Chapter 2. CASH FLOW

Objectives:

- To calculate the values of cash flows using the standard methods..
- To evaluate alternatives and make reasonable suggestions.
- To simulate mathematical and real content situations.

Examined study results::

- Will understand the scope of annuity.
- Will simulate mathematical and real content situations.
- Will provide the calculation-based recommendations

Student achievement assessment criteria:

- Correct use of concepts.
- Proper use of formulas.
- Correct intermediate and final answers.
- Correct answers to questions

Note 1. Unless it is indicated individually, we assume that a year consists of 365 days.

Note 2. When calculating values the numbers will often need to be rounded. Usually we will perform the calculation by using the precision of simply 2 decimal points or simply the whole numbers; moreover, we will write an equality sign within the range of this accuracy. For example, instead of A = 2.333333 we will write A = 2.33.

Repeat the concepts: Nominal interest rate, effective interest rate, interest conversion period, compound interest, efficient interest rate, capitalization of interest, discounting, precision accumulation (discounting) method, discount factor, the future and present capital values.

2.1 General concepts

Definition. The sequence of payments, which is made at any point in time during a specified time interval, will be called cash flow (abbreviated as CF).

The specified time interval is called the CF period. Payments can be made at the beginning or the end of a time interval. The amount of a payment may be constant or may vary over the changing time. This section will mainly focus on CFs, when payments are at the end of time interval or at the beginning of the time interval. When all the payment intervals are equal these CFs are usually referred to as periodic payments, shortened as (PP). In case of periodic payments, when payments are fixed and interest rate for CF at the relevant period of time is constant, these payments are referred to as an annuity, while CF as the annuity method.

Example. Let us suppose that the person made a contract which indicates that every first day of each month the indicated amount will be transferred from his account to the account of the gas company (the direct debit method). In this case, we have CF which is not an annuity, since the transferred periodic amount is fixed. Meanwhile, if the insurance company is awarded the contract which states that if at the end of each year (within a certain period) the amount of, say, 2000 will be transferred to an account, the annuity method can be analysed.

The time interval between the consecutive payments is called an interval of payments. Intervals of payments for CF can be fixed, but may be different as well. In the case of an annuity the intervals of payments are fixed and these intervals are called the payment period.

Definition. The CF period will be called the time interval from the beginning of the first payment interval to the end of the last payment interval.

Definition. CF will be referred to as a *simple CF*, if its interest conversion period coincides with a payment interval. CF will be called a *complex one*, if the interest conversion period does not coincide with the length of a payment interval. In both cases, the CF is divided into: 1) the ordinary CF; 2) the pdue CF (paid) CF ; 3) the deferred CF; 4) the infinite CF (or perpetuity) CF).

Definition. CF is called *ordinary* one (also often referred to as postnumerando), if every payment is made at the end of a payment interval.

Definition. CF is called the paid *due* one (also often referred to as prenumerando), if every payment is made at the beginning of a payment interval.

Definition. CF is called the deferred one, if the first payment is made no earlier than the end of a second payment interval.

Definition. CF is called the infinite (lifetime (perpetuity)) payment, if payments lasts indefinitely. During the analysis of the CFs two values the final (total) value of the CF and the current value of CF is of particular importance. In the Figure 1.1, various modifications (types) of the regular and complex CFs are indicated. The modifications of CF can be seen while reading any sequence, indicated by arrows. For example, the regular conventional, the regular instructed deferred, or the complex conventional lifetime cash flows, etc. may be considered.

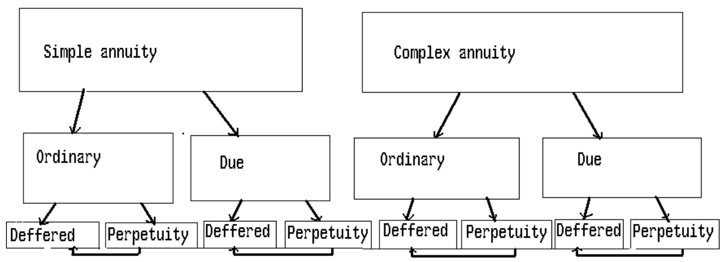

fig 1.1

Definition. The amount of all payments, including interest, will be called the future vale or total amount of a cash flow. This value (sum) will be marked by S. Sometimes the letter S will be used with an index.

Note. In the case when the annuity method is used, this together with the sum of all payments is referred to as to the lump sum (amount or future value) of an annuity.

Example. Suppose that you have a contract with an insurance company for the period of five years which states that at the end of each month your own chosen amount will be transferred to their account. The interest applied to this amount can be either fixed or variable. The amount accumulated in the account after five years will be referred to as the final value (future value) of CF.

Definition. The present (discounted) value of CF will be called a sum of the discounted values of all payments, when discounting is performed with the interest rate (which may depend on time) indicated in the contract. We will start the cash flow analysis from the simplest cases, i.e. when the payments are fixed, the payment period is constant and the effective interest rate for the period payment is also fixed.

2.2 Simple annuity. The future and present values

This section will explore the CF, where the payments of a flow are fixed and the interest rate for the CF-period is also fixed. This case of a cash flow will be called an annuity. We will discuss various modifications of an annuity while calculating the current and future values of a flow.

Definition. An annuity is called an *simple* one, when the periods for the interest payment and conversion are the same.

The main problem which will be explored is the determination of the present and future value of a cash flow. Furthermore, we will assume that the money value during an annuity period, or in other words, the interest rate which can be invested is not equal to zero. Each modification of an annuity will be correlated to the future and present values. An annuity is a separate case of a CF; thus, all the concepts that have been determined above, are used to examine of the issues of an annuity.

Note. We will consider the situations where payments are fixed and each payment R is a subject to capitalization; as a result, while solving the tasks of determining the current (present) and future values, we will employ the compound interest methods for the determination of values.

Legend:

n the number of payment periods (and hence the number of payments);

$i = \frac{r}{m} -$ the effective interest rate;

m the number of conversions of the interest rate periods per year;

k the number of payments per year.

Note. In case of an simle annuity m = k. We will indicate general formulas used for the calculation of the total amount of an annuity S and the present value of an annuity A.

Let us look at the task of an simple - ordinary annuity. Assume that payments a made at the end of the payment period and n payments are made during the annuity period, while the size of each payment is R. It is known that the first payment R participates in the accumulation for the time intervals of order $n1$, or in this case we have $n1$ conversion periods. As an interest is compound the final value $S_1 = (1+i)^{n-1}R$. will accrue from the payment R during these periods. With the help of an analogous reasoning, we obtain that at the end of the second period the performed payment R, together with interest, will be equal to the amount of $(1+i)^{n-2}R$, etc., and the payment of the k-th period, together with an interest, will amount to $(1+i)^{n-k}R$. The final payments finish the process. Then, the total amount of an ordinary annuity is:

$$S = R + R(1+i) + R(1+i)^2 + \cdots + R(1+i)^{n-1}.$$

Using the formula for the sum of a geometric progression

$$1 + a + a^2 + \cdots + a^{n-1} = \frac{a^n - 1}{a - 1}, \quad a \neq 1,$$

we obtain that

$$S = R\Big(\frac{(1+i)^n - 1}{(1+i) - 1}\Big) = R\Big(\frac{(1+i)^n - 1}{i}\Big).$$

In the financial literature, the expression found in the brackets of the last equality is denoted in the following manner:

$$s_{n\rceil i} := \frac{(1+i)^n - 1}{i}.$$

Based on the latter note, a regular calculation formula for the final value of an ordinary annuity is rewritten as follows:

$$S = Rs_{n\rceil i}. \tag{1}$$

Example. Suppose that at the end of each six months the person transfers 100 to his account. What amount of money will be found in his account after two years if he has made four payments? The interest is equal to 6Based on the formula (1) we obtain that:

$$S = Rs_{n\rceil i} = 100 s_{4\rceil 0.03} = 100 \cdot 4.183627 = 418.36.$$

Definition. The present (discounted) value of an ordinary annuity will be called the sum of values of the all the periodic payments R.

Please note that $A-$ the present value of an annuity, n the number of payments, i the effective interest rate, R the payment size. Then, the present value of an ordinary annuity is:

$$A = \frac{R}{(1+i)} + \frac{R}{(1+i)^2} + \cdots + \frac{R}{(1+i)^n}.$$

We see that this is the sum of n members of the geometric progression, the first member of which equals to $R(1+i)^{-1}$, while the denominator of the progression is equal to $(1+i)^{-1}$. Then

$$A = \frac{\frac{R}{(1+i)}\big(1-(1+i)^{-n}\big)}{1-(1+i)^{-1}} = \frac{R(1-(1+i)^{-n})}{(1+i)-1}.$$

Having restructured the last correlation, we obtain that

$$A = \frac{R(1-(1+i)^{-n})}{i}.$$

The expression found in brackets is usually denoted by the symbol

$$a_{n\rceil i} := \frac{1-(1+i)^{-n}}{i}.$$

It is easy to understand that

$$a_{n\rceil i} := s_{n\rceil i} \cdot (1+i)^{-n}.$$

Thus, the present value of an annuity, or in other words, of all the payments, can be determined taking the following formula into account

$$A = Ra_{n\rceil i}.$$

We would like to draw an attention to the fact that the present value A of an annuity may be associated with the future value S of an ordinary annuity in the following way: the entity (creditor), who lends a sum A at the effective market interest rate i, could earn the future value $S = A(1+i)^n$ by putting the money to a bank account. On the other hand, if he lends money to another entity who makes fixed payments R to the creditor and these payments are made after a fixed time interval, the received amount of money can be again reinvested by the creditor at the moments, when the payments have been received with the same interest rate, and should acquire the future value from the entire sum of these payments

$$S = R\Big(\frac{(1+i)^n - 1}{i}\Big).$$

Having equated two last equalities we obtain that

$$A(1+i)^n = R\Big(\frac{(1+i)^n - 1}{i}\Big).$$

Ha ving solved the tasks in respect of A, we get

$$A = \frac{R(1-(1+i)^{-n})}{i}.$$

Example. The family has decided to buy a car, using the possibility of leasing. For this reason, for four years the family will have to pay 400 at the end of each month. The agreed interest rate is 12%, the interest is converted every month. What is the cars current value (the cost of a new car)?

We have that $R = 400, i = 0,01, n = 12 \cdot 4 = 48$. Then

$$A = 400a_{48\rceil 0.01} \approx 400 \cdot 37.973959 = 15189.58.$$

Tasks for the practice

1. In order to save money for their retirement the person transfer 500 to a bank account at the end of each month. The Bank pays the interest of 7%, which is also compounded monthly. What amount of money will be found in an account after twelve years?

2. Parents have signed a contract with an insurance company and save money for the studies of their son, by transferring a fixed amount of money to an account at the end of each six months. The contract states that the Bank will pay the annual interest of 5% for eighteen years, which will be converted every six months. It is known that at the point of maturity the amount of 60000 will be found in an account. 1) What is the interest share in this saved amount? 2) What common nominal value is transferred by the parents during these years to their account?

3. For sixteen years, at the end of each quarter, the person puts 500 to the Credit Union account. The Credit Union pays the interest of 10%, which is converted on a quarterly basis. Determine the amount that should be put in the account at the present moment for the same future value to be collected in the account during the same period under an analogous cash value?

4. You have to pay 600 for the car rent at the end of each month for the period of five years. The interest rate is equal to 12a) It is known that you will use the car for five years. How much does this car cost at the moment? b) How much interest will you pay to a bank over the period of five years?

5. Determine at what interest rate the amount of 200000 could be found in the account after 15 years if at the end of each quarter 2000 is put to it. 130

6. Determine what time is needed for the leased boat to be redeemed if the lease interest rate is 6

2.3 Annuity due. The future and present values

We will consider an ordinary annuity, when payments are made at the beginning of the payment period.

Note. In the cases annuity due the accumulated amount and the present value will be marked by an asterisk at the top.

With the help of an analogous reasoning, as well as in the case of the analysis of an ordinary annuity, we will determine the sum of payments including the accrued interest, paying attention to the fact that all the payments "accumulate the interest" one period longer than in the case of an ordinary annuity. We have that

$$S^* = R(1+i) + R(1+i)^2 + \cdots + R(1+i)^n = R(1+i)\Big(\frac{(1+i)^n - 1}{i}\Big).$$

Using the formula for the sum of a geometric progression, we obtain that

$$S^* = R(1+i)s_{n\rceil i}. \tag{2}$$

Example. At the beginning of each quarter the person transfers the amount of 100 to their bank account. The bank pays the interest of 8 percent which are converted on a quarterly basis. What amount will be found in an account after 9 years?

While applying the formula (2) we obtain that

$$S = R(1+i)s_{n\rceil i} = 100 \cdot 1,002(s_{36\rceil 0,02}) = 5303,43.$$

Let us analyze the task for the present value calculation of an annuity. Having noticed that at each payment a degree of the discount factor is by one unit smaller than in the case of an ordinary annuity, we link the present value with the payment amount by the following correlation:

$$A^* = R + \frac{R}{(1+i)} + \cdots + \frac{R}{(1+i)^{n-1}} = \Big(\frac{R}{(1+i)} + \frac{R}{(1+i)^2} + \cdots + \frac{R}{(1+i)^n}\Big)(1+i).$$

Since

$$\frac{1-(1+i)^{-n}}{i} = a_{n\rceil i},$$

the formula of the present value can be rewritten in the following way:

$$A^* = R(1+i)a_{n\rceil i}.$$

Example. Find the present value of an instructed annuity for all the payments, if the payments (the values of each of them is 1000) are made every quarter for the period of five years, the interest rate is equal to 8% and the interest is converted every quarter

We have that $R = 1000, i = 0,02, n = 20$. Then

$$A^* = R \cdot 1,02 \cdot a_{n\rceil i} = 1020 \cdot 1,02a_{20\rceil 0,02} = 16678,46.$$

Tasks for the practice

1. At the beginning of each month the person transfers 300 to his account. The Bank pays the interest of 8%, which is converted every month. What amount of money will accumulate in the account after ten years?

2. For four years at the beginning of each month 800 must be paid for the purchased car. The interest rate is 5.75%; the interest is converted on a monthly basis. a) How much one has to pay for the car right away? b) How much will one pay in total during the period of four years? c) What are the financing costs?

3. A TV cost 1600. The contract states that this amount will be repaid by the consumer over three years, making even payments at the beginning of each month. Determine a constant level of payments if the interest rate is 7.5%, the interest is converted every month.

4. An entrepreneur pays for the credit, the nominal value of which is 250000, by the fixed payments of 1500 at the beginning of each quarter. How long will it take for the entrepreneur to pay a loan when the interest rate is 12.75% and the interest is converted every quarter?

5. At what nominal interest rate in the case of the fixed payment of 500 to an account at the beginning of each month, the amount of 100000 will accumulate in the account within the period of ten years?

2.4 Deferred ordinary and due annuities.

We would like to remind that a deferred annuity is a sequence of payments that starts later than the end of the first payment period. Let us get acquinted with the concepts used in the context of the deferred annuity. The time interval between the beginning of the first payment period and the end of the last payment period is called the payment period of a deferred annuity, the entire contract period is called the period of a deferred annuity, and the time interval between the contract signing and the beginning of the first payment period the period of deferred payments. Suppose that l is the number of deferred payments, and n is the number of payment periods of a deferred annuity. Then $n + l$ is the total number of periods during the period of a deferred annuity. The future value of a deferred annuity is observed when the number of payment periods is n, and the the number of deferred payment period is l, is marked by the symbol $S_n(l)$. Similarly, the current value of a deferred annuity will be marked by the symbol $A_n(l)$. If deferred payments are paid, the present value of a deferred annuity will be denoted by an additional asterisk added to these symbols. It is easy to understand that the future value is calculated by an analogy as well as in the case of non-deferred payments and depends only on the point in time when the payments were begun. As a result $S_n(l) = S_n$. If an annuity is due, then $S_n^*(l) = S_n^*$. The present value of a deferred annuity, when the number of payment periods is n, the number of deferred periods is l, and the period rate is i, is set on the basis of the following arguments. First of all we determine the present value of n payments for the beginning of payment periods. We have that

$$A_n = Ra_{n\rceil i}.$$

Then, the calculation formula of the present value of a deferred conventional annuity is obtained by discounting the latter value for the number of deferred payment periods l :

$$A_n(l) = Ra_{n\rceil i}(1+i)^{-l}.$$

If an annuity is instructed, then

$$A_n^*(l) = Ra_{n\rceil i}(1+i)^{-l+1}.$$

Example. A.B.covers the debt by the payments of 200 for three years; the payments are made at the end of each month. The payments are deferred. They are started at the end of the sixth month. Find the amount of this annuity (debt) if the interest rate is 12 percent, and the interest is converted on a monthly basis. We would like to note that the amount of a debt is the present value of a deferred annuity. We have that$R = 200, l = 5, n = 36, i = 0.01$. Then

$$A = Ra_{n\rceil i}(1+i)^{-l} = 200 \cdot a_{36\rceil 0.01}(1.01)^{-5} = 200 \cdot 30.107505 \cdot 0.951466 \approx 5729.$$

Example. Assume that the debt of 20000 is paid by 20 payments, which are made on a quarterly basis, at the end of each quarter. Determine the size of each payment, if the first payment is made after two years from now, and the interest rate of 20 percent is converted every quarter. We have that $A = 2000, l = 7, n = 20$. With the calculation formula of the present value of a deferred annuity we obtain that

$$R = \frac{A}{a_{n\rceil i}(1+i)^{-l}} = \frac{2000}{a_{20\rceil 0.05}(1.05)^{-7}} \approx \frac{2000}{12.46221 \cdot 0.7107} = 225.8.$$

Example. The amount of 200000 is taken from the account at the start of each quarter, beginning with the 10th year from now and finishing with the 22nd year from now. Identify the amount which needs to be put in an account at the moment to secure these payments if the interest rate is 10%, and the interest is converted on a quarterly basis. We have that $R = 200000, i = 0.025, n = 48$. The number of deferred periods $l = 40$. 1) We have that

$$A_n = \left(\frac{a_{48\rceil 0,025}}{1,025}\right) 200000 \cdot 1,025 = 5693497.$$

Then

$$A_n^*(l) = 1.025^{-40} \cdot 5693497 = 2$$

Tasks for the practice

1. The company has the amount of 600000 in its bank account for the payment of salaries. Determine for how many months it can withdraw 15000 from the account at the end of each month, if the company starts withdrawing the above-mentioned amounts after six years from now, and the interest rate during the whole contract is 8

2. What amount of money needs to be put in an account now in order after nine years the payments of 4000 could be received ten years in a row at the end of six months? It is known that the banks interest rate is 6%, the interest is converted every six months.

3. A.B. has a life insurance agreement with an insurance company for the period of fifteen years. During this period the payments have been deferred for five years. It has been agreed that at the beginning 133 of each month the amount of 450 should be put in an account, and the accumulated final value will be 100000. Set a nominal interest rate, if during the entire contract the interest is converted every month.

4. It is known that the person retires twelve years from now. The bank suggested the person to put a fixed amount of money with the interest rate of 12% to his account now; the interest will be converted every six months with the condition that he will receive the amount of 2500 after twelve years twenty years in a row, at the beginning of each six months. What amount should be put by the person in the account now in order to realize the intentions?

5. Grandparents put the amount of 10000 in their bank account with the interest rate of 10.5%, which is converted every month, for nine years with the condition that at the end of this period their grandson will receive 500 at the end of each month. Determine how many months will the grandson receive these payments?

6. Determine at what nominal interest rate the person will receive the amount of 2000 after five years at the beginning of each month for exactly 25 years if he has purchased the annuity of 200000

2.5 Ordinary infinite (perpetuity) annuity.

Suppose that you have purchased the shares of any company. Then the dividends for shares will be paid regularly, in the case the company will not go bankrupt or you will not you sell your shares. Let us formalize this situation. Please note that we will call an perpetuity annuity the sequence of periodic payments, which starts at a fixed moment of time, and lasts indefinitely. As well as in other cases of an annuity, two infinite annuity types can be distinguished: a) an ordinary lifetime annuity; b) a paid lifetime annuity. It is easy to understand that it is impossible to find the future value of an infinite annuity, however, the present value is always available when the additional conditions, i.e. the amount of payments, a nominal interest rate and payment frequency (per year), are known. Since we are considering an ordinary annuity, the number of interest conversion periods coincides with the number of payments per year. The symbol of $A(\infty)$ will mark the present value of an annuity, R the amount of periodic payments, i - the effective interest rate. If payments starts after one period from now (a conventional annuity), then

$$A(\infty) = \frac{R}{(1+i)} + \frac{R}{(1+i)^2} + \cdots + \frac{R}{(1+i)^n} \ldots = \frac{R}{(1+i)(1-\frac{1}{1+i})} = \frac{R}{i}.$$

Thus $R = A(\infty)i$.

It is easy to understand that if an annuity is not infinite, but includes a number of payment periods, we can apply the following formula for this finite annuity, with a small error

$A \approx \frac{R(\infty)}{i}$.

Example. The amount of 50000 has been put in an account in order at the end of each year a fixed amount of money would be paid. Determine the amount of payments if the contract is awarded an annual interest rate of 11%. We have that $A = 50000, i = 0.11$. Then $R = 50000 \cdot 0.11 = 5500$. If payments are made as soon as the contract is concluded (a paid annuity), then the present value of an annuity is calculated in the following way:

$$A^* = R + \frac{R}{i} = \frac{R(1+i)}{i}.$$

Example. Suppose that the land is rented for the payments of a perpetuity annuity $R = 1250$, which are made at the beginning of a month. Find the value of land, if the cash value is 13.5%, the money is converted on a monthly basis. We have that $R = 1250, i = 0.01125. Then A = 1250 +\ 112361$.

Example. What amount of money A has to be accumulated in the pensioner insurance fund at the moment of starting to pay him a pension, if the expected interest rate is $r = 0.12$, the interest is compounded monthly ($i = 0.01$), the paid amount is 1000, and the payments are made until the end of his life? We have that the fixed amount is$R = 1000$. Then at the start of payments the accumulated amount must be equal to

$$A = \frac{R}{i} = 100000.$$

Then, if an annuity is a fixed conventional and deferred for the period l, its present value is calculated in the following way:

$$A = \frac{R}{i(1+i)^l}.$$

If the annuity is fixed, paid and deferred for the period l, in this case a formula of the current value will be as follows:

$$A = \frac{R}{i(1+i)^{l-1}}.$$

Tasks for the practice

1. The Bank set up a fund, which is intended for the monthly payments of scholarships the value of which is 500000. Set the fund balance, if the contract stipulates that the fund will last forever, the agreed interest rate is 8a) if an annuity is ordinary; b) if an annuity is due; c) if an annuity is ordinary and deferred for six months; d) if an annuity is due and deferred for a year.

2. The person put the amount of 10000 with the interest rate of 10

3. The land is rented by paying a fixed amount at the beginning of each month. Determine the regular payments if the present value of the leased land is 10000, and the interest rate of 8.5% is compounded every month.

2.6 Complex annuity

We have looked at cash flows, when an interest and a payment periods coincide, or when the general analysis has been performed (the CF case); in this case the effective interest rate of that moment has been considered during the payment. Such an annuity has been called an ordinary annuity. Now let us explore the periodic payments when a payment period and an interest conversion period are different, and while analyzing a general case, during a certain payment period an interest conversion (compound) period is different from a payment period. This kind of periodic payments are called complex payments or a complex cash flow (CCF). This section will explore formulas of the future and present value of various modifications of CCF. First of all, we will discuss the annuity method. Let n be the total number of periodic payment and c the number of interest conversion periods per one payment period. Then the total number of interest conversion periods s is $s = nc$. A reader is suggested to pay an attention to the parameter c that plays a special role in calculating a complex annuity.

Example. Determine the amount which will result in a savings account, if at the end of each six months for the period of three years the amount of 1000 is put in an account, when the interest conversion period is a year, and the interest rate is 12%? We see that the interest conversion period and the payment period does not coincide, as a result this is a complex annuity. Furthermore, payments are made at the end of the period, and therefore an annuity is ordinary. A maturity is at the end of the third year and the final payment is made within this point of time and (as all the payments) equals to 1000. Let us consider the influence of deposits on the final amount from the "other end". Note that in this case the precision method of calculating the compound interest is applicable. The fifth deposit is made after 2.5 years, and up to the end of the third year it remained in the account for a half of the conversion period; moreover, the contribution of this final deposit to the total amount is equal to $1000(1.120, 5) = 1058.3$. Fully parallel, the contribution of the fourth payment, which remained for the conversion period of n = 1 until in the final period, is $1000(1.121) = 1120$, the contribution of the third deposit is $1000(1.12^{1,5}) = 1185.3$, the contribution of the second deposit is $1000(1.12^2) = 1254.4$, and finally the contribution of the first deposit is the largest and equal to the amount of $1000 \cdot 1.12^{2,5} = 1327.5$. The sum of these amounts presents the final value of the entire annuity:

$$S_n = 1000 + 1000(1.12^{0.5}) + 1000(1.12^1) + 1000(1.12^{1.5}) + 1000(1.12^2) + 1000 \cdot 1.12^{2.5} = 6945.5.$$

Example. Set the account balance after four years if it is known that at the end of each year 10000 are put in the account, the interest rate is 12%, and the interest is converted on a quarterly basis. We will round the result to the whole numbers.

We have four payments made at the end of the period. The periodic payment is 10000. The interest rate per the conversion period is equal to 0.03, and a total of 16 conversion periods are observed. The maturity is after four years. The last payment is made at the end of the fourth year and is equal to 10000. The third payment is made after three years, and before the maturity four conversion periods are applied to this payment. Thus, the contribution of this final payment to the final value of an annuity is $10000(1.03)^4 = 10120$. With the help of an analogous reasoning, we obtain that the contribution of the second payment is $10000(1.03)^8 = 12667$, the contribution of the first payment is $10000(1.03)^{12} = 14257$. Then the final value of an annuity is

$$S = 10000 + 10000(1,03)^4 + 10000(1,03)^8 + 10000(1,03)^{12} = 11255 = 37044.$$

These examples illustrate the possible payment frequencies in respect of conversion periods. In the case of a complex annuity it may appear that: 1) An interest period is longer than a payment period. In this case each payment interval includes a part of the conversion period. 2) An interest period is shorter than a payment period. In this case more than one interest period are included in the payment period. Legend c the number of interest conversion periods included the payment interval, (not necessarily a whole number); m the number of interest conversion periods per year; k the number of payment periods per year; Then

$$c = \frac{m}{k}$$

p the efficient interest rate per payment periods. If an annuity is the simple one, then $p = i$. Assume that the payments are made on a quarterly basis $k = 4$, and the interest is converted every month $m = 12$. Then $c = \frac{12}{4} = 3$. If the payment number is $k = 12$, and the interest is converted every six months $m = 2$, then $c = \frac{2}{12} = \frac{1}{6}$. Let p be the efficient interest rate of the payment period. Then this rate is linked to the effective interest i rate by the following correlation

$$p = (1 + i)^c 1.$$

Example. The Bank pays s compound interest of 12%, which is converted on a quarterly basis. Suppose that at the end of each month A.B. put 2500 to his account. Find the efficient interest rate for the payment period. We have that $c = 1;\ i = 0.03$. Then

$$1 + p = (1.03)^{\frac{1}{3}} = 1.0099, \text{ or p} = 0.0099.$$

2.7 Complex ordinary due and deferred annuities

The following formula

$$p = (1 + i)^c - 1,$$

used to define the rate p allows to change a complex annuity with the ordinary one to. In other words, we correlate the interest rate which is equivalent to the effective rate by the payment period. This correlation let us to to use all the known formulas applied in the case of an ordinary annuity for the rewrite of them in the case of a complex annuity. With the help of an analogous reasoning, as in the case of an ordinary annuity, we obtain that the calculation formula for the future value is as follows:

$$S_n^c = \Big(\frac{(1+p)^n - 1}{p}\Big)R =: R \cdot s_{n\rceil p},$$

here R the fixed payment, n the total number of payments. The calculation formula for the present value of a ordinary complex annuity is as follows:

$$A_n^c = \Big(\frac{1 - (1+p)^{-n}}{p}\Big)R =: R \cdot a_{n\rceil p}.$$

In the case of a complex annuity due the present and future values are formed fully parallel:

$$S_n^{c^*} = (1 + p)S_n^c,$$

here S_n^{c*} is the future value of a instructed annuity. Or

$$S_n^{c^*} = (1 + p)\Big(\frac{(1+r)^n - 1}{p}\Big)R =: R(1 + p) \cdot s_{n\rceil p}.$$

Using the similar arguments we obtain the calculation formula for the present value of an instructed annuity is as follows:

$$A_n^{c^*} = (1 + p) \cdot A_n^c.$$

Or

$$A_n^{c^*} = \Big(\frac{1 - (1+p)^{-n}}{p}\Big)(1 + p)R =: R(1 + p) \cdot a_{n\rceil p},\quad p = (1 + i)^c - 1.$$

Example. Determine the balance in the savings account after five years, if at the beginning of each year the amount of 20000 is added to the account. The contractual interest rate is 15%, which is converted every quarter. We have $R = 20000, n = 5, c = 4, i = 0.0375$. Having counted $p = 1.0375^4 1 = 0.1586504$ we obtain that

$$S_5^{c^*} = \Big(\frac{(1.1586504)^5 - 1}{0.1586504}\Big)1.1586504 \cdot 20000 \approx 158939.$$

Example. While paying the loan within the period of three years at the beginning of each quarter the amount of 1600 is paid. Determine the amount of the loan, if the cash value is 16.5%, the money is converted every month. We have $R = 1600, n = 12, m = 12, k = 4, c = 3, i = 0.01375$. The efficient rate for the quarter $p = 1.01375^4 1 = 4.18198$. Then

$$A_3^{c^*} = \Big(\frac{1 - (1.0418198)^{-12}}{0.0418198}\Big)1.0408198 \cdot 1600 = 15480.2.$$

Suppose that the number of deferred periods is l, n is the number of payment periods, c is the number of interest periods during the payment period, and R is the periodical payment. The future and present value of a deferred annuity is marked by the symbols

$$S_n^c(l), \quad \text{ir} \quad A_n^c(l),$$

respectively. As in the case of an ordinary deferred annuity, the deferred period has no effect on the final value, as a result $S_n^c(i) = S_n^c$.

With the help of an analogous reasoning, as in the case of an ordinary annuity, we obtain that the present value of a deferred-complex annuity, is equal to the following equation when during of the deferred period the number is l,

$$A_n^{c^*}(l) = \Big(\frac{1 - (1+p)^{-n}}{p}\Big)(1+p)^{-l}R =: R \cdot a_{n\rceil p}, \quad p = (1+i)^c - 1.$$

Then the general formula of an instructed deferred annuity is as follows:

$$A_n^{c^*}(l) = \Big(\frac{1 - (1+p)^{-n}}{p}\Big) \cdot \Big((1+p)^{-l}\Big)(1+p)R$$
$$=: R(1+p) \cdot a_{n\rceil p}(1+p)^{-l}, \quad p = (1+i)^c - 1.$$

Example. A.B. plans to continue adding the amount of 925 to a bank savings account for 12 years at the beginning of each quarter. What is the amount generated in the account of A.B. at the end of the contract, if the interest rate is 12We have $R = 925, n = 4 \cdot 12 = 48; c = 3, i = 0,01$. Then the efficient rate for the payment period is $p = 1,01^3 1 = 0,0303$. We obtain that

$$A_{48}^{c^*}(40) = \Big(\Big(\frac{1 - (1,0303)^{-48}}{0,0303}\Big)1,0303 \cdot 925\Big) \cdot \frac{1 - (1,0303)^{-40}}{0,0303} = 7255,72.$$

3.8 Complex Perpetuity Annuity

Definition. A complex lifetime annuity will be called an infinite sequence of periodic payments, when payments begin at a fixed moment of time and runs continuously; moreover, the lengths of the payment interval and the interest conversion do not match. Note that the methodology for the derivation of these formulas does not differ from the simple interest case, but in this case the efficient interest rate of the payment period is used instead of the effective interest rate. Let A^c, R be the present value of an annuity and the amount of periodic payments, respectively. In the case of an infinite conventional annuity we have that

$$R = A^c p, \quad A^c = \frac{R}{p}, \quad p = (1+i)^c - 1.$$

Example. What amount of money should be put aside today, if the interest rate of 12% is converted on a quarterly basis, so that from today at the end of every year the payment of 2500 could be received? We have $R = 2500, c = 4, i = 0,03$. Then $p = 1.03^4 1 = 0.125508$. Thus, an initial investment should be as follows:

$$A = \frac{2500}{0,125508} \approx 19920.$$

In the case the payments, the amount of which is R, are received immediately, then, based on an analogy of an ordinary annuity, we receive that the present value of an annuity can be calculated in the following way:

$$A^* = R(\frac{p+1}{p}).$$

Example. What is the present value of a perpetuity annuity, if the regular payments of 750 are made at the beginning of each month, and the interest rate of 14.5We have $R = 750, i = 0.0725; c = \frac{1}{6}$. Then $p = 1,01^{\frac{1}{6}} 1 = 0.0117337$. The present value of a lifetime annuity is 750

$$A^{*^c} = 750 + \frac{750}{0,0117337} = 64668,46.$$

3.9 General formulas of cash flow

We will consider the task of the simply annuity, when the effective interest rate (for the payment moment) is a function of time, i.e. $i = f(t)$, and the amount of payment also depends on time $R = R(t)$. While solving the task of an annuity, we will link all the payments to the current moment of time, i.e. the moment of time when the contract is concluded. In this case, we make an assumption that despite the changing time the cash value is stable and is equal to the present interest rate, while the payments are also even for all the points in time. During the analysis of the task of CFs one can also follow the other assumption that the cash value changes over time and the payments can also be different for different moments in time. During the analysis of CFs, when interest rate and payments depend on time we cannot use the known formulas, as the relevant sequence is not a geometric progression. Let us analyze the conventional CF sequence, i.e. the sequence, when the interest is compounded at the moment of payment and once during this period. The length of the i-th time interval will be marked by the symbol t_i, where i is a natural number. For example, if the contract was signed on 15/02/2010, the first payment was made on 15/04/2010 and the second payment was made on 07/15/2010 we assume that $t_1 = 1/6, t_2 = 0.25$. Furthermore, let $f(t_i)$ be the effective interest rate at the $i-th$ point in time, r the number of interest compounding periods (precise, including the rational number, as well) at the $i-$th time interval. Let us examine this situation in further details. After the end of the first interval of time the payment $S_1 = R(t_1)$ was made; as a result, currently the account balance is exactly the same. After the end of the second period the account balance is $S_2 = R(t_2) + R(t_1)(1 + f(t_2))^{n_2}$. After the end of the third period, the balance is

$$S_3 = R(t_3) + S_2(1 + f(t_3))^{r_2} = R(t_3) + R(t_2)(1 + f(t_3))^{r_3} + R(t_1)(1 + f(t_2))^{r_2}(1 + f(t_3))^{r_3}$$

etc.

$$S_n = R(t_n) + R(t_{n-1})(1+f(t_n))^{r_n} + R(t_{n-2})(1+f(t_{n-1}))^{r_{n-1}}(1+f(t_{n-2}))^{r_{n-2}} + \dots$$
$$+R(t_1)(1+f(t_2))^{r_2}(1+f(t_3))^{r_3}\dots(1+f(t_n))^{r_n}.$$

Using the generalized symbols of summation and multiplication, the latter correlations can be rewritten in the following manner:

$$S_n = \sum_{i=1}^{n} R(t_i) \prod_{j=i}^{n-1} (1+f(t_{j+1}))^{r_{j+1}},$$

we assume that the result of a product is equal to 1, if the index above the multiplication sign is smaller than the one below the sign. In the case of a paid annuity the calculation formula of the future value is as follows:

$$S_n^* = R(t_n)(1+f(t_n))^{r_n} + R(t_{n-1})(1+f(t_{n-1}))^{r_{n-1}}(1+f(t_n))^{r_n} + \dots$$
$$+R(t_1)(1+f(t_1))^{r_1}(1+f(t_2))^{r_2}\dots(1+f(t_{n-1}))^{r_{n-1}}(1+f(t_n))^{r_n}.$$

Or shortly

$$S_n = \sum_{i=1}^{n} R(t_i) \prod_{j=i}^{n} (1+f(t_j))^{r_j}.$$

Let us look at the task of the present value calculation in a general case. Let s_i be the number (possibly rational) of the interest conversion periods for the i-th period of time attributable to time interval $T = t_1 + t_2 + \dots + t_i$. With the help of an analogous reasoning as in the case of an annuity, we obtain that if CFs are ordinary, then the first payment $R(t_l)$ is discounted at the effective interest rate, which could be found in the market at the time interval t_l, i means $f(t_l)$. Thus, the present value of this payment will be $A_1 = R(t_1)/(1+f(t_1))^{s_1}$. Then the second payment $R(t_2)$ is discounted at the time interval $t_l + t_2$ with the interest rate, which could be found in the market at the moment of the second payment, i.e. $f(t_2)$. As a result, the present value of this payment will be

$$A_2 = R(t_2)/(1+f(t_2))^{s_1+s_2},$$

etc. The $n-$th payment $R(t_n)$ is discounted at the effective interest rate which could be found in the market at the $n-$th period of, i.e. $f(t_n)$. As a result, the present value of this payment will be

$$A_n = R(t_n)/(1+f(t_n))^{s_1+s_2+\dots+s_n}.$$

If the present value of a CF is the total amount of all the present values, then

$$A := A_n = \frac{R(t_1)}{(1+f(t_1))^{s_1}} + \frac{R(t_2)}{(1+f(t_2))^{s_1+s_2}} + \dots + \frac{R(t_n)}{(1+f(t_n))^{s_1+\dots+s_n}}.$$

Using the abbreviated formula, the latter phenomena can be overwritten in the following manner:

$$A = \sum_{i=1}^{n} \frac{R(t_i)}{(1+f(t_i))^{s_1+\dots+s_i}}.$$

In the case of an annuity instructed, with the help of an analogous reasoning, we obtain that

$$A^* = R(t_0) + \frac{R(t_1)}{(1+f(t_1))^{s_1}} + \dots + \frac{R(t_{n-1})}{(1+f(t_{n-1}))^{s_1+\dots+s_{n-1}}}$$

or

$$A^* = \sum_{i=1}^{n} \frac{R(t_i)}{(1+f(t_i))^{s_1+\cdots+s_{i-1}}}.$$

Note that if the period T is not a multiple of t, during the discounting the precision method is applied. Let us discuss the task of the deferred CF. As well as in the case of an annuity and holding that the deferred time interval is t, the number of the deferred interest conversion periods is l, and the number of the deferred CF payment periods is n, we obtain that the future value of a CF is $S_n(l) = S_n$, while the present value of the deferred CF is

$$S_n(l) = S_n,$$

and present value of deferred CF is

$$A_n(l) = \frac{\sum_{i=1}^{n} \frac{R(t_i)}{(1+f(t_i))^{s_1+\cdots+s_i}}}{(1+f(t))^l}.$$

We would like to remind that l is a number of conversion periods at the moment t in the time interval $[0, t]$ which can be rational. With the help of an analogous reasoning, we obtain the future and present values of the paid deferred periodic payments.

$$S_n^*(l) = S_n^*, \text{ ir } A_n^*(t) = \frac{\sum_{i=0}^{n-1} \frac{R(t_i)}{(1+f(t_i))^{s_i}}}{(1+f(t))^l},$$

here t is the time interval in which no payments have been made, $t = 0$ is the initial moment of time. Let us consider a task of a perpetuity annuity in the case of an annuity is ordinary and paid. In the case of an ordinary annuity we obtain that the total present value of all payments can be recorded in the following way:

$$A(\infty) = \sum_{i=1}^{\infty} \frac{R(t_i)}{(1+f(t_i))^{s_i}}.$$

If an annuity is paid then

$$A^*(\infty) = R(0) + \sum_{i=1}^{\infty} \frac{R(t_i)}{(1+f(t_i))^{s_i}},$$

here $R(0)$ is an initial payment at the commence of the contract.

Let us examine a few examples of the payments varying in a special way. We will consider a case where payments are determined by a geometric progression sequence, and each subsequent payment changes in the percentage R, and the interest rate is i, which is accumulated on an annual basis. Let$R_k, k = 1, 2, ...,$ be the k-th payment. Then, a payment sequence can be written in the following manner:

$$R_1 = R,\ R_2 = (1+r)R,\ \ \ldots, R_k = (1+r)^k R, \ldots.$$

The discounted payment sequence can be written as follows:

$$A(\infty) = \frac{R}{(1+i)}\Big(1 + \frac{1+r}{(1+i)} + \cdots + \big(\frac{1+r}{(1+i)}\big)^2 + \cdots + \big(\frac{1+r}{(1+i)}\big)^n + \ldots\Big).$$

By applying the formula of a geometric a sequence of an infinite sum we obtain the present formula of a fixed annuity:

$$A = \frac{R}{i - r}.$$

Note. We would like to draw the attention of a reader that in this case$r$ can be considered both a positive and negative value, i.e. any payment may be increased or decreased by a constant percentage.

Suppose that payments are not even; in addition, in respect of the $k1$th payment, the k-th payment increases by the value of rk and the periodic (effective) rate i. By marking the n-th payment by the symbol $P_n, n = 1, 2, ...$, we obtain the following sequence of payments:

$$P_1,\ P_2 = P_1(1 + r_1),$$

$$P_3 = P_1(1 + r_1)(1 + r_2),\ \ldots P_k = P_1(1 + r_1)\ldots(1 + r_k), \ldots.$$

Then the present discounted value of payments is equal to the following line:

$$A = \sum_{k=1}^{\infty} \frac{P_k}{(1 + i)^k} = \sum_{k=1}^{\infty} \frac{P_1(1 + r_1)\ldots(1 + r_k)}{(1 + i)^k}.$$

We suggest a reader to determine when this line converges.

Note. The last two formulas have been concluded in the case of a ordinary CF. If a CF is paid, then the right-hand sides of these formulas have to be multiplied by the value of $1 + i$. Consider the situation where the amount of payment changes. Let us assume that a payment period and an interest period coincide. Suppose that the amount of the n-th payment is $P_{n+1} = P_n(1 + r), n = 0, 1, 2, ...$ Assume that the payments form a geometric progression $P_0, P_l, ...,$. Thus, we have the following sequence $P_0,\ P_1 = (1 + r)P_0,$

$$P_2 = (1 + r)^2 P_0, = \ldots, P_n = (1 + r)^{n-1} P_0, \ldots.$$

It is clear that this is an indefinite sequence. Consider the first n members of this sequence holding that payments are made at the end of the period. When calculating the present value of the payments we obtain that

$$A = \frac{P_1}{(1 + r)} + \frac{P_2}{(1 + r)^2} + \cdots + \frac{P_n}{(1 + r)^n} = \frac{P}{1 + i}\big(1 + \frac{1 + r}{1 + i} + \frac{(1 + r)^2}{(1 + i)^2} + \cdots + \frac{(1 + r)^{n-1}}{(1 + i)^{n-1}}\big).$$

From the last relation we obtain

$$A = P\Big(1 - \frac{\left(\frac{1+r}{1+i}\right)^n}{i - r}\Big).$$

This equation is calculated by a regular periodic payment, when an initial deposit is P, and other deposits increase by 100

Self-control exercises

Simple annuity

1. At the end of each quarter A.B. transfers the amount of 2000 to their account. The Bank pays the interest of 13%, which is also converted on a quarterly basis. What amount of money will accumulate in the account after twelve years?

Ans: 224135

2. Parents have signed a contract with the bank and save money for the studies of their son by transferring a fixed amount of money to an account at the end of every six months. The contract states that the Bank will pay the annual interest of 12% for fifteen years, and the interest will be converted every six months. It is known that at the point of maturity the amount of 150000 will be found in the account. Determine a share of the interest within this accrued amount.

Ans: 224135.

3. In order to secure the sufficient funds for his retirement A.B. have been transferring the amount of 250 at the end of each month to his account for fifteen years. Then the saved amount had been kept in the bank account for ten years. The terms of the contract: the interest rate of 12

a) Determine the account balance after 25 years;

b) What is a nominal amount by A.B.;

c) What is a share of the interest within the account balance?

Ans: a) 412202, b) 45000, c) 367202.

4. For sixteen years, at the end of each quarter, the person puts the amount of 3750 to the Credit Union account. The Credit Union pays the interest of 17%, which is converted on a quarterly basis. Determine the present value of the accrued value after sixteen years.

Ans: 40300.7.

5. You have to pay 600 for the car rent at the end of each quarter for the period of five years. The interest rate is equal to 17.6%; the interest rate is converted every quarter.

a) How much would you have to pay for the use of the car for five years if you decided to pay the entire amount at the moment?

b) How much interest will you pay to the bank over the period of five years

Ans: a) 78728.3, b) 41271.7.

6. When purchasing an apartment A.B. agrees to pay the amount of 2537.4 with an annual interest of 15a) How much does the apartment cost at the conclusion of the contract. b) What is the amount of the interest paid throughout the entire payment period?

Ans: a) 120000, b) 9285.9.

7. Determine at what interest rate the amount of 200000 could accumulate after 15 years, if at the end of each quarter the amount of 2000 is put to an account.

Ans: 6.5%.

8. A.B. transfers the amount of 3000 to their account at the beginning of each month. The Bank pays the interest of 12

Ans: $\approx$ 395937.

9. Using the possibility of leasing, John purchases a motor boat, the original price of which is 12500. The contract states that this amount will be repaid within four years by making the even payments at the beginning of each month. Set the size of the fixed payments if the interest rate is 16.5%, the interest is converted on a monthly basis.

Ans: 352.61.

10. At what nominal interest rate the amount of 183070 will accumulate in an account within the period of ten years, if the fixed payments of 2500 are made to the account at the beginning of each quarter?

Ans: 11%.

11. Let us assume that you transfer the amount of 750 at the beginning of every month for ten years. Determine for how long you could withdraw the amount of 2600 from your account at the beginning of each month after a decade, if the total contract interest rate is 12%, and the interest is converted on a monthly basis.

Ans: 109.5 months.

12. Determine the effective interest rate of leasing, if the value of the contract is 1350000, the debt is paid over seven years by the payments, each of which amounts to 150000 and are made every six months at the beginning of every six months.

Ans: 16.165%.

13. Suppose that at the moment you have the amount of 160000 in your bank account. Determine for how long you can withdraw the amount of 10000 from your account at the end of each month, if you start withdrawing the above-mentioned amounts six years from now, and the interest rate during the validation period of the contract is 12%, the interest is converted every month.

Ans: 54.5 months.

14. What amount of money should be invested at the moment in order the payments of 1250 could be received after nine years for six years in a row at the end of each quarter? It is known that the banks interest rate is 10%, which is converted on a quarterly basis.

Ans: 8822.9.

15. A.B. concluded a contract with a life insurance company for eight years. It was agreed that the end of each quarter they have to transfer the amount of 750 to the account. If during the entire contract period the interest rate is 18%, which is converted on a quarterly basis, determine:

1) The account balance at the end of the contract.

2) What is a lump sum to be paid by the first payment for the contract to be exercised in a normal mode, assuming that the first three payments were missed?

Ans: 1) 12591.67 2) 3208.64.

16. Parents would like their daughter to receive the amount of 800 from the fund during her medicine studies at the end of each month. It is known that the studies of the daughter will start after seven years and they will last for ten years. Identify the amount of money which should be transferred to the account at the moment for these plans to be realized, if the interest rate is 12% and the interest is converted every month?

Ans: 24173.

17. It is known that the person retires twelve years from now. The bank suggested them to put a fixed amount of money with the interest of 10

Ans: 21204. 33.

18. The company leases the office space for the payments of 1250 made at the beginning of each month. The market interest rate is 13.5146

Ans: 112361.

19. The entrepreneur established a Fund of the Nominal Art Scholarships. Determine what annual payments could be allocated from the fund, if the interest rate is 11.5% and the payments start after four years.

Ans: 200000.

20. Determine the present market price of the hotel, if it is known that an average income amount to 175000 per month, the market interest rate is 15.6%, and interest is converted every month.

Ans: 13461538.

21. The person purchased the real estate with a value of 468071 and after three years he plans to get the income of 120000 every month, at the end of each month, for the entire life. Determine the interest rate of this investment project, if the interest is converted every month.

Ans: $\approx$ 18%.

Complex annuity.

1. Anthony must pay the amount of 3750 on a quarterly basis to cover his loan. It is known that the loan has been taken for eight years with the annual interest of 12%, which is converted every month. Determine what is he total amount paid after eight years if:

(a) the payments are made at the end of each quarter?

(b) at the beginning of each quarter?

Ans: (a) 197923 (b) 203921.

2. Determine the account balance after twelve years if the amount of 145 is transferred to this account every month, when the banks interest rate is 15% and the interest is converted every six months. Analyze the following two cases:

(a) transfers are made at the beginning of each month?

(b) at the end of each month?

Ans: (a) 55875 (b) 56553.

3. Every month the amount of 15 is added to your sons savings account, and under the contract, the compound interest 12% is paid, which is also converted on a quarterly basis. For how long will you have to wait until the amount of 5000 accumulates in the account, if the transfers are made:

(a) at the beginning of each month?

(b) at the end of each month?

Ans: (a) n = 148.05 (month) (b) n = 147.3 (month).

4. The amount of 32000 was taken as a loan for the repair of the appartment, and this loan will be repaid by the method of a conventional annuity by paying the amount of 8200 every quarter for fifteen years. It is known that the paymentd for the loan has been deferred for ten

years. Determine the nominal interest rate, if it is known that the interest is converted every six months.

Ans: 17.699%.

5. Identify what is the nominal interest rate of the contract if it is known that the interest is converted on a quarterly basis, and in addition, it is known that the payments of 250 are made every six months (at the end of every six months) for eight years from the account, where the amount of 8400 can be found?

Ans: 16.3%.

6. Determine what a single payment made at the moment would be equivalent to the fixed payments of 3500 which are made every six months with the 14% interest rate which is converted in on a quarterly basis and the payments are made:

(a) at the end of every six months for 15 years?

(b) the beginning of every six months for 10 years?

(c) at the end of every six months for 18 years, if the payments are deferred for four years?

(d) at the beginning of every six months for 9 years, if the payments are deferred for three years?

(e) at the end of every six months for the entire life?

(f) the beginning of every six months the entire life?

Ans: (a) 42902, 5 (b) 39345 (c) 18914 (d) 24740 (e) 49140 (f) 52640.

7. Identify the amount of money which now needs to be invested in s pension fund in order you could receive the payment of 800 at the end of each month for twenty years, if the interest rate of the entire analyzed period (of the contract period) is 12%, which is converter every six months and the payments are deferred for fifteen years?

Ans: 12885.42.

8. The debt payment eual to 4000000 has been deferred for three years. The debt has been covered for seven years by paying a fixed amount at the end of each month. Determine the size of these payments if the interest rate is 17%, which is converted on a quarterly basis.

Ans: 133804.

9. Determine the savings account balance after three years if the contract provides that every six months a person transfers the amount which exceeds the prior amount by 10% during the first year, every quarter during the second year, and every two months during the third year; moreover, the contract indicates that the annual interest rate will be determined by the formula

$$f(t) = \frac{4t+3}{t+1}, \quad t \in [0,3],$$

the interest is compounded every six months. The initial payment is 1000; payments will be made at the beginning of the period.

10. Suppose that the person returns the loan of 20000 by the regular payments, which are made at the end of every six months, while the interest is converted every two months and is determined according to the formula

$$f(t) = 5\sin\frac{\pi t}{4} + 6, \quad t \in [0,4].$$

Determine the amount of the loan.

Homework exercises

1. Payments of 600 are made into a fund at the end of every three months for twelve years. If the fund earns interest at 12% compounded quarterly

(a) how much will be the balance in the fund after twelve years?

(b) how much of the balance is deposits?

(c) how much of the balance is interest?

2. A loan of 5000 is to be repaid in equal quarterly payments over two and a half years. If interest is 12% compounded quarterly, how much is the quarterly payment?

3. Mr. A.B. borrowed 152000 which he agreed to repay in monthly payments of 2000 each. If interest on the loan is 10% compounded semi-annually, how long will it take Mr.Arnold to pay off the loan?

4. A leasing agreement with a cash value of 50000 requires monthly payments of 1000 for five years. If the first payment is due three years from the date of the agreement, what is the effective rate of interest on the lease if interest is compounded monthly?

5. What is the term of a mortgage of 35000.00 repaid by monthly payments of 475.00 if interest is 13.5% compounded monthly?

6. An income averaging annuity of 1250.00 per quarter for six years is paid from an initial retirement gratuity of 21000.00. What nominal rate of interest is paid by annuity?

7. Anne received 50000 from her mother's estate. She wants to set aside a part of her inheritance for her retirement nine years from now. At that time she would like to receive a pension supplement of 700.00 at the end of each month for 25 years. If the first payment is due one month after her retirement and interest is 10% compounded monthly, how much must Anne set aside?

8. Find the amount and the present value of semi-annual payments of 540.00 for seven and a half years if interest is 13.5% compounded semi-annually and the payments are made

(a) at the end of every six months

(b) at the beginning of every six months

9. A collateral mortgage can be discharged by making payments of 368.00 at the end of each month for fifteen years. If interest is 13% compounded monthly, what was the original principal borrowed?

10. Mr. Deed deposited 100.00 in a trust account at the date of his son's birth and every three months thereafter. If interest paid is 14% compounded quarterly, what will the balance in the trust account be before the deposit is made on the son's 21st birthday?

11. A debt of 40000.00 is to be repaid in instalments due at the end of each month for seven years. If the payments are deferred for three years and interest is 17% compounded quarterly, what is the size of the monthly payments?

12. A fund to provide an annual scholarship of 4000.00 is to be set up. If the first payment is due in three years and interest is 11% compounded quarterly, what sum of money must be deposited in the scholarship fund today?

Homework Exercises. Complex annuity

1. Find the total amount and present value of an annuity of 2000 payable at the beginning of each half-year if the interest rate is 18 percent compounded monthly.

2. Find the total amount and present value of an annuity due of 8000 quarterly for 5 years if the interest rate is 12 percent compounded monthly.

3. A loan of 220000 is to be repaid by semiannual payments at the beginning of each 6-month time period over a some time interval. Find the term of the annuity if the interest rate is 8 percent compounded monthly an the semiannual payments equals 20000.

4. A person wishes to accumulate 10000 during a 5-years period by making monthly deposits into fund earning interest at 16% compounded quarterly. Find the monthly deposit if payments are made at the beginning of each month.

5. A.B. wishes to accumulate 10000 over a 4-years period by making annual deposits into a fund earning interest at 10% compounded quarterly. Find the annual deposits if payments are made at the beginning of each year.

6. For how long can withdrawals of 3000 be made from a fund of 100000 invested at $11,25\%$ compounded annually if withdrawals are made

(a) at the end of every three months?

(b) at the beginning of every six months?

(c) at the end of each month but deferred for fifteen years?

(d) at the beginning of every three months but deferred for five years?

7. What is the term of a mortgage of 500000 repaid by monthly payments of 5000 if interest is 5% compounded semiannually?

8. What is the rate of interest compounded quarterly at which payments of 500 made at the beginning of every six months accumulate to 100000 in ten years?

Repeat the concepts:

What do we call: 1) the periodic payment; 2) what periodic payment is called an annuity; 3) an annuity is called (a simple) a complex one if; 4) (a simple) a complex annuity is called the ordinary one, if ...; 5) (a simply) a complex annuity is called the due one if ...; 6) a complex annuity is called the deferred due if ...; 7) (a simple) a complex annuity is called the deferred-due one if ...; 8) (a simply) a complex annuity is called the ordinary-perpetuity one if ...; 9) (an ordinary) a complex annuity is called the pepetuity due one if...