$$R = \frac{B}{a_{n\rceil i}} = B\Big(\frac{i}{(1-(1+i)^{-n}}\Big).$$

2. Principal outstanding at end of kth period:

$$B_k = Ra_{n-k\rceil i} = R\Big(\frac{1-(1+i)^{-(n-k+1)}}{i}\Big).$$

3. Interest in kth payment:

$$I_k = Ra_{n-k+1\rceil i}.$$

4. Principal contained in kth payment:

$$P_k = R(1 - ia_{n-k+1\rceil i}).$$

5. Total interest paid:

$$\sum_k I_k = R(n - a_{n\rceil i}) \ \text{ or } \ nR - A.$$

The annuity formula

$$B = R\Big(\frac{1-(1+i)^{-n}}{i}\Big)$$

can be solved for n to give the number of periods of a loan:

$$\frac{Bi}{R} = 1-(1+i)^{-n}.$$

Thus

$$1-\frac{Bi}{R} = (1+i)^{-n}$$

Taking logarithm both sides and solving equation by n we obtain

$$n = \frac{\ln\left(\frac{R}{R-Bi}\right)}{\ln(1+i)}.$$

Anuity due

1. Periodical payments:

$$R = \frac{B}{(1+i)a_{n\rceil i}} = B\Big(\frac{i}{(1-(1+i)^{-n}}\Big).$$

2. Outstanding loan at the end of $k-$th payment period (balance):

$$B_k = Ra_{n-1-k\rceil i} = R\Big(\frac{1-(1+i)^{-(n-k-1)}}{i}\Big),\ k=0,\ldots,n-1.$$

When $k=0$, we obtain balance of the loan after the first payment (at the beginning of the first payment interval).

3. Amount of interest in $k-$ th payment period:

$$I_k = iRa_{n-k\rceil i}.$$

4. Repaid loan at the end of $k-$th period:

$$P_k = R(1-ia_{n-k\rceil i}).$$

5. Total amount of interest:

$$I = R(n-(1+i)a_{n\rceil i})\ \text{ arba }\ nR-B.$$

Example A.B. amortizes a loan 30000 for a new home obtaining a 20− year mortgage at the rate of 9 percent compounded monthly. Find (a) the monthly payment, (b) the interest in the first payment, and (c) the principal repaid in the first payment.

We have $i=\frac{0.09}{12}=0.0075,\ \ n=12\cdot 20=240$. Then the monthly payment

$$R = \frac{30000}{a_{240\rceil 0.0075}} = 30000\Big(\frac{0.0075}{(1-(1.0075)^{-240}}\Big) \approx 269.92.$$

The interest portion of the first payment is $I_1 = 30000\cdot 0.0075 = 225$. Thus the principal repaid in the first payment is $269.92-225=44.92$

Example A.B. purchases a TV system for 1500 and agrees to pay it off by monthly payments of 75. If the store charges interest at the rate of 12 percent compounded monthly, how many months will it take to pay off the debt?

We have that $R=75,\ \ i=0.01,\ \ B=1500$. Thus

$$n = \frac{\ln\left(\frac{75}{75-1500\cdot 0.01}\right)}{\ln(1.01)} = \frac{\ln 1.25}{\ln 1.01} \approx 22.4.$$

Thus we obtain 22.4 months. reality there will be 23 payments; however, the final payment will be less than 75.

We give an example where we consider another structure of the amortization schedule.

Suppose that A.B. is borrowing 7000 to by car. The loan plus interest is to be repaid in equal quarterly installments made at the end of each quarter during 2- years interval. Let the interest rate be 16 percent compounded quarterly. First we determine the quarterly payment R.

Applying formula of present value of ordinary annuity we deduce

$$7000 = R \cdot a_{8\rceil 0.04}.$$

Solving R yields

$$R = \frac{7000}{6.732745} = 1039.69.$$

Thus, the borrowed will make eight payments of 1039.69 each or $8 \cdot 1039.69 = 8317.52$, to repay 7000 loan. Thus, the interest is

$$8317.52 - 7000 = 1317.52, \quad I = 1317.52.$$

In what follows we use more detail amortization schedule, which is given above. Such a schedule normally shows the payment number, the amount paid, the interest paid, the principal repaid and outstanding debt balance.

No	I	R	P	M	B
1	280	1039.69	759.69	759.69	6240.31
2	249.61	1039.69	790.08	1549.77	5450.23
3	218.01	1039.69	821.68	2371.45	4628.55
4	185.14	1039.69	854.55	3226	3774
5	150.96	1039.69	888.73	4114.73	2885.27
6	115.41	1039.69	924.28	5039.01	1960.99
7	78.44	1039.69	961.25	6000.26	999.74
8	39.99	1039.69	999.70	6999.96	0.04.
Total	1717.56	–	–	70000	0

Table 2

Studying the last table note that amount principal reduction for a period is the difference between the payment R and the interest for that period. The equity column is the cumulation of the principal reductions. The balance column may be determined by either of two methods:

1. As the difference between the amount of the loans and the equity;

2. As difference between the previous period's balance and the principal reduction for the given period.

Let us summarize some basic concepts from previous sections. For any financial transaction, the value of an amount of money changes with time as a result of the application of interest. Thus, to accumulate or bring forwards a single payment R for n periods at an interest rate i per period, we multiply R by $(1+i)^n$. To bring back a single payment R for n periods at an interest rate of i per period, we multiply R by $(1+i)^{-n}$. To accumulate or bring forward an annuity of n payments of R each, we multiply R by $s_{n\rceil i.}$ To bring back an annuity of n payment of R dollars each, we multiply R by $a_{n\rceil i}$.

Example A.B. borrowed 15000 from SEB bank at 16% compounded quarterly. The loan agreement requires payment of 2500 at the end of every three months. Construct an amortization schedule.

No	I	R	P	M	$Balance$
0	0	0	0	0	15000
1	600	2500	1900	1900	13100
2	524	2500	1976	3876	11124
3	444.96	2500	2055.04	5931.04	9068.96
4	362.76	2500	2137.24	8068.28	6931.72
5	277.27	2500	2222.73	10391.01	4708.99
6	188.36	2500	2311.64	12702.65	2397.35
7	95.89	2500	2397.35	15000	0

Table 3

Tasks for the practice

1. A new flat cost 106,000 to a person. When buying the flat, the person knocked down the price by 12% and agreed to repay the entire amount in 12 years by paying equal instalments at the end of each quarter. The interest rate is 16% and the interest rate is compounded every 6 months. Calculate the following:

1) (a) The amount of the fixed instalment.

(b) How much will A.B. still owe after 8 years?

() How much will they pay to completely repay the loan?

(d) How much interest will they pay?

2) Complete the same task if the payments are made at the beginning of the payment period.

3) Complete the same task if the payments are made at the end of the payment period, as provided in the conditions, but by deferring them by four years.

3.2 Amortization of loan deferred anuity case

Suppose that ordinary anuity with deferred l deferred periods and n payments periods. The total deferred annuity number of periods is $n+l$.

Using general present value formula we get $R = \frac{A_n(l)(1+i)^l}{a_{n\rceil i}}$.

We note that filling amortization schedula the first l payments are empty. Thus in this case we write $R=0$. Assume that R_t $t=1,\ldots,n+l$ are $t-$ th payment made at the end of period.

Then

$$R_k = \begin{cases} 0, \; k \le l, \\ R, \; k \ge l+1 \end{cases}.$$

1. $k-$ th $(k \ge 0)$ period book value are $B_k = B_{k-1} - P_k$; or

$$B_k = \begin{cases} Ra_{n\rceil i}(1+i)^{k-l}, \; l = 1, \dots l \\ Ra_{n+l-k\rceil i}, \; k = l+1, n+l. \end{cases}$$

2. Amount of interest in $k-$ th payment period:

$$I_l = \begin{cases} iB(1+i)^k, \quad k = 1, \dots, l \\ iRa_{n+l-k+1\rceil i} \; k = l+1, \dots, n+l. \end{cases}$$

3. Principal contained in $k-$ th payment:
$P_k = R - I_k, \; k = 1, \dots, n+l.$
4. Principal outstanding at end $k-$ th period:

$$M_k = M_{k-1} + P_k, \quad k = 1, \dots, n+l \;\; \text{or} \;\; M_k = B - B_k, \; k = 1, \dots, n+l.$$

Example Farmer loan 100000 for nine year with 10%. Loan was deferred for 4 year. Make amortization schedule.

We have $B = 100000, \; i = 0,1$. Then for the formula

$$R = \frac{A_5(4) \cdot 1,1^4}{a_{5\rceil 0,1}}$$

we obtain $R = 38622,58$. Then

Nr	R	I	P	M	B
0	0	0	0	0	100000
1	0	10000	-10000	-10000	110000
2	0	11000	-11000	-21000	121000
3	0	12100	-12100	-33100	133100
4	0	13310	-13310	-46410	146410
5	$38622,58$	14641	$23981,58$	$-22428,41$	$122428,42$
6	$38622,58$	$12242,84$	$26379,74$	$3951,33$	$96048,68$
7	$38622,58$	$9604,868$	$29017,712$	$32969,042$	67031
8	$38622,58$	$6703,1$	$31919,48$	$64888,822$	$35111,52$
9	$38622,58$	$3511,152$	$35111,428$	$100000,25$	$0,092$
Σ	$-$		100000	10000	0

Table 4

3.3 Amortization (complex annuity)

We consider the situation when the length of the payment interval is different from the length of the interest conversion period the equal debt payments form a complex annuity. The amortization of such debs has the same principles which was discussed above in case simple annuities. The payments are made at the end of the payment intervals are obtained by formula

$$A_n^c = R(\frac{1-(1+p)^{-n}}{p})R = Ra_{n\rceil p}, \quad \text{here } p = (1+i)^c - 1.$$

Consider the example and construct amortization schedule.

Example A debt of 30000 with interest at 12% compounded quarterly is to be repaid by equal payments at the end of each year for 7 years.

1) Compute the size of the yearly payments;

2) Construct an amortization schedule.

We have that

$A_n^c = 30000, \quad n = 7, \quad c = 4, \quad i = 0.03$. Further $p = 1.03^4 - 1 = 0.01255088. \quad R = \frac{30000}{4.4851295} = 6688.77$.

Then an amortization schedule:

No	I	R	P	M	B
1 3765.26	2923.51	6688.77		2923.51	27076.49
2	3398.34	6688.77	3290.43	6213.94	23786.06
3	2985.36	6688.77	3703.41	9917.35	20082.65
4	2520.55	6688.77	4168.22	14085.57	15914.43
5	1997.40	6688.77	4691.37	18776.94	11223.06
6	1408.59	6688.77	5280.18	24057.12	5942.88
7	745.88	6688.77	5942.88	30000.00	0
Total	16821.38	46821.38	–	–	

Table 5

Tasks for the Practice

1. A person has purchased a car that cost 66,000. It has been agreed that the debt will be repaid in equal annual instalments in 6 years. The interest rate is 10%, the interest is compounded quarter. Create a loan amortization table.

2. A new flat cost 1,560,000. A person has agreed to repay the entire amount in 12 years by paying equal instalments at the beginning of each quarter. The interest rate is 6% and the interest is re-calculated every quarter.

Determine: (a) How much will A.B. still owe after 8 years?

(b) Fill in the row of the amortisation table for the payments of year 6.

(c) Complete the same task when payments are made at the end of the payment period, as provided in the conditions, but by deferring them by 4 years.

3. A.B. has borrowed 8,500 with 18% interest, which are re-calculated every quarter for 8 years. Equal instalments are made every month, at the end of each quarter.

(a) Calculate the size of the monthly instalments.

(b) Calculate the repaid interest until the payment 16 inclusive.

(c) Which part of the loan (in per cent) was repaid by payment

4. A.B borrowed 140,000 with 12(a) How many payments will need to be made until the debt is repaid?

(b) How much interest will be paid with payment 6?

(c) What amount of the loan will be repaid with payment 10?

(d) Create a partial loan repayment table, which would have the first three and the last three payment rows.

3.4 Repay of the debt in case of the simple interest rates

Consider the problem of the repayment of loans when all repaid are made by equal parts of loan plus interest in case simple rates.

Amortization of loan (method S1) Consider the following problem: repaid the debt at the end of loans period. In this case the debt is repaid at the end of loan period and the payments consists from the interest rB.

We have $I_k = rB, \quad R_k = rB, k = 1, \dots n-1$ and $R_n = B(1+r)$.

The schedule of the loans is such

k	I	R	P	B
0	0	0	0	B
1	rB	Br	0	B
2	rB	0	rB	B
...	...	...	...	...
n-1	rB	rB	0	B
n	rB	$(1+r)B$	B	0
totals	rBn	$B + Bnr$	aB	

Table 6

Amortization of loan (method S2)

Suppose that a bank loans B. As were considered above the debt is to be repaid by equal payments of $\frac{a}{n}$ each at the end of each n period plus interest from the initial amount of the debt. Assume that the bank charges interest rate of r percent in year. Then all payments R are equal to $\frac{B}{n} + rB$ Essentially, amount B is an annuity of n payments which we give in the following table:

k	I	R	P	B
0	0	0	0	B
1	rB	$B(r+\frac{1}{n})$	$\frac{B}{n}$	$B(1-\frac{1}{n})$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
n-1	rB	$B(r+\frac{1}{n})$	$\frac{B}{n}$	$B(1-\frac{n-1}{n})$
n	rB	$B(r+\frac{1}{n})$	$\frac{B}{n}$	0
totals	rBn	$B+Bnr$	$\frac{B}{n}$	

Table 7

Amortization of loan (method S3 or linear method)

Suppose that a bank loans B. This amount plus interest is to be repaid by equal payments of $\frac{B}{n}$ each at the end of each n period plus interest from the remainder of the loans. Further, let us assume that the bank charges interest rate of r percent in year. Essentially, for B the bank is an annuity of n payments which we give in the following table:

No	I	R	P	B
0	0	0	0	B
1	rB	$B(r+\frac{1}{n})$	$\frac{B}{n}$	$B(1-\frac{1}{n})$
2	$rB(1-\frac{1}{n})$	$B(r(1-\frac{1}{n})+\frac{1}{n})$	$\frac{B}{n}$	$B(1-\frac{2}{n})$
3	$rB(1-\frac{2}{n})$	$B(r(1-\frac{2}{n})+\frac{1}{n})$	$\frac{B}{n}$	$B(1-\frac{3}{n})$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
n-1	$rB(1-\frac{n-2}{n})$	$B(r(1-\frac{n-2}{n})+\frac{1}{n})$	$\frac{B}{n}$	$\frac{B}{n}$
n	$r\frac{B}{n}$	$B(\frac{r}{n}+\frac{1}{n})$	$\frac{B}{n}$	0
totals	$rB(\frac{n+1}{2})$	$-$	B	$B+Br\frac{n+1}{2}$

Table 8

here $I-$ interest, $P-$ equal parts of loan, $R-$ payment at the end of period, $B-$ remainder of the loan (balance).

We have that

$$B_k = B\big(1-\frac{k}{n}\big),\ k=1,2,\dots$$

is decreasing arithmetical sequence with difference $-\frac{ra}{n}$ and first term rB. It is clear, that an interest of the k th period $t_k - t_{k-1}$ is

$$I_k = rB(1-\frac{(k-1)}{n}),\ k=0,1,\dots,n.$$

Applying the formula for the sum of the n terms of arithmetical sequence we then deduce, that total amount of the interest is $\frac{rB+\frac{rB}{n}}{2}\cdot n=\frac{n+1}{2}\cdot B$. The sequence of the payments $R_k=\frac{B}{n}+I_k$ is decreasing arithmetic sequence with difference $-\frac{Br}{n}$. Then the total amount of the all payments is

$$B+\frac{n+1}{2}\cdot rB.$$

Tasks for the practice

1. A loan of 20,000, taken for 5 years, must be repaid by instalments every 6 months. The ordinary interest rate of the loan is 12%.

Create a loan repayment table: 1) Using method Pl; 2) Using method P2; 3) Using method P3.

2. When implementing an investment project, an amount of 100,000 was borrowed for 20 years with the ordinary interest of 6%. A part of the loan or the interest will be repaid at the end of each quarter. 1) Determine what the costs of financing this project would be if the following methods were applied: a) Pl; b) P2; c) P3.

2) What is the balance value of the debt at the end of year 10? a) Applying method Pl; b) applying method P2; c) applying method P3.
3) How much interest was paid on the loan until the end of year 12 inclusively: a) applying method Pl; b) applying method P2; c) applying method P3.

3. When expanding a business, an entrepreneur has taken a loan of 500,000 for 10 years with the interest of 6%. The loan is repaid every quarter.
1) Determine what fixed instalments would need to be paid at the end of each quarter if:
a) The method P3 (linear) was used to repay the loan;
b) The simple ordinary annuity method was used to repay the loan;
c) Compare the costs of both financing methods.
2) Create an amortisation table of the last two years if method P3 is applied for repaying the loan.

4. A three-year bill, whose interest rate is 8% is repaid by quarterly payments, which cover the interest, in equal instalments. The loan is repaid with the last payment, which is conducted at maturity. Create the amortisation table for covering the bill.

3.5 Sinking funds

Definition The interest bearing fund which payments are made at periodic time intervals to provide a desired sum of money at a specified future point in time is called sinking fund.

Sinking funds usually involve large sums of money used by both the private sector and the public sector to repay loans, finance future capital acquisitions, provide for the replacement of depreciable plant and equipment and recover investments in depletable natural resources.

The main problem in dealing with sinking funds is that of determining size of the periodic payments, which will accumulate to known a future amount. These payments form an annuity in which the accumulated value is known.

Depending on whether the periodic payments are made 1) at the end or 2) at the beginning of each payment period, the annuity formed is an ordinary annuity or an annuity due. Depending on whether or not the payment interval is equal in the length a to the interest conversion period, the annuity formed is a simple annuity or a complex annuity.

We introduce formulas defining amounts of sinking funds. In what follows
$R-$ size of periodic payments;
$i-$ rate of the payment period;
$n-$ number of converse periods.

1) For the sinking funds in case simple annuity with payments at the end of each payment intervals, the amount of the sinking fund find by

$$S_n = \Big(\frac{(1+i)^n - 1}{i}\Big)R =: R \cdot s_{n\rceil i};$$

2) For the sinking funds in case simple annuity with payments at the beginning of each payment intervals, the amount of the sinking fund find by

$$S_n^* = \Big(\frac{(1+i)^n - 1}{i}\Big)(1+i)R =: R(1+i) \cdot s_{n\rceil i}$$

3) For the sinking funds in case complex annuity with payments at the end of each payment intervals, the amount of the sinking fund find by

$$S_n^c = \Big(\frac{(1+p)^n - 1}{p}\Big)R =: R \cdot s_{n\neg p},$$

4) For the sinking funds in case complex annuity with payments at the beginning of each payment intervals, the amount of the sinking fund find by

$$S_n^{c*} = (1+p)\Big(\frac{(1+p)^n - 1}{p}\Big)R =: R(1+p) \cdot s_{n\rceil p}.$$

here, both cases 3) 4), $p = (1+i)^c - 1$, c is the number of interest conversion periods per payment interval.

Suppose, that a person decided to accumulate a sum of money by making periodic deposits into a fund. At the end of a specified time period the deposits plus the interest earned equal the desire accumulated amount. Such a fund is called a sinking fund.

General formulas

Ordinary annuity

1. Periodical payments:

$$R = \frac{S}{s_{n\rceil i}} = S\Big(\frac{i}{(1+i)^n - 1}\Big).$$

2. Balance at the end of $k-$t period:

$$S_k = Rs_{k\rceil i} = R\Big(\frac{(1+i)^k - 1}{i}\Big), \quad k = 1, 2, \dots n.$$

3. Interest in $k-$th period:

$$I_k = iRs_{k-1\rceil i}, \quad k = 1, 2, \dots n.$$

4. Total amount:

$$R(s_{n\rceil i} - n) \text{ arba } S - nR.$$

Annnuity due

Nr indicate number of period and R is done at the beginning of this period and balance S_k - at the end of this period.

1. Periodical payments:

$$R = \frac{S}{(1+i)s_{n\rceil i}} = A\Big(\frac{i}{(1+i)((1+i)^n - 1)}\Big).$$

2. $k-$th period balance:

$$S_k = R(1+i)s_{k\rceil i} = R(1+k)\Big(\frac{(1+i)^k - 1}{i}\Big), \quad k = 0, 1, 2, \dots n.$$

3. Interest in $k-$th period:

$$I_k = iS_{k-1}, \quad k = 0, 1, \dots n-1.$$

4. Total amount of interest:

$$R((1+i)s_{n\rceil i} - n) \text{ arba } S - nR.$$

Example Consider a contractor foreseeing the need for a new truck 4 years from now. The price of the truck is forecast to be 20000. The contractor wishes to accumulate this amount by setting aside semiannual payments of R each for 4 years. Each payment of this sinking fund earns interest at 10 percent compounded semiannually. The contractor must determine the semiannual payment R.

Since the semiannual payment constitute an annuity with a amount of 20000, then $S = R \cdot s_{n\rceil i}$. Solving for R yields

$$R = \frac{20000}{s_{8\rceil 0.05}} \approx 2094.44.$$

Thus, the semiannual payments $R = 2094.44$ plus interest will accumulate to $S = 20000$. Note that the contractor will make eight payments of 2094.44 each, or $8 \cdot 2094.44 = 16755.52$. Therefore, the interest earned is $20000 - 16755.52 = 3244.48$.

This result are summarized in the table, which is called a schedule of the sinking fund, which in more details shows all the process of the payments. In the sinking fund schedule 1) payment number, 2) periodic payment, 3) interest earned by the fund, 4) the increase in the fund and 5) accumulated balance will be shown.

Construct a sinking fund schedule for the last example above. We have that

$R = 2094.44$; $n = 8, \quad i = 0.05$.

Set $No-$ payment number;

P- periodic payment;

$I-$ interest earned by the fund;

B – *Balance* accumulated balance

$\Delta-$ increasing of the balance in this time moment.

No	R	I	Δ	B
1	2094.44	0	0	2094.44
2	2094.44	104.72	2199.16	4293.60
3	2094.44	214.68	2309.12	6602.72
4	2094.44	330.14	2424.58	9027.30
5	2094.44	451.37	2545.81	11573.11
6	2094.44	578.66	2673.10	14246.21
7	2094.44	712.31	2806.75	17052.96
8	2094.44	852.65	2947.09	20000.05
Totals	16755.52	3244.53	20000.05	

Table 8

Example A.B. company wants to provide for replacement of equipment estimated 60000 seven years from now. To do so the company set up a sinking fund into which the company will pay equals sums of money at the beginning of each of the next seven years. Interest paid by the fund is 11.5% compounded annually.

1) Find the size of annual payment into the fund;

2) What is a total payment into the fund by A.B.?;

3) How much of the fund will be interest?

We have that $S_n^* = 60000, \quad n = 7, \quad i = 0.115$. Then

1)

$$60000 = 1.115 \cdot R \cdot s_{7\rceil 0.115}.$$

From the last equality we deduce $R = 5416.42$.

2) The total paid into the fund by A.B. will be $7 \cdot 5416.42 = 37914.94$.

3) The interest earned by the fund will be $60000 - 37914.94 = 22085.06$.

No.	P	I	Δ	B
1	5416.42	622.89	6039.31	6039.31
2	5416.42	1317.41	6733.83	12773.14
3	5416.42	2091.8	7508.22	20281.36
4	5416.42	2955.24	8371.66	28653.02
5	5416.42	3917.99	9334.41	37987.43
6	5416.42	4991.44	10407.86	48395.29
7	5416.42	6188.35	11604.77	60000.06
Totals	37914.94	22085.12	60000.06	

Table 9

We deal with the sinking funds which accumulate future value using linear method (simple interest case.)

We assume, that interest is calculated from the balance value, and accumulated interest not capitalized. The balance is increased in beginning of payment period. In this case we have the following schedule of the sinking funds:

Nr.	R	I	Δ	B
1	R	0	R	R
2	R	rR	$R(1+r)$	$2R$
...	...	...	...	...
n	R	$rR(n-1)$	$R(1+r(n-1))$	nR
Σ	nR	$r\frac{(n-1)n}{2}R$	nR	S

Table 10

Here accumulated total amount is

$$S = r\frac{(n-1)n}{2}R + nR.$$

Task for the practise

1. A person has decided to save. The bank has offered the linear method for 10 years by paying instalments every quarter. The interest rate is 10%.

1) Determine what instalments the person will have to pay every quarter if during the period they hope to accumulate 200000.

2) Formulate the 5th and 6th rows of payments of the amortisation table.

3) What percentage of interest will be included in this amount?

Self-control exercises

1. Create the general formulas, based on which it would be possible formulate the row of any payment period in the amortisation table, when:

a) The annuity is ordinary paid-up;

b) The annuity is ordinary paid-up and deferred by payment k.

2. A.B. has purchased a car that cost 36000. They knocked the price down by 4,000 and agreed that the debt will be repaid in equal instalments within 15 years, by paying the instalments at the end of each quarter. The interest rate is 14

(a) Determine the size of the fixed instalment;

(b) How much will they still owe after 10 years?

(c) How much they will pay in total after 15 years?

(d) How much interest will they pay?

Ans: (a) 1282.84 (b) 18232.24 () 80970.40 (d) 44970.40.

3. A debt of 1000000 with the interest rate of 15%, which are re-calculated every year, is repaid at the end of each year within a period of 7 years. Create a loan amortisation table. Determine the size of the annual instalments, the total amounts paid and the costs of the loan.

Ans: Instalments 240360; total amount 1682525; costs 682525.

4. A.B. has borrowed 920000 with the interest rate of 13

Ans: Paid in total 1494316; interest paid 574316.

5. A.B. has borrowed 8500000 with the interest rate of 18%, which is re-calculated every 8 years. Equal instalments are also made every quarter, at the end of each quarter. (a) Calculate the size of the quarterly payments.

(b) Calculate the interest paid until payment 16, inclusive;

(c) What part of the loan was repaid with payment 20?

Ans: (a) 506287 (b) 266724 (c) 285683.

6. A.B has borrowed 2400000 with the interest of 17%, which are re-calculated every 6 months. The debt is repaid by instalments of 250000 at the end of each half of the year. (a) How many payments will need to be made until the debt is repaid?

(b) How much interest will be paid with payment 6?

(c) What amount of the loan will be paid with payment 10?

(d) Create a partial loan repayment table, which would include the first three and the last three payment rows and the last balance row.

Ans: (a) n = 20.750427 (b) 180832 (c) 95857 (d) Total 5189503; 2789503; 2400000.

7. A debt of 2500000 is repaid by instalments of 350,000, which are made at the end of each half of the year. The interest of 21% is re-calculated every 6 months: (a) How many payments will need to be done until the loan is repaid? (b) What will be the amount of the last payment?

Ans: (a) n = 13.884418 (b) 311309.

8. A debt of 1750000 is repaid by equal instalments of 285000 at the end of each year. The interest of 14(a) Determine how many payments will need to be made until the debt is repaid.

(b) Determine the costs of the loan of the first three years.

(c) Which part of the loan will be repaid in year 7?

(d) Create a debt amortisation table by indicating the first three and the last three loan repayment years and the balance row.

Ans: (a) n = 16.294188 (b) 746405 (c) 70775 (d) balance row 4647884; 2897884; 1750000.

9. A loan has been taken for 5 years, it has been agreed that it will be repaid by instalments every 6 months, when the simple interest rate is 12%. Create a loan repayment table: 1) Using method P1. 2) Using method P2. 3) Using method P3. 10. When implementing an investment project, an amount of 100,000 was borrowed for 20 years with the simple interest of 6%. A part of the loan or the interest will be paid at the end of each quarter. 1) Determine what the costs of financing this project would be if the following methods were applied: a) P1; b) P2; c) P3.

2) What is the debt balance value at the end of year 10?

a) applying method P1; b) applying method P2; c) applying method P3.

3) How much interest will be paid on the loan until the end of year 12 inclusive?

a) applying method P1; b) applying method P2; c) applying method P3.

11. An entrepreneur has borrowed a loan of 500000 for 10 years with the interest rate of 6%; the interest rate is re-calculated every quarter. 1) Determine what fixed instalments would need to be paid at the end of each quarter if: a) Method P3 (linear) was applied to repay the loan; b) The ordinary annuity method was applied to repay the loan; 2) Compare the financing costs of both methods; 3) Create the amortisation table of the last two years if method P3 is applied to repay the loan.

12. A.B. at the beginning of each month transfers 900 to an account in which at the end of the instalments they expect to accumulate 72,500. The interest rate is 12%, the interest is re-calculated ever month. (a) How many payments will need to be done until the desired result is achieved? (b) What will be the amount of the last instalment?

Ans: (a) n = 160.53831 (b) 485.59.

13. After purchasing a boat that cost 330000 via leasing, in the future every quarter 43000 will have to be paid. Payments are deferred by three years. The value of money during this entire period is 20%. The interest is recalculated every quarter. (a) How many payments will need to be made until the debt is repaid? (b) What is the amount of the last leasing fee?

Ans: : (a) n = 21.888956 (b) 38.328.

14. A transport company, in order to renew the vehicle park, seeks to accumulate 11000000 in 5 years. At the end of each half of the year, from the profit they transfer a fixed amount to the cumulative account. The interest rate of the account is 17.5%. The interest rate is calculated every 6 months. (a) Determine the size of the fixed amount. (b) What will the balance of the account be after payment 3? (c) Determine the amount of interest that will accumulate after making payment 6.

Ans: (a) 732.706 (b) 2396063 (c) 381784.

It is necessary to know the following: Loan repayment methods, comparing methods by determining their effectiveness, creating loan amortisation tables, calculating the values of any independently chosen rows (in the amortisation tables), creating cumulative fund tables and calculating the values of any period, analysed in the cumulative fund tables, conducting calculations in the case of simple interest and annuity.

Homework exercises

1. Make general formulas for the amortization of loan and sinking funds in case annuity due.

2. A.B. borrowed 140000 for replacement of equipment. The debt is repaid in instalments of 12000 made at the end of every three months.

(a) If interest is 10% compounded quarterly, how many payments are needed?
(b) How much will Comfort Swim owe after two years?
(c) How much of the 12th payment is interest?
(d) How much of the principal will be repaid by the 20th payment?
(e) Construct a partial amortization schedule showing details of the first three payments, the last three payments and totals.

3. A debt of 100000 is repaid in equal monthly instalments over four years. Interest is 15% compounded quarterly.

(a) What is the size of the monthly payments?
(b) What will be the total cost of borrowing?
(c) What is the outstanding balance after one year?
(d) How much of the 30th payment is interest?
(e) Construct a partial amortization schedule showing details of the first three payments, the last three payments and totals.

4. A contract worth 80000 provides benefits of 20000 at the end of each year. The benefits are deferred for ten years and interest is 9% compounded quarterly.

(a) How many payments are to be made under the contract? (b) What is the size of the last benefit payment?

5. Mr.A.B borrowed 800000 from his Credit Union. He agreed to repay the loans by making equal quarterly payments for five years (at the end of quarter). Interest rate is 15%.

(a) What is the size of the quarterly payment?
(b) How much will the loan cost him?
(c) How much will Mr.A.B owe after three years?
(d) How much interest will he pay in his 16th payment?
(e) How much of the principal will he repay by his 14th payment?
(f) Prepare a partial amortization schedule showing details of the first three payments, Payments 8, 9, 10, the last three payments and totals. Use method S3 (Linear)

6. A sinking fund of 100000 is to be created by equal annual payments at the beginning of each six month for seven years. Interest earned by the fund is 17.5% compounded annually.

(a) Compute the annual deposit into the fund. (b) Construct a sinking fund schedule showing totals.

7. A sinking fund of 100000 is to be created by equal annual payments at the end of each six month for seven years. Interest earned by the fund is 17.5% compounded annually.

(a) Compute the annual deposit into the fund. (b) Construct a sinking fund schedule showing totals.

8. Find balance of the sinking fund after 10 years if at the beginning of each year are deposite 5000. Interest earned by the fund is 22%. Make amortization schedula if case linear method

a) S1;
b) S2;
c) S3.