AN INTRODUCTION TO THE SELBERG CLASS OF L-FUNCTIONS

by

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1. Classical L-functions

The first example of a complex variable L-function is the famous *Riemann zeta function* $\zeta(s)$, introduced by Riemann in 1859 and defined for $s = \sigma + it$ with $\sigma > 1$ by the absolutely convergent Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The Riemann zeta function was introduced to study the distribution of prime numbers, and in particular to detect the asymptotic behaviour as $x \to \infty$ of the prime numbers counting function

$$\pi(x) = \sum_{p \leq x} 1.$$

The basic connection between $\zeta(s)$ and the primes is given by the Euler product

$$\zeta(s) = \prod_p \Big(1 - \frac{1}{p^s}\Big)^{-1} \qquad\qquad \sigma > 1,$$

a simple but very interesting identity since the primes appear explicitly only on the right hand side. Thanks to the Euler product, the relation between $\zeta(s)$ and $\pi(x)$ can be made explicit by a classical Fourier transform argument, thus getting

$$\pi(x) \log x \sim \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} \mathrm{d}s. \tag{1.1}$$

Clearly, in order to deduce the asymptotic behaviour of $\pi(x)$ from (1.1), we need to know some analytic properties of $-\frac{\zeta'}{\zeta}(s)$. In particular, we require some information on the polar structure of $-\frac{\zeta'}{\zeta}(s)$ or, equivalently, on the distribution of poles and zeros of $\zeta(s)$.

The fundamental analytic properties of $\zeta(s)$ are as follows.

$\star$ $\zeta(s)$ has meromorphic continuation to the whole complex plane $\mathbb{C}$ and its only singularity is a simple pole at $s = 1$.

$\star$ Writing

$$\Phi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$$

($\Gamma(s)$ is the Euler Γ-function), $\zeta(s)$ satisfies the functional equation

$$\Phi(s) = \Phi(1-s).$$

$\star$ $\zeta(s)$ has polynomial growth on vertical strips, that is

$$\zeta(\sigma + it) = O(|t|^c) \qquad\qquad |t| \to \infty$$

uniformly for $a \le \sigma \le b$, where $c = c(a,b)$.

$\star$ $\zeta(s) \neq 0$ for $\sigma > 1$ by the Euler product, and hence by the functional equation the zeros of $\zeta(s)$ in the half-plane $\sigma < 0$ are simple and located at the points $s = -2, -4, -6, ...$; such zeros are called the trivial zeros. The other zeros of $\zeta(s)$ are called the non-trivial zeros, are located inside the critical strip $0 \le \sigma \le 1$ and are symmetric with respect to the critical line $\sigma = \frac{1}{2}$ and to the real axis.

$\star$ The non-trivial zeros counting function

$$N(T) = \sharp\{\rho = \beta + i\gamma : \zeta(\rho) = 0, 0 \le \beta \le 1, 0 \le \gamma \le T\}$$

satisfies

$$N(T) \sim \frac{T \log T}{2\pi}.$$

$\star$ $\zeta(s) \neq 0$ on the line $\sigma = 1$. Moreover, $\zeta(s)$ has zero-free regions to the left of $\sigma = 1$, the simplest being of the following form: $\zeta(\sigma + it) \neq 0$ for

$$\sigma > 1 - \frac{c}{\log(|t| + 2)} \tag{1.2}$$

for some $c > 0$. Better zero-free regions are known at present, but all are asymptotic to the line $\sigma = 1$ as $|t| \to \infty$.

From the integral representation (1.1) and the above analytic properties we can deduce the famous Prime Number Theorem, proved independently by Hadamard and de la Vallée-Poussin in 1896:

$$\pi(x) \sim \frac{x}{\log x}.$$

Stronger forms of the Prime Number Theorem are known; for instance, from the zero-free region (1.2) we can get

$$\pi(x) = \mathrm{li}(x) + O(xe^{-c\sqrt{\log x}})$$

for some $c > 0$, where

$$\mathrm{li}(x) = \int_2^x \frac{\mathrm{d}t}{\log t}$$

is the integral logarithm function. However, due to the shape of the known zero-free regions, no error term of type $O(x^\theta)$ with $\theta < 1$ is available at present.

The famous Riemann Hypothesis, probably the most important open problem of contemporary mathematics, states that all non-trivial zeros lie on the critical line. Hence the Riemann Hypothesis gives the best possible zero-free region $\zeta(s) \neq 0$ for $\sigma > \frac{1}{2}$, from which the essentially best possible form of the Prime Number Theorem

$$\pi(x) = \mathrm{li}(x) + O(x^{1/2} \log x)$$

follows.

We refer to the classical book of H.Davenport [15] for an excellent exposition of the basic theory of the Riemann zeta function and its applications to the distribution of primes. We also refer to Weil [79] for a beautiful account of the prehistory of the zeta functions.

Since the appearance of the Riemann zeta function, many other L-functions have been introduced in the theory of numbers, and in other branches of mathematics as well. Here is a very synthetic list.

⋆ The *Dirichlet L-functions*

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s},$$

where χ is a character of the multiplicative group $\mathbb{Z}_q^*$ (the coprime residue classes modulo a positive integer q), were introduced by Dirichlet in 1837, hence about twenty years before Riemann's work. However, Dirichlet dealt with the $L(s,\chi)$'s as real variable functions, and the basic complex variable theory of the Dirichlet L-functions was established after Riemann's fundamental paper. The analytic properties of the Dirichlet L-functions are quite similar to those of the Riemann zeta function, and in fact $\zeta(s)$ is the special case corresponding to the character (mod 1). The Dirichlet L-functions were originally introduced to prove that the prime numbers are equidistributed in the arithmetic progressions $a \pmod q$ with $(a,q)=1$, for any fixed modulus q. Clearly, the functions $L(s,\chi)$ are of *arithmetic* nature. We refer to Davenport [15] for the basic theory of the Dirichlet L-functions.

⋆ The *Hecke L-functions* are defined for $\sigma>1$ by

$$L_K(s,\chi)=\sum_{I}\frac{\chi(I)}{N(I)^s},$$

where K is an algebraic number field, I runs over the non-zero ideals of the ring of integers of K, $N(I)$ denotes the norm of I and χ is a Hecke character of finite or infinite order. The functions $L_K(s,\chi)$ are a far reaching generalization of the Dirichlet L-functions. In fact, when $K=\mathbb{Q}$ the Hecke L-functions reduce to the Dirichlet L-functions. Moreover, when χ is trivial the function $L_K(s,\chi)$ reduces to the important special case of the *Dedekind zeta function*

$$\zeta_K(s)=\sum_{I}\frac{1}{N(I)^s}.$$

The analytic behaviour of the Hecke L-functions is similar to the Dirichlet L-functions, although the functional equation has a more complicated shape and is definitely more difficult to prove.

A different type of L-functions associated with algebraic number fields is provided by the *Artin L-functions* $L(s,K/k,\rho)$. Here K/k is a Galois extension of number fields with Galois group G, and ρ is a finite dimensional representation of G. The Artin L-functions are defined for $\sigma>1$ by certain Euler products, and their analytic properties are eventually deduced from the analytic properties of the Hecke L-functions. In fact, the Artin reciprocity law states if K/k is abelian, then $L(s,K/k,\rho)$ coincides with a suitable Hecke L-function. Moreover, the Artin-Brauer theory of group characters implies that every function $L(s,K/k,\rho)$ is a product of integer powers of abelian Artin L-functions. As a consequence, the Artin L-functions can be expressed as products of integer powers of Hecke L-functions, hence they have meromorphic continuation to $\mathbb{C}$, possibly with infinitely many poles. However, the famous *Artin conjecture* predicts that every function $L(s,K/k,\rho)$ is holomorphic on $\mathbb{C}$, apart possibly for a pole at $s=1$. The other analytic properties of the Artin L-functions are similar to those of the Hecke L-functions.

The Hecke and Artin L-functions, clearly of *algebraic* nature, provide quite a lot of information on the structure of algebraic number fields. We refer to Heilbronn [23] for the basic theory of Hecke and Artin L-functions.

⋆ The *Hecke L-functions associated with modular forms* are defined for σ sufficiently large by the Dirichlet series

$$L_f(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s},$$

where $f(z)$ is a holomorphic modular form and $a(n)$ are its Fourier coefficients. Under suitable restrictions and normalizations, the functions $L_f(s)$ satisfy analytic properties similar to those of the Riemann zeta function. For suitable choices of $f(z)$ (the Eisenstein series), such normalized L-functions give raise the Dedekind zeta functions of imaginary quadratic fields. There is an interesting "operation" between L-functions associated with modular forms, namely the *Rankin-Selberg convolution*. Roughly speaking, given two modular forms $f(z)$ and $g(z)$ with Fourier coefficients $a(n)$ and $b(n)$ respectively, the Rankin-Selberg convolution is defined by the Dirichlet series

$$L_{f\times\overline{g}}(s) = \sum_{n=1}^{\infty} \frac{a(n)\overline{b(n)}}{n^s}.$$

Under suitable restrictions and normalizations, and modulo a certain "fudge factor", the Rankin-Selberg convolution has analytic properties similar to the Riemann zeta function. A similar, and in a way more fundamental, "operation" is the *m-symmetric product L-function* of two modular forms. In this case, the analytic properties are known at present only for small values of the integer m. A related class of L-functions are the *Maass L-functions* associated with non-holomorphic modular forms. The definition of such functions is quite complicated, hence we skip it. We only remark that the known analytic properties of the Maass L-functions are similar to the Hecke L-functions, but the state of the art is more rudimentary in this case.

Around the mid of the last century, a deep interpretation of the Hecke and Maass L-functions in terms of representations was established. Roughly speaking, such L-functions were associated with automorphic representations of GL(2) over the rational field. This theory then evolved into the theory of *automorphic L-functions*, associated with automorphic representations of GL(n) over number fields. The theory of automorphic L-functions is very deep both from technical and conceptual viewpoints, and is not fully understood at present. For instance, analytic continuation and functional equation of the automorphic L-functions have been established, and the above mentioned Rankin-Selberg convolution and m-symmetric power L-functions are now interpreted as the L-functions associated with the tensor product and the m-symmetric power of representations, respectively. However, many deep conjectures remain open, and in particular the amazing *Langlands program*. The Langlands program is a very deep unifying program which, roughly speaking, predicts that the L-functions of arithmetic, algebraic and geometric (see below) nature are in fact members of the class of automorphic L-functions. An important "special case" of the Langlands program in the Shimura-Taniyama conjecture, asserting that the L-functions associated with elliptic curves correspond to suitable L-functions associated with modular forms. Such a conjecture has been first proved in important special cases by A.Wiles (see Wiles [80] and Taylor-Wiles [74]) as a key step in the proof of Fermat Last Theorem, and then in full generality by Wiles' followers.

The nature of the above L-functions is of course *automorphic*, and we refer to Hecke [22], Iwaniec [26] and Bump [8] for the basic theory of such functions (see also the recent survey by Gelbart-Miller [19]). We conclude the synthetic list of L-functions by remarking that L-functions of *geometric* nature, *i.e.* attached to geometric objects like elliptic curves and varieties, have been introduced as well, and we refer to Silverman [70] for an introductory presentation. Moreover, we refer to Chapter 5 of the recent book by Iwaniec-Kowalski [27] for an excellent introduction to the classical L-functions presented in this section.

Although the nature of the above L-functions is apparently *different*, once suitably normalized they share the following important *common* properties (in some cases only *conjecturally*):

- $\star$ ordinary Dirichlet series, absolutely convergent for $\sigma > 1$;
- $\star$ meromorphic continuation to $\mathbb{C}$, with at most a pole at $s = 1$;

⋆ functional equation of Riemann type with multiple Γ factors, relating s with $1-s$;

⋆ coefficients are $O(n^\varepsilon)$ for every $\varepsilon > 0$;

⋆ Euler product.

We will see that there is probably a very deep unifying theory behind such common properties which, in a sense, represents an analytic counterpart of the Langlands program.

2. Basic theory of the Selberg class

The following two natural questions arise at this point:

⋆ what is *in general* an L-function ?

⋆ are *all* L-functions already known ?

Clearly, the second question depends on the first one. In a way, an answer to the first question was given by Selberg [69], defining the **Selberg class** $\mathcal{S}$ of L-functions. Writing $\overline{f}(s) = \overline{f(\overline{s})}$ and assuming, as usual, that *an empty product equals* 1, the Selberg class is axiomatically defined as follows: $F \in \mathcal{S}$ if

(i) (ordinary Dirichlet series) $F(s) = \sum\limits_{n=1}^{\infty} a_F(n)n^{-s}$, absolutely convergent for $\sigma > 1$;

(ii) (analytic continuation) there exists an integer $m \geq 0$ such that $(s-1)^m F(s)$ is an entire function of finite order;

(iii) (functional equation) $F(s)$ satisfies a functional equation of type $\Phi(s) = \omega\overline{\Phi}(1-s)$, where

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s) = \gamma(s)F(s),$$

say, with $r \geq 0$, $Q > 0$, $\lambda_j > 0$, $\Re\mu_j \geq 0$ and $|\omega| = 1$;

(iv) (Ramanujan conjecture) for every $\varepsilon > 0$, $a_F(n) \ll n^\varepsilon$;

(v) (Euler product) $\log F(s) = \sum\limits_{n=1}^{\infty} b_F(n)n^{-s}$, where $b_F(n) = 0$ unless $n = p^m$ with $m \geq 1$, and $b_F(n) \ll n^\vartheta$ for some $\vartheta < \frac{1}{2}$.

Other axiomatic classes of L-functions have been proposed in the literature, see *e.g.* Piatetski-Shapiro [60] and Carletti-Monti Bragadin-Perelli [9]; however, the axioms of the Selberg class appear to be more satisfactory. Moreover, the problems raised by Selberg are definitely very interesting. As we shall see, such problems are of a rather different nature with respect to the classical problems on L-functions, in the sense that they deal with the L-functions *as a class*.

Examples of members of $\mathcal{S}$ are the Riemann zeta function, the Dirichlet L-functions, the Hecke L-functions associated with algebraic number fields and, under suitable restrictions and normalizations, the Hecke L-functions associated with holomorphic modular forms. The other L-functions listed in Section 1 are also in $\mathcal{S}$, provided certain classical conjectures hold. In particular, the Artin L-functions belong to $\mathcal{S}$ if the Artin conjecture holds, while the automorphic L-functions are in $\mathcal{S}$ provided the Ramanujan conjecture holds true.

Here are few comments on the five axioms defining the Selberg class. By axiom *(i)*, the functions in $\mathcal{S}$ are ordinary Dirichlet series. This is an important point since, as we shall see, the picture would change if *general Dirichlet series* are allowed. We recall that the general Dirichlet series are of the form

$$\sum_{n=1}^{\infty} \frac{a(n)}{l_n^s}$$

where l_n is an increasing sequence of positive real numbers tending to ∞. Restricting the frequences l_n to be integers, as in axiom *(i)*, carries some arithmetical information.

Axiom *(ii)* allows $s = 1$ to be the only pole of functions in $\mathcal{S}$, but most probably the picture does not change much if finitely many poles on the line $\sigma = 1$ are allowed.

The function $\gamma(s)$ in axiom *(iii)* is called γ-**factor**, and its factors $\Gamma(\lambda_j s + \mu_j)$ are the Γ-**factors**. The form of the γ-factor of a given $F \in \mathcal{S}$ is clearly *not unique*. For instance, application of the Legendre duplication formula for the Γ-function changes its shape, as the following example shows:

$$(\frac{\pi}{2})^{-s/2}\Gamma(\frac{s}{4})\Gamma(\frac{s}{4}+\frac{1}{2})\zeta(s) = (\frac{\pi}{2})^{-(1-s)/2}\Gamma(\frac{1-s}{4})\Gamma(\frac{1-s}{4}+\frac{1}{2})\zeta(1-s).$$

In other words, writing $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_r)$ and $\boldsymbol{\mu} = (\mu_1, ..., \mu_r)$, the **data** $(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$ of $F \in \mathcal{S}$ are *not uniquely defined* by $F(s)$. This gives rise to the notion of **invariant**, *i.e.* an expression defined in terms of the data of $F(s)$ which is uniquely determined by $F(s)$ itself. We will soon see an important example of invariant.

Probably, axiom *(iv)* can be weakened to "for every $\varepsilon > 0$ there exists a positive integer $M = M(\varepsilon)$ such that $a_F(n) \ll n^\varepsilon$ for $(n, M) = 1$" without changing much the picture. The advantage of this form of axiom *(iv)* rests on the fact that a similar bound can be proved for the coefficients $a_F^{-1}(n)$ and $b_F(n)$. In other words, assuming this form of axiom *(iv)* and denoting by $c(n)$ any of the coefficients $a_F(n)$, $a_F^{-1}(n)$ and $b_F(n)$, one has that for every $\varepsilon > 0$ there exists a positive integer $M = M(\varepsilon)$ such that $c(n) \ll n^\varepsilon$ for $(n, M) = 1$, and $c(n) \ll n^\vartheta$ for some $\vartheta < \frac{1}{2}$. Moreover, it is interesting to note that axiom *(iv)* is *crucial* for the Riemann Hypothesis. In fact, Jurek Kaczorowski constructed the following simple example of L-function satisfying axioms *(i)*, *(ii)*, *(iii)* and *(v)* but violating the Riemann Hypothesis. Let χ be a primitive Dirichlet character with $\chi(-1) = -1$ and write $G(s) = L(2s - 1/2, \chi)$. $G(s)$ is absolutely convergent for $\sigma > 3/4$, satisfies a functional equation with $\lambda = 1$ and $\mu = 1/4$, and has an Euler product allowing the choice $\vartheta = 1/4$. Taking $0 < \delta < 1/4$ and writing $F(s) = G(s - \delta)G(s + \delta)$, thanks to the above properties it is easy to see that $F(s)$ satisfies all axioms but the Ramanujan conjecture, and has no zeros on the critical line for suitable choices of δ.

Axiom *(v)* implies in particular that the coefficients $a_F(n)$ are *multiplicative*. Hence the standard Euler product

$$F(s) = \prod_p F_p(s) \qquad\qquad F_p(s) = \sum_{m=0}^{\infty} a_F(p^m)p^{-ms}$$

holds; $F_p(s)$ is the p-**Euler factor** of $F(s)$. Moreover, the seemingly harmless condition $\vartheta < \frac{1}{2}$ has in fact a relevant role. For instance it implies that $F_p(s) \neq 0$ for $\sigma > \vartheta$ for every prime p, and this will be crucial at several places. Moreover, if such a condition is relaxed and values of ϑ greater than $\frac{1}{2}$ are allowed, then examples of functions satisfying axioms *(i)*,...,*(v)* and violating the Riemann Hypothesis are easily constructed. A simple example is

$$f(s) = (1 - 2^{a-s})(1 - 2^{b-s}) \text{ with } a + b = 1 \text{ and } a > \tfrac{1}{2}.$$

Note that the five axioms of the Selberg class are not completely independent (for example, axiom *(v)* implies that $F(s)$ is an ordinary Dirichlet series). We refer to Molteni [47] for further pathological examples arising if parts of the axioms are dropped.

We finally remark that axioms *(i)*, *(ii)* and *(iii)* are more of *analytic* nature, while axioms *(iv)* and *(v)* are more of *arithmetic* nature. Therefore, we define the **extended Selberg class** $\mathcal{S}^\sharp$ to be the class of the non identically vanishing functions satisfying axioms *(i)*, *(ii)* and *(iii)*. Clearly, $\mathcal{S}^\sharp \supset \mathcal{S}$, and we shall see that $\mathcal{S}^\sharp$ still carries some of the properties of $\mathcal{S}$. We also remark that many of the definitions involving $\mathcal{S}$ carry over in an obvious way to the case of $\mathcal{S}^\sharp$.

The standard analytic properties of the functions $F \in \mathcal{S}$ are easily obtained by means of the classical arguments used to study the Riemann zeta function. Let $F \in \mathcal{S}$. We define the **polar order** m_F of $F(s)$ to be the least value of m in axiom *(ii)*, and

$$d_F = 2\sum_{j=1}^{r} \lambda_j$$

is the **degree** of $F(s)$. It is easy to see that

$$d_\zeta = 1, \qquad d_{L(\cdot,\chi)} = 1, \qquad d_{\zeta_K} = [K:\mathbb{Q}], \qquad d_{L_K(\cdot,\chi)} = [K:\mathbb{Q}], \qquad d_{L_f} = 2 \tag{2.1}$$

and similarly for the other classical L-functions. The function $\Psi(s) = s^{m_F}(1-s)^{m_F}\Phi(s)$ is an entire function of order 1, and the Lindelöf μ-function $\mu_F(\sigma)$ satisfies $\mu_F(\sigma) = 0$ for $\sigma \geq 1$ and, by the functional equation, $\mu_F(\sigma) = d_F(\frac{1}{2} - \sigma)$ for $\sigma \leq 0$. This shows in particular that the degree in an *invariant*, and hence $\mathcal{S}$ can be partitioned as

$$\mathcal{S} = \bigcup_{d \geq 0} \mathcal{S}_d,$$

where

$$\mathcal{S}_d = \{F \in \mathcal{S} : d_F = d\}.$$

From the Euler product we have that $F(s) \neq 0$ for $\sigma > 1$, hence by the functional equation we have the familiar notions of **critical strip** and **critical line**, *i.e.* the strip $0 \leq \sigma \leq 1$ and the line $\sigma = \frac{1}{2}$, respectively. The zeros of $F(s)$ located at the poles of the γ-factor $\gamma(s)$, *i.e.* at $\rho = -\frac{\mu_j + k}{\lambda_j}$ with $k = 0, 1, 2, ...$ and $j = 1, ..., r$, are called the **trivial zeros**, and are the only zeros of $F(s)$ in the half plane $\sigma < 0$. The case $\rho = 0$, if present, requires special attention in view of the possible pole of $F(s)$ at $s = 1$. The other zeros, located inside the critical strip, are called the **non-trivial zeros**. We cannot *a priori* exclude the possibility that $F(s)$ has a trivial and a non-trivial zero at the same point, on the line $\sigma = 0$. Moreover, writing

$$N_F(T) = |\{\rho = \beta + i\gamma : F(\rho) = 0,\ 0 \leq \beta \leq 1,\ 0 \leq \gamma \leq T\}|,$$

the analog of the Riemann-von Mangoldt formula holds in the form

$$N_F(T) = \frac{d_F}{2\pi} T \log T + c_F T + O(\log T),$$

where c_F is a certain constant depending on $F(s)$. This shows once again that the degree d_F is an invariant (as well as c_F).

For details and further discussions on the matters above we refer to Selberg [69], Conrey-Ghosh [12], Murty [51] and the survey papers Kaczorowski-Perelli [34], Kaczorowski [29] and Perelli [57] and [58].

Roughly speaking, the problems about the Selberg class are of two types.

⋆ *Classical problems*: these are the extension to $\mathcal{S}$ of the problems about the classical L-functions, the most important being the Riemann Hypothesis. In fact, Selberg [69] conjectured that the Riemann Hypothesis holds for every function $F \in \mathcal{S}$, *i.e.*

Conjecture 2.1. (GRH) *Let $F \in \mathcal{S}$. Then $F(s) \neq 0$ for $\sigma > \frac{1}{2}$.*

We remark at this point that the knowledge about the distribution of zeros of the functions in $\mathcal{S}$ is definitely poorer than in the case of the classical L-functions. For example, it is not yet known in general if $F(1+it) \neq 0$ for every $t \in \mathbb{R}$.

⋆ *Structural problems*: these are the problems on the stucture of $\mathcal{S}$ *as a class*. The classification of the functions in $\mathcal{S}$, the independence properties of the functions in $\mathcal{S}$, the study of the invariants in $\mathcal{S}$, the countability and rigidity conjectures for $\mathcal{S}$ are important examples of structural problems.

In this survey we focus on the structural problems for the Selberg class. Such problems, in part raised by Selberg himself, deal with L-functions from a somewhat diffferent perspective with respect to the classical problems, and their solution will eventually lead to a deeper understanding of the nature of L-functions.

We start with the classification of the functions in the classes $\mathcal{S}$ and $\mathcal{S}^\sharp$ with degree smaller than 1, since such results are needed later in this section. The basic result, Theorem 3.1 below, has apparently been proved first by Richert [63] and then independently by Bochner [1] and Conrey-Ghosh [12]. Further proofs have been given by Molteni [44] and Kaczorowski-Perelli [37], [40].

Theorem 3.1. ([63], [1]. [12]) *$\mathcal{S}_d^\sharp = \emptyset$ for $0 < d < 1$.*

A key point in the proof of Theorem 3.1 (common to several of the above proofs) is showing that the Dirichlet series of every function in $\mathcal{S}_d^\sharp$ with $0 \leq d < 1$ is absolutely convergent over $\mathbb{C}$. This contradicts $\mu_F(\sigma) > 0$ for $\sigma \leq 0$, provided $0 < d < 1$. For $d = 0$, the functional equation then shows that

$$F(s) = \sum_{n|q_F} \frac{a_F(n)}{n^s} \tag{2.2}$$

with $q_F = Q^2 \in \mathbb{N}$. Thus, in particular, the functions in $\mathcal{S}_0^\sharp$ are Dirichlet polynomials. For $q \in \mathbb{N}$ and $|\omega| = 1$, let $\mathcal{S}_0^\sharp(q, \omega)$ be the set of $F \in \mathcal{S}_0^\sharp$ with given ω and $q_F = q$, and let

$$V_0^\sharp(q, \omega) = \mathcal{S}_0^\sharp(q, \omega) \cup \{0\}.$$

Moreover, let $d(n)$ denote the divisor function. The above simple argument leads to

Theorem 3.2. ([33]) *Let $F \in \mathcal{S}_0^\sharp$. Then $q_F \in \mathbb{N}$ and $F(s)$ has the form (2.2). Moreover, q_F and ω are invariants, thus $\mathcal{S}_0^\sharp$ is the disjoint union of the subclasses $\mathcal{S}_0^\sharp(q, \omega)$ with $q \in \mathbb{N}$ and $|\omega| = 1$. Further, for any such q and ω, $V_0^\sharp(q, \omega)$ is a real vector space of dimension $d(q)$.*

We refer to Steuding [73] for a different characterization of the functions $F \in \mathcal{S}_0^\sharp$. Starting from (2.2), a simple argument based on the Euler product further shows

Theorem 3.3. ([12]) $\mathcal{S}_0 = \{1\}$.

We already noticed that every function in the Selberg class has a standard Euler product, *i.e.* it can be expressed as a product of its p-Euler factors. It may happen that two distinct functions $F, G \in \mathcal{S}$ have equal p-Euler factors for certain primes p. Denote by $E_{F,G}$ the set of such primes. The "exceptional set" $E_{F,G}$ can be pretty large, as the following example shows. Let χ_1 and χ_2 be distinct primitive Dirichlet characters (mod q) such that $\chi_1(a) = \chi_2(a)$ for some a coprime to q. Then the corresponding exceptional set contains the primes $p \equiv a$ (mod q). Hence, in particular, $E_{F,G}$ can have positive density.

On the other hand, a well known result in representation theory, called the *Strong Multiplicity One Theorem* (see Piatetski-Shapiro [59]), implies that if the p-Euler factors of two automorphic L-functions are equal for all but finitely many primes, then the two L-functions are equal. The analog of such a result for the Selberg class is called the **multiplicity one** property, and has been proved by Murty-Murty [53].

Theorem 3.4. ([53]) *Let $F, G \in \mathcal{S}$. If $F_p(s) = G_p(s)$ for all but finitely many primes p, then $F(s) = G(s)$.*

The proof amounts to the observation that by the functional equation $F(s)/G(s)$ is entire and non-vanishing, hence the result follows by Hadamard's theory. The same argument shows that the assumption $F_p(s) = G_p(s)$ can be replaced by the weaker requirement that $a_F(p^m) = a_G(p^m)$ for $m = 1, 2$. It would be desirable to remove the condition involving the squares, as suggested by the following conjecture.

Conjecture 3.1. (strong multiplicity one) *Let $F, G \in \mathcal{S}$. If $a_F(p) = a_G(p)$ for all but finitely many primes p, then $F(s) = G(s)$.*

We will describe a rather sharp unconditional result in this direction in a later section.

Clearly, the classes $\mathcal{S}$ and $\mathcal{S}^\sharp$ are *multiplicative semigoups* and the degree is *additive*, in the sense that $d_{F_1F_2} = d_{F_1} + d_{F_2}$. Moreover, given an *entire* $F \in \mathcal{S}^\sharp$ and $\theta \in \mathbb{R}$ we define the **shift** $F_\theta(s)$ as $F_\theta(s) = F(s + i\theta)$. Clearly, $F_\theta \in \mathcal{S}$ if $F \in \mathcal{S}$, and the same holds for $\mathcal{S}^\sharp$. Further, 1 is the only constant function in $\mathcal{S}$, and also the only invertible element of $\mathcal{S}$. A function $F \in \mathcal{S} \setminus \{1\}$ is **primitive** if $F(s) = F_1(s)F_2(s)$ with $F_1, F_2 \in \mathcal{S}$ implies $F_1(s) = 1$ or $F_2(s) = 1$; in other words, primitive functions are the irreducible elements of the semigrup $\mathcal{S}$. In view of Theorems 3.1 and 3.3, every function of degree < 2 is primitive, hence $\zeta(s)$ and the $L(s, \chi)$'s with primitive χ are primitive. Other examples of primitive functions are provided by a suitable class of normalized L-functions associated with holomorphic modular forms. These are degree 2 functions, and the proof requires a deeper knowledge of the structure of $\mathcal{S}_1$, see Section 6.

Primitive functions play an important role in the theory of the Selberg class. As a first result, Theorems 3.1, 3.3 and a simple induction on the degree give

Theorem 3.5. ([12], [51]) *Every $F \in \mathcal{S}$ can be factored as a product of primitive functions.*

In other words, every $F \in \mathcal{S}$ has a factorization of type

$$F(s) = \prod_{j=1}^{k} F_j(s)^{e_j} \tag{2.3}$$

with $e_j \in \mathbb{N}$ and $F_j(s)$ primitive and distinct. A related natural conjecture is

Conjecture 3.2. (unique factorization, UF) *Factorization into primitive functions is unique.*

If the UF conjecture holds, then (2.3) is called the **standard form** of $F(s)$.

The Rankin-Selberg convolution method shows that the L-functions associated with holomorphic modular forms satisfy a kind of orthogonality relation. Precisely, under suitable normalizations and restrictions, the function $L_{f\times\overline{g}}(s)$ defined in Section 1 has a simple pole at $s = 1$ if $f(z) = g(z)$ and is entire otherwise. A similar result holds in general for the irreducible automorphic L-functions. Motivated by such properties, Selberg formulated the following fundamental conjecture.

Conjecture 3.3. (Selberg orthonormality conjecture, SOC) *Let $F, G \in \mathcal{S}$ be primitive functions and $\delta_{F,G} = 1$ if $F(s) = G(s)$, $\delta_{F,G} = 0$ otherwise. Then as $x \to \infty$*

$$\sum_{p \le x} \frac{a_F(p)\overline{a_G(p)}}{p} = (\delta_{F,G} + o(1)) \log\log x.$$

In order to appreciate the depth of the Selberg orthonormality conjecture, we list few simple but interesting consequences in Theorem 3.6 below. We first recall the the *Dedekind conjecture* asserts that $\zeta(s)$ divides $\zeta_K(s)$ for every algebraic number field $K/\mathbb{Q}$. This is well known in the case of normal extensions by the Aramata-Brauer theorem (see [23]). Moreover, given $F \in \mathcal{S}$ we define the real number n_F, if it exists, by

$$\sum_{p\leq x} \frac{|a_F(p)|^2}{p} = (n_F + o(1)) \log\log x. \tag{2.4}$$

Further, we denote as usual by $\sigma_a(F)$ the abscissa of absolute convergence of $F \in \mathcal{S}^\sharp$. We have

Theorem 3.6. ([12], [51], [53]) *Assume* SOC *and let e_j be as in* (2.3). *Then*

i) the UF *conjecture holds;*

ii) $n_F = \sum_{j=1}^k e_j^2$, and hence $F \in \mathcal{S}$ is primitive if and only if $n_F = 1$;

iii) $\zeta(s)$ is the only primitive function in $\mathcal{S}$ with a pole at $s = 1$, and hence the Dedekind conjecture holds;

iv) $F(1 + it) \neq 0$ for every $t \in \mathbb{R}$, for every $F \in \mathcal{S}$;

v) the strong multiplicity one conjecture holds;

vi) $\sigma_a(F) = 1$ for every $F \in \mathcal{S} \setminus \{1\}$.

We already remarked that at present is not unconditionally known if $F(1 + it) \neq 0$, $t \in \mathbb{R}$, for every $F \in \mathcal{S}$. However, it is not surprising that this follows from SOC. In fact, the standard proofs of the non-vanishing of L-functions on the 1-line are based on the properties of the Rankin-Selberg convolution. We also remark that under SOC (in fact, under UF) the usual notions of *coprimality* and of *greatest common divisor* are easily defined in $\mathcal{S}$. From *ii)* of Theorem 3.6 it is quite clear that in the case of *any* $F, G \in \mathcal{S}$, SOC becomes

$$\sum_{p\leq x} \frac{a_F(p)\overline{a_G(p)}}{p} = (\sum_{j=1}^{l} f_j g_j + o(1)) \log\log x,$$

where

$$F(s) = H(s) \prod_{j=1}^{l} F_j(s)^{f_j} \qquad\qquad G(s) = K(s) \prod_{j=1}^{l} F_j(s)^{g_j},$$

the functions $F_j(s)$ are primitive and distinct, and $H(s)$, $K(s)$ are coprime and not divisible by the $F_j(s)$'s.

We remark here that the proof of the assertion on page 6 of Murty [51] that UF implies the Dedekind conjecture (unfortunately reported as Proposition 4.2 in Kaczorowski-Perelli [34]) appears to be incorrect. In fact (using the notation in [51]) assuming only UF we do not see how to exclude, for example, that $\zeta_K(s)$ is primitive and $F(s) = \zeta(s)H(s)$ with a primitive $H \in \mathcal{S}$ vanishing at $s = 1$.

Another interesting consequence of SOC, based on the Artin-Brauer theory and on Chebotarev density theorem, is

Theorem 3.7. ([51]) SOC *implies the Artin conjecture.*

We recall that the Artin conjecture states that the Artin L-functions $L(s, K/k, \rho)$ are entire if ρ is irreducible and non-trivial. Moreover, the argument in the proof of Theorem 3.7 shows also that such functions are *primitive*. It is interesting to note how a conjecture concerning an axiomatic class of L-functions has a strong consequence on a classical conjecture. The argument in the proof of Theorem 3.7, coupled with work of Arthur-Clozel on solvable extensions, can be suitably adapted to show that SOC implies the Langlands reciprocity conjecture for solvable extensions of $\mathbb{Q}$, see Murty [51].

One may ask if primitivity can be characterized by the functional equation. Apparently this is not the case, as shown by an example in Molteni [47] of a degree 2 functional equation with a non-primitive solution in $\mathcal{S}$ (a Dedekind zeta function of a real quadratic field) and, assuming the Takhtajan-Vinogradov conjecture on the dimension of the space of Maass forms, a primitive solution as well. We state here a problem about the shifts of primitive functions.

Problem 3.1. *Show that $F_\theta(s)$ is primitive for all $\theta \in \mathbb{R}$ if $F \in \mathcal{S}$ is primitive.*

There is an easy proof of this statement if axiom *(ii)* of the Selberg class is weakened to allow a finite number of poles on the line $\sigma = 1$ (note that *every* function in such a larger class can be shifted). In fact, suppose that $F(s)$ is primitive, while $F_\theta(s) = F_1(s)F_2(s)$ is a non-trivial factorization for some $\theta \in \mathbb{R}$. Then $F(s) = F_\theta(s - i\theta) = F_1(s - i\theta)F_2(s - i\theta)$, a contradiction. In the framework of the Selberg class $\mathcal{S}$, the problem arises from the situation, which we cannot *a priori* exclude, that $F_\theta(s)$ is entire while $F_1(s)$ has a pole and $F_2(s)$ has a zero at $s = 1$. This situation is of course impossible under SOC.

The Selberg orthonormality conjecture can be regarded as a rather strong form of independence of the functions in $\mathcal{S}$. The unique factorization conjecture, which follows from SOC, is also an independence statement in $\mathcal{S}$. We may therefore ask if the simplest form of independence, namely the *linear independence*, holds in $\mathcal{S}$. We recall that a Dirichlet series $D(s)$, absolutely convergent in some right half-plane, is called *p-finite* if there exists a positive integer M such that the coefficients $c(n)$ of $D(s)$ vanish whenever n has a prime factor not dividing M. In this case, the arithmetic function $c(n)$ is called p-finite as well. We denote by $\mathcal{F}$ both the ring of p-finite Dirichlet series and the ring of p-finite arithmetic functions; note that $\mathcal{F}$ contains all Dirichlet polynomials.

Theorem 3.8. ([30]) *Distinct functions in $\mathcal{S}$ are linearly independent over $\mathcal{F}$.*

In particular, distinct functions of $\mathcal{S}$ are linearly independent over $\mathbb{C}$. We remark that Theorem 3.8 is basically a result on multiplicative arithmetic functions. We call *equivalent* two multiplicative functions $f(n)$ and $g(n)$ if $f(p^m) = g(p^m)$ for all integer $m \geq 1$ and all but finitely many primes p. The main step in the proof of Theorem 3.8 is showing that pairwise non-equivalent multiplicative functions are linearly independent over $\mathcal{F}$. This is in fact an analogue of Artin's well known result that distinct characters are linearly independent, and the proof is similar. Theorem 3.8 follows then by Theorem 3.4, which ensures that the coefficients of distinct functions in $\mathcal{S}$ are pairwise non-equivalent multiplicative functions.

We remark that Theorem 3.8 is a special case of a more general result, see Kaczorowski-Molteni-Perelli [31]. In fact, its proof can be suitably modified to show the linear independence of functions in a larger class, including the derivatives of all orders and the inverses of the functions in $\mathcal{S}$. Moreover, such a class also contains the Artin and the automorphic L-functions, which are not yet known to belong to $\mathcal{S}$. See also Molteni [46] for further results.

It is well known that the Prime Number Theorem is equivalent to $\zeta(1 + it) \neq 0$ for $t \in \mathbb{R}$. Although the non-vanishing on the 1-line is at present a conditional result in the general setting of the Selberg class, the analog of the above-mentioned equivalence can be proved unconditionally in $\mathcal{S}$. Let $\Lambda_F(n)$ be the **generalized von Mangoldt function**, defined by

$$-\frac{F'}{F}(s) = \sum_{n=1}^{\infty} \Lambda_F(n) n^{-s},$$

i.e. $\Lambda_F(n) = b_F(n) \log n$, and let

$$\psi_F(x) = \sum_{n \leq x} \Lambda_F(n).$$

It is expected that the prime number theorem (PNT) holds in the form

$$\psi_F(x) = m_F x + o(x)$$

for every $F \in \mathcal{S}$, where m_F is the polar order of $F(s)$ defined in Section 2. Writing

$$\psi_{F\times\overline{F}}(x) = \sum_{n\leq x} |\Lambda_F(n)|^2,$$

a simple consequence of axioms *(iv)* and *(v)* is that $\psi_{F\times\overline{F}}(x) \ll x^{1+\varepsilon}$, and hence the bound $\psi_F(x) \ll x^{1+\varepsilon}$ holds unconditionally.

Theorem 3.9. ([39]) *Let $F \in \mathcal{S}$. Then* PNT *holds if and only if $F(1+it) \neq 0$ for every $t \in \mathbb{R}$.*

The proof is based on a weak density estimate for the zeros of $F(s)$ close to the 1-line and on a simple almost periodicity argument. From Theorems 3.6 and 3.9 we see that SOC *implies* PNT. However, the argument in the proof of Theorem 3.9 allows to obtain a sharper result. To this end we introduce the following much weaker version of SOC.

Conjecture 3.4. (normality conjecture, NC) *Let $F \in \mathcal{S} \setminus \{1\}$. Then* (2.4) *holds with $n_F > 0$, and $n_F \leq 1$ if $F(s)$ is primitive.*

We have

Theorem 3.10. ([39]) *Assume* NC *and let $F \in \mathcal{S}$. Then $F(1+it) \neq 0$ for every $t \in \mathbb{R}$.*

In view of Theorems 3.9 and 3.10, NC *implies* PNT. We recall that

$$\sum_{p\leq x} \frac{1}{p} \sim \log\log x$$

is a weaker statement than the Prime Number Theorem, and an analogous assertion holds for other classical L-functions as well. Hence, NC for a *given* $F \in \mathcal{S}$ is, in general, weaker than PNT for the *same* function. Therefore Theorem 3.10 is a simple example of the philosophy that general properties of a *family* of L-functions can be used to derive stronger consequences for *individual* members of the family.

Now we turn to a discussion of the factorization problem in $\mathcal{S}^\sharp$. In order to extend the notion of primitive function to the class $\mathcal{S}^\sharp$, we need to know the invertible functions in $\mathcal{S}^\sharp$. Clearly, the non-zero complex constants belong to $\mathcal{S}^\sharp$, and it is easy to see that these are the only invertible elements of $\mathcal{S}^\sharp$. Hence we say that a non-constant $F \in \mathcal{S}^\sharp$ is **$\sharp$-primitive** if $F(s) = F_1(s)F_2(s)$ with $F_1, F_2 \in \mathcal{S}^\sharp$ implies that $F_1(s)$ or $F_2(s)$ is constant. The problem of the factorization into primitive functions can therefore be raised for $\mathcal{S}^\sharp$ as well. The analogous property for $\mathcal{S}$ depends on the following three facts: the degree is additive, there are no functions with degree $0 < d_F < 1$ and $\mathcal{S}_0 = \{1\}$. The first two facts hold for $\mathcal{S}^\sharp$ as well, but $\mathcal{S}_0^\sharp$ is definitely more complicated than $\mathcal{S}_0$. Therefore, the proof of Theorem 3.5 does not carry over to the case of $\mathcal{S}^\sharp$. However, the argument can be suitably modified to prove

Theorem 3.11. ([38]) *Every $F \in \mathcal{S}^\sharp$ can be factored as a product of $\sharp$-primitive functions.*

The proof is based on the notion of **almost-primitive** function, that is a function $F \in \mathcal{S}^\sharp$ such that $F(s) = F_1(s)F_2(s)$ implies $d_{F_1} = 0$ or $d_{F_2} = 0$. The main part of the proof of Theorem 3.11 is devoted to the following characterization of almost-primitive functions: *if $F \in \mathcal{S}^\sharp$ is almost-primitive, then $F(s) = P(s)G(s)$ with $P(s)$ $\sharp$-primitive and $d_G = 0$.* In turn, such a characterization is based on a uniform estimate for the number of zeros of the Dirichlet polynomials of $\mathcal{S}_0^\sharp$. Theorem 3.11 follows then from the above characterization by a double induction, first on the degree (giving the factorization into almost-primitive functions) and then on the integer q_F in Theorem 3.2 (giving the factorization of the functions of $\mathcal{S}_0^\sharp$ into $\sharp$-primitive functions). We will see in the next section that such an integer q_F is a special instance of the general notion of *conductor* in $\mathcal{S}^\sharp$.

We remark that *the analog of* SOC *does not hold for* $\mathcal{S}^\sharp$. Indeed, let χ_1, χ_2 be two primitive Dirichlet characters with the same modulus and parity, and let $F(s) = L(s,\chi_1) + L(s,\chi_2)$ and $G(s) = L(s,\chi_1)$. Thanks to Theorems 3.1 and 3.2 and to the description of $\mathcal{S}_1^\sharp$ in Section 6, in view of Theorem 3.8 we have that $F(s)$ and $G(s)$ are $\sharp$-primitive, but it is easily checked that SOC does not hold for $F(s)$ and $G(s)$. In view of this, we conclude the section with two problems.

Problem 3.2. *Does* UF *hold for* $\mathcal{S}^\sharp$?

Problem 3.3. *Is it true that a primitive* $F \in \mathcal{S}$ *is also* $\sharp$*-primitive* ?

We conclude this section with a problem on the characterization of divisibility in $\mathcal{S}$. In view of the Hadamard product, a function in $\mathcal{S}$ is essentially determined by its zeros. Denoting by Z_F the **set of zeros** of $F \in \mathcal{S}$ *counted with multiplicity*, we raise the following

Problem 3.4. *Let* $F, G \in \mathcal{S}$*. Show that* $F(s)$ *divides* $G(s)$ *in* $\mathcal{S}$ *if and only if* $Z_F \subset Z_G$*.*

We refer to Molteni [47] and [45] for closely related results.

3. Invariants

We already pointed out in Section 2 that, due to the application of suitable identities satisfied by the Γ-function, the shape of the γ-factor $\gamma(s)$ of $F \in \mathcal{S}^\sharp$ is not uniquely determined by $F(s)$. We also remarked that this fact gives rise to the notion of *invariant, i.e.* an expression defined in terms of the data of $F(s)$ which is uniquely determined by $F(s)$ itself. Moreover, we already met an important invariant, namely the degree d_F.

Although their *shape* may change considerably, γ-factors are essentially *uniquely determined as functions.* In fact we have

Theorem 4.1. ([12]) *Let* $\gamma(s)$ *and* $\gamma'(s)$ *be two* γ*-factors of* $F \in \mathcal{S}^\sharp$*. Then there exists a constant* $c_0 = c_0(\gamma, \gamma') \in \mathbb{C}$ *such that* $\gamma(s) = c_0\gamma'(s)$*.*

The proof follows by Hadamard's theory, observing that $h(s) = \gamma(s)/\gamma'(s)$ is entire and non-vanishing thanks to the functional equation.

In view of Theorem 4.1, in order to study the invariants we need to detect the operations which transform a γ-factor $\gamma(s)$ of a function $F \in \mathcal{S}^\sharp$ into another γ-factor of the same function. It turns out that such a transformation can be performed by repeated applications to $\gamma(s)$ of two basic formulae in the theory of the Γ-function, namely the *Legendre-Gauss multiplication formula*

$$\Gamma(s) = m^{s-\frac{1}{2}}(2\pi)^{\frac{1-m}{2}} \prod_{k=0}^{m-1} \Gamma(\frac{s+k}{m}) \qquad m = 2, 3, ... \tag{3.1}$$

and the *factorial formula*

$$\Gamma(s+1) = s\Gamma(s).$$

We also need some definitions. Two positive real numbers α, β are $\mathbb{Q}$-*equivalent* if $\alpha/\beta \in \mathbb{Q}$. We denote by h_F, the γ**-class number**, the number of $\mathbb{Q}$-equivalence classes arising from the

λ-coefficients $\lambda_1, ..., \lambda_r$ of a γ-factor of $F \in \mathcal{S}^\sharp$. Moreover, we say that $F(s)$ is **reduced** if it has a γ-factor with $0 \leq \Re\mu_j < 1$ for $j = 1, ..., r$; such a γ-factor is also called *reduced*. It turns out that h_F *is an invariant*, and that $F(s)$ *is reduced if and only if all its* γ*-factors are reduced*, so these are reasonable definitions.

Recalling that c_0 is the constant in Theorem 4.1 we have

Theorem 4.2. ([35]) *Let* $\gamma(s)$ *and* $\gamma'(s)$ *be two* γ*-factors of* $F \in \mathcal{S}^\sharp$*. Then* $\gamma(s)$ *can be transformed into* $c_0\gamma'(s)$ *by repeated applications of the multiplication and factorial formulae. Moreover, the factorial formula can be avoided if* $h_F = 1$ *or if* $F(s)$ *is reduced.*

We refer to Section 4 of Vignéras [77], as well as to and Serre's appendix there, for related results. It is clear that applications of the multiplication formula to a γ-factor give rise to another γ-factor, and do not change the $\mathbb{Q}$-equivalence classes. Applications of the factorial formula are a bit more involved. Basically, such a formula is used to *reduce* a γ-factor, *i.e.* to write it as the product of a reduced γ-factor, called the *reduced part*, times a product of suitable linear factors. Such linear factors are then re-absorbed into the Γ-factors by further applications of the factorial formula, provided suitable consistency conditions hold. Although examples of non-reduced γ-factors are easily produced, see for instance the case of L-functions associated with holomorphic modular forms (suitably nomalized to meet the axioms of $\mathcal{S}$), according to the following conjecture we expect $h_F = 1$ to be the general case.

Conjecture 4.1. (γ-class number conjecture) *Every* $F \in \mathcal{S}^\sharp$ *has* $h_F = 1$.

We will see in Section 6 motivations for this and for the following *much stronger* conjecture.

Conjecture 4.2. ($\boldsymbol{\lambda}$-conjecture) *Every* $F \in \mathcal{S}^\sharp$ *has a* γ*-factor with* $\lambda_j = \frac{1}{2}$ *for* $j = 1, ..., r$.

Therefore, we expect that the factorial formula is not necessary in the transformation of the γ-factors. However, at the present state of the knowledge, we cannot in general avoid using it, and here is an example:

$$\Gamma(s)\Gamma(\sqrt{2}s + 1) = \sqrt{\frac{2}{\pi}}2^s\Gamma(\frac{s}{2} + 1)\Gamma(\frac{s+1}{2})\Gamma(\sqrt{2}s).$$

Note that there are two $\mathbb{Q}$-equivalence classes, and that the pole at $s = 0$ comes, in the two sides of the identity, from Γ-factors belonging to different classes. This is the typical situation requiring application of the factorial formula.

In view of the identity $\gamma(s) = c_0\gamma'(s)$ in Theorem 4.1, the proof of Theorem 4.2 rests on a detailed analysis of the structure of the following general Γ-identity

$$\prod_{j=1}^{N} \Gamma(\lambda_j s + \mu_j) = e^{as} R(s) \prod_{j=1}^{M} \Gamma(\lambda'_j s + \mu'_j), \tag{3.2}$$

where $a \in \mathbb{C}$ and $R(s)$ is a rational function. Clearly, $R(s)$ arises from applications of the factorial formula. The structure of (3.2) is studied by means of the analysis of the poles of both sides. This leads to a *transformation algorithm* for γ-factors, which we briefly outline. Let $\gamma(s)$ and $\gamma'(s)$ be as in Theorem 4.2. Then $\gamma(s)$ is transformed into $c_0\gamma'(s)$ as follows.

Step 1. (*reducing*) Apply the factorial formula to reduce $\gamma(s)$ and $\gamma'(s)$.

Step 2. (*grouping*) Group the Γ-factors of the reduced parts and the corresponding linear factors according to $\mathbb{Q}$-equivalence classes. The $\mathbb{Q}$-equivalence classes arising from $\gamma(s)$ and $\gamma'(s)$ are the same, and identity $\gamma(s) = c_0\gamma'(s)$ induces suitable sub-identities of type (3.2) between the pairs of groups with the same $\mathbb{Q}$-equivalence class.

Step 3. (*equating*) Apply the multiplication formula to each pair of groups, to obtain new pairs of groups with the property that all the Γ-factors in the same pair of groups have the same λ-coefficient. In such a situation, in each pair of groups the Γ-factors coming from $\gamma(s)$ are a permutation of those coming from $\gamma'(s)$.

Step 4. (*transforming*) Perform on the Γ-factors coming from $\gamma(s)$ the inverse of the operations performed in steps 3, 2 and 1 on the Γ-factors coming from $\gamma'(s)$, thus transforming $\gamma(s)$ into $c_0\gamma'(s)$.

A more combinatorial argument leading to a simple proof of Theorem 4.2 is provided by Wirsing [81].

The proof of Theorem 4.2 involves also the notion of *exact covering system*, *i.e.* a family (M, l_j, m_j), $j = 1, ..., M$, with the property that for every integer n there exists a unique j such that $n \equiv l_j \pmod{m_j}$. As a byproduct of the arguments in the proof, we can get the following complete description of all γ-factors of the Dirichlet L-functions. Of course, other known L-functions can be treated analogously. Let $\chi \pmod q$ be a primitive Dirichlet character. Then *all γ-factors of $L(s,\chi)$ are of the form*

$$Q^s \prod_{j=1}^{M} \Gamma\Big(\frac{s}{2m_j} + \frac{2l_j + a(\chi)}{2m_j}\Big),$$

where (M, l_j, m_j) is any exact covering system,

$$Q = \Big(\frac{q}{\pi} \prod_{j=1}^{M} m_j^{1/m_j}\Big)^{1/2}$$

and $a(\chi) = \frac{1+\chi(-1)}{2}$.

In order to give a characterization of the invariants by means of Theorem 4.2, we need to give a more formal definition of invariant. An expression depending on the "variables" $(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$ is called a *parameter*. An **invariant** is a parameter depending only on $F(s)$ and not on the particular choice of the data of $F(s)$, for every $F \in \mathcal{S}^\sharp$. In other words, a parameter $I(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$ is an invariant if $I(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega) = I(Q', \boldsymbol{\lambda}', \boldsymbol{\mu}', \omega')$ for any pair of data $(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$, $(Q', \boldsymbol{\lambda}', \boldsymbol{\mu}', \omega')$ of $F(s)$, for every $F \in \mathcal{S}^\sharp$. Parameters and invariants will sometimes be denoted by $I(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$. A generic invariant will be denoted by I, and when referred to a function $F(s)$ will be denoted by I_F or $I(F)$. An invariant I is called **numerical** if $I_F \in \mathbb{C}$ for every $F \in \mathcal{S}^\sharp$.

We say that a parameter is *stable by multiplication formula* if $I(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega) = I(Q', \boldsymbol{\lambda}', \boldsymbol{\mu}', \omega')$, where $(Q', \boldsymbol{\lambda}', \boldsymbol{\mu}', \omega')$ are the new data obtained by application of the multiplication formula to a Γ-factor. Similarly we say that a parameter is *stable by factorial formula*, although this case is a bit more subtle since we always apply the factorial formula to a pair of Γ-factors satisfying a consistency condition. In fact, by the factorial formula we have

$$\Gamma(\lambda s + \mu)\Gamma(\lambda' s + \mu') = \frac{\lambda}{\lambda'}(\lambda' s + \frac{(\mu - 1)\lambda'}{\lambda})\Gamma(\lambda s + \mu - 1)\Gamma(\lambda' s + \mu'),$$

and assuming the *consistency condition*

$$\frac{\mu - 1}{\lambda} = \frac{\mu'}{\lambda'} \tag{3.3}$$

we get

$$\Gamma(\lambda s + \mu)\Gamma(\lambda' s + \mu') = \frac{\lambda}{\lambda'}\Gamma(\lambda s + \mu - 1)\Gamma(\lambda' s + \mu' + 1). \tag{3.4}$$

The above notions of stability will be clarified below, where we will list several important examples of invariants. From Theorem 4.2 we immediately obtain

Corollary 4.1. ([35]) *A parameter is an invariant if and only if it is stable by multiplication and factorial formulae.*

Here is a short list of important invariants of $\mathcal{S}^\sharp$, as well as some remarks; see Kaczorowski-Perelli [35], [36].

$\star$ The H**-invariants** $H_F(n)$. For a non-negative integer n let

$$H_F(n) = 2\sum_{j=1}^{r} \frac{B_n(\mu_j)}{\lambda_j^{n-1}},$$

where $B_n(z)$ denotes the n-th Bernoulli polynomial. Since $B_0(z) = 1$, $B_1(z) = z - \frac{1}{2}$, $B_2(z) = z^2 - z + \frac{1}{6}$, ..., we have for instance

$$H_F(0) = 2\sum_{j=1}^{r} \lambda_j = d_F \qquad \text{(the degree)}$$

$$H_F(1) = 2\sum_{j=1}^{r} (\mu_j - \frac{1}{2}) = \xi_F = \eta_F + i\theta_F \qquad \text{(the } \xi\text{-}\textbf{invariant}\text{)}.$$

We sketch the proof that the $H_F(n)$ are invariants, hence clarifying Corollary 4.1. Let $\Gamma(\lambda s+\mu)$ be one of the Γ-factors of $F(s)$. After application of the multiplication formula (3.1) to such a Γ-factor, we have to prove that

$$\frac{B_n(\mu)}{\lambda^{n-1}} = \sum_{j=0}^{m-1} \frac{B_n(\frac{\mu+j}{m})}{(\frac{\lambda}{m})^{n-1}} \qquad n \geq 0,\ m \geq 1,$$

and this follows from the following identity for Bernoulli polynomials

$$B_n(z) = m^{n-1} \sum_{j=0}^{m-1} B_n(\frac{z+j}{m}) \qquad n \geq 0,\ m \geq 1.$$

Therefore the $H_F(n)$ are stable by multiplication formula. In order to check that the $H_F(n)$ are stable by factorial formula as well, let $\Gamma(\lambda s + \mu)$ and $\Gamma(\lambda' s + \mu')$ be two Γ-factors of $F(s)$ and apply the factorial formula as in (3.4). Consequently, we have to prove that

$$\frac{B_n(\mu)}{\lambda^{n-1}} + \frac{B_n(\mu')}{\lambda'^{n-1}} = \frac{B_n(\mu-1)}{\lambda^{n-1}} + \frac{B_n(\mu'+1)}{\lambda'^{n-1}} \qquad n \geq 0,$$

and this follows from the identity

$$B_n(z+1) = B_n(z) + nz^{n-1} \qquad n \geq 0,$$

under the consistency condition (3.3). Hence the $H_F(n)$ are invariants by Corollary 4.1. Note that the H-invariants are *additive*, *i.e.* $H_{FG}(n) = H_F(n) + H_G(n)$.

We already saw in Section 2 the meaning of the degree d_F in terms of the function $F(s)$. Note that the degree of the functions in (2.1) is always an *integer*; in Section 6 we will state a fundamental conjecture about the degree, namely the *degree conjecture*. Concerning the ξ-invariant ξ_F, its real part η_F is called the **parity** of $F(s)$, while its imaginary part θ_F is the **shift**, not to be confused with the shift $F_\theta(s)$ introduced in Section 3. Observe that the shift θ_F is usually 0 for the classical L-functions ($\mu_j \in \mathbb{R}$ in many cases). Observe also that the Hecke L-functions $L_K(s,\chi)$, with χ character of infinite order, provide non-trivial examples of $\theta_F = 0$, due to the fact that χ is normalized. We refer again to Section 6 for the meaning of the invariants η_F and θ_F, at least for degree 1 functions. For general n we raise the following problem about H-invariants:

Problem 4.1. *Give a meaning in terms of $F(s)$ to every invariant $H_F(n)$, $n \geq 2$.*

In Kaczorowski-Perelli [36] an asymptotic expansion of $\log \gamma(s)$ is given that involves the $H_F(n)$. However, Problem 4.2 asks for a more explicit meaning for such invariants, possibly without explicit reference to the functional equation.

$\star$ The **conductor** q_F. We already defined in the previous section the *conductor* in the case of functions of degree 0, and we saw that it is an integer and an invariant. In the general case of $F \in \mathcal{S}^\sharp$ we define

$$q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}.$$

As before, it is easy to show that the conductor is stable by multiplication and factorial formulae, hence it is an invariant by Corollary 4.1. Moreover, it is easy to check that

$$q_\zeta = 1, \qquad q_{L(\cdot,\chi)} = q, \qquad q_{\zeta_K} = |D_K|, \qquad q_{L_K(\cdot,\chi)} = |D_K| N(\mathfrak{f}), \qquad q_{L_f} = N,$$

where q is the modulus of the primitive Dirichlet character χ, D_K is the discriminant of K, $N(\mathfrak{f})$ is the norm of the conductor $\mathfrak{f}$ of the primitive Hecke character χ and N is the level of the holomorphic modular form $f(z)$. Hence the conductor q_F appears to be the right extension to $\mathcal{S}^\sharp$ of the various classical notions of conductor. Note that the conductor is *multiplicative*, *i.e.* $q_{FG} = q_F q_G$. Note also that the above functions belong to $\mathcal{S}$, and their conductor is an *integer*. In fact, we have

Conjecture 4.3. (conductor conjecture) *Every $F \in \mathcal{S}$ has $q_F \in \mathbb{N}$.*

Probably this conjecture does not hold for $\mathcal{S}^\sharp$, and counterexamples can possibly be found among the L-functions associated with the Hecke groups $G(\lambda)$.

$\star$ The **root number** ω_F^*. The root number of $F \in \mathcal{S}^\sharp$ is defined by

$$\omega_F^* = \omega e^{-i\frac{\pi}{2}(\eta_F+1)} \Big(\frac{q_F}{(2\pi)^{d_F}}\Big)^{i\theta_F/d_F} \prod_{j=1}^{r} \lambda_j^{-2i\Im\mu_j}.$$

Once again, it is easy to show that the root number is stable by multiplication and factorial formulae, hence it is an invariant by Corollary 4.1. The root number ω_F^* comes out naturally in certain computations, and is of course closely related to ω for the classical L-functions. Here are two problems about ω_F^*.

Problem 4.2. *What is the meaning of ω_F^* ? Is ω_F^* the correct definition of "root number" ?*

Problem 4.3. *Is it true that ω_F^* is always an algebraic number for $F \in \mathcal{S}$?*

Problem 4.2 is related with the definition of ω_F below. Moreover, Problem 4.3 has a negative answer in the case of $\mathcal{S}^\sharp$, as we will see in Section 6.

A set $\{I_j\}_{j\in J}$ of *numerical* invariants is called a set of **basic invariants** if the I_j characterize the functional equation of *every* $F \in \mathcal{S}^\sharp$, in the sense that if $I_j(F) = I_j(G)$ for all $j \in J$ then $F(s)$ and $G(s)$ satisfy the same functional equation, for any $F, G \in \mathcal{S}^\sharp$. In principle, such a set should be called a **global** set of basic invariants, since we will also deal with **local** sets of basic invariants, characterizing the functional equation of a *given* function $F \in \mathcal{S}^\sharp$.

Theorem 4.3. ([36]) *The H-invariants $H_F(n)$, $n \geq 0$, the conductor q_F and the root number ω_F^* are a global set of basic invariants.*

The proof is based on the fact that the function

$$K_F(z) = 2z \sum_{j=1}^{r} \frac{e^{z\mu_j/\lambda_j}}{e^{z/\lambda_j} - 1} = -2z \sum_{\rho} e^{\rho z}, \tag{3.5}$$

where the last sum is over the poles of a γ-factor of $F \in \mathcal{S}^\sharp$, has the power series expansion

$$K_F(z) = \sum_{n=0}^{\infty} \frac{H_F(n)}{n!} z^n,$$

hence the γ-factors of $F(s)$ and $G(s)$ differ by a factor e^{as+b} if the H-invariants are equal. Assuming further that conductors and root numbers are equal, it is not difficult to show that $F(s)$ and $G(s)$ satisfy the same functional equation.

Clearly, if we drop the condition that basic invariants are numerical invariants, then finite global sets of basic invariants are easily detected, for instance $\{K_F(z), q_F, \omega_F^*\}$. However, Jurek Kaczorowski and Giuseppe Molteni pointed out that there exist global sets of basic invariants with cardinality 1. The argument is, roughly speaking, as follows. The set of the functional equations of axiom *(iii)* (modulo the "equivalent" functional equations in the sense of Theorem 4.2) has the cardinality of the continuum, and hence there exists an injective mapping ϕ from such functional equations to $\mathbb{R}$. Given $F \in \mathcal{S}^\sharp$, define the numerical invariant I_F as the value of the mapping ϕ at the functional equation satisfied by $F(s)$. Clearly, such an invariant forms a global set of basic invariants. Of course, the invariants coming from this argument are not explicit, but more explicit versions can be obtained by refining the argument. However, such invariants are quite artificial, while the invariants in Theorem 4.3 are definitely more interesting.

Another problem related with invariants is determining an **invariant form** of the functional equation, where all data are invariants. Clearly, such an invariant form provides in particular a local set of basic invariants. We deal with this problem by constructing a special (essentially) invariant form of the functional equation, which we call the *canonical functional equation*. The motivation comes from the fact that the λ-coefficients in the standard functional equation of the classical L-functions are all equal to $\frac{1}{2}$ (or easily transformed to $\frac{1}{2}$). Roughly speaking, the canonical functional equation plays this role in the general case of $\mathcal{S}^\sharp$.

To construct the canonical functional equation, we split the function $K_F(z)$ in (3.5) into $\mathbb{Q}$-equivalence classes as

$$K_F(z) = \sum_{j=1}^{h_F} K_j(z)$$

and define the **canonical exponents** Λ_j by

$$\Lambda_j = \max\{\Lambda \in \mathbb{R} : (e^{z/\Lambda} - 1)K_j(z) \text{ is entire}\}.$$

The canonical exponents exist, are positive and distinct, and are invariants, see Kaczorowski-Perelli [36]. Moreover, every $F \in \mathcal{S}^\sharp$ has a **balanced** γ-factor, *i.e.* of the form

$$\gamma(s) = Q^s \prod_{j=1}^{h_F} \prod_{k} \Gamma(\lambda_j s + \mu_{j,k})$$

with all ratios Λ_j/λ_j equal. Such ratios are positive integers, and their minimum over all balanced γ-factors of $F(s)$ is called the **reduction index** l_F, clearly an invariant; see [36].

Given positive integers K_j $(j = 1, ..., h_F)$ and complex numbers $\mu_{j,k}$ with $\Re\mu_{j.k} \geq 0$ $(j = 1, ..., h_F,\ k = 1, ..., l_F K_j)$ to be specified later, we write

$$\begin{aligned} Q_F &= \Big(q_F(2\pi)^{-d_F} l_F^{d_F} \prod_{j=1}^{h_F} \Lambda_j^{-2K_j\Lambda_j}\Big)^{1/2} \\ \omega_F &= \omega_F^* e^{i\frac{\pi}{2}(\eta_F+1)}\Big(\frac{q_F}{(2\pi)^{d_F}}\Big)^{-i\theta_F/d_F} l_F^{-i\theta_F} \prod_{j=1}^{h_F}\prod_{k=1}^{l_F K_j} \Lambda_j^{2i\Im\mu_{j,k}} \\ \gamma_F(s) &= Q_F^s \prod_{j=1}^{h_F}\prod_{k=1}^{l_F K_j} \Gamma(\frac{\Lambda_j}{l_F}s + \mu_{j,k}). \end{aligned} \tag{3.6}$$

Theorem 4.4. ([36]) *Every $F \in \mathcal{S}^\sharp$ uniquely determines positive integers K_j such that*

$$\gamma_F(s)F(s) = \omega_F \overline{\gamma_F}(1-s)\overline{F}(1-s), \tag{3.7}$$

where $\gamma_F(s)$ and ω_F are given by (3.6) *and the $\mu_{j,k}$'s are uniquely determined* (mod $\mathbb{Z}$) by $F(s)$. *Moreover, the $\mu_{j,k}$'s are uniquely determined by $F(s)$ if $h_F = 1$ or if $F(s)$ is reduced, and $l_F = 1$ in the latter case.*

The functional equation in (3.7) is called the **canonical functional equation**, and in view of Conjecture 4.1 we expect that (3.7) is in invariant form. The non-uniqueness of the $\mu_{j,k}$ when $h_F > 1$ comes from possible applications of the factorial formula to Γ-factors belonging to different $\mathbb{Q}$-equivalence classes. The proof of Theorem 4.4 is quite technical; we refer to Kaczorowski-Perelli [36] for the proof and for an algorithm for the computation of the canonical functional equation from a given one.

Assuming that $h_F = 1$, a γ-factor is balanced if and only if all its λ-coefficients are equal, hence by the definition of l_F we have that *the canonical functional equation has the minimum number of* Γ*-factors among the γ-factors with all λ-coefficients equal.* This clarifies somewhat the meaning of the Λ_j and of l_F in the case of balanced γ-factors: the Λ_j are the "largest possible" λ-coefficients and l_F somehow measures the "reduction" of γ-factors, attainig its minimum ($l_F = 1$) in the reduced case.

The standard functional equation of $\zeta(s)$ and $L(s,\chi)$, χ primitive Dirichlet character, coincides with the canonical one. This holds for the L-functions $L_f(s)$ as well. The canonical functional equation of $\zeta_K(s)$ is obtained from the standard one by applying the Legendre duplication formula to the Γ-factors with λ-coefficient equal to 1, in those cases where both $\frac{1}{2}$ and 1 are present as λ-coefficients. Note that all the classical L-functions have a balanced γ-factor with λ-coefficient equal to $\frac{1}{2}$ or 1. A related problem is

Problem 4.4. *Is it true that the canonical functional equation of the classical L-functions has λ-coefficient always equal to $\frac{1}{2}$ or* 1 ?

In other words: is it true that all the balanced γ-factors of the classical L-functions have λ-coefficient not larger than 1 ?

Coming to the local sets of basic invariants, with the notation in Theorem 4.4 let

$$r_F = l_F \sum_{j=1}^{h_F} K_j, \qquad g_F = 2^{2h_F-1} r_F - 2h_F + 1.$$

Using the function $K_F(z)$ in (3.5) and the canonical exponents, by Theorems 4.3 and 4.4 we get

Theorem 4.5. ([36]) *(i) A local set of basic invariants of $F \in \mathcal{S}^\sharp$ is provided by h_F, r_F*

$$q_F, \qquad \omega_F^* \quad \text{and the} \quad H_F(n) \quad \text{with} \quad n \leq g_F. \tag{3.8}$$

(ii) Assuming the $\boldsymbol{\lambda}$-conjecture, a local set of basic invariants of $F \in \mathcal{S}^\sharp$ is provided by the invariants in (3.8) *with g_F replaced by d_F.*

As a consequence, we expect that q_F, ω_F^* *and the H-invariants with $n \leq d_F$ characterize the functional equation of $F \in \mathcal{S}^\sharp$*. In Section 6 we will see that this is in fact the case for the degree 1 functions. We remark that *(ii)* of Theorem 4.5 is best possible, in the sense that for every integer $d \geq 1$ there exist $F, G \in \mathcal{S}_d^\sharp$ for which the invariants in (3.8) with g_F replaced by $d-1$ are equal, but $F(s)$ and $G(s)$ satisfy different functional equations. Examples are provided by suitable products of shifted Dirichlet L-functions, see Kaczorowski-Perelli [36].

A fundamental problem in the theory of the Selberg class is describing the admissible values of the numerical invariants, that is the set of values that numerical invariants attain at the functions of $\mathcal{S}$ and $\mathcal{S}^\sharp$:

Problem 4.5. *Given a numerical invariant $I : \mathcal{S}^\sharp \to \mathbb{C}$, describe $I(\mathcal{S})$ and $I(\mathcal{S}^\sharp)$.*

For some invariants there are good conjectures about admissible values, see for example Conjectures 4.1 and 4.3, Problem 4.3 and the degree conjecture in Section 6.

We end this section by a first measure theoretic result on Problem 4.5; more precise results of this type will be obtained in Section 6. We denote by $\mathbb{R}^+$ and by $\mathbb{C}^+$ the positive real numbers and the complex numbers with non-negative real part, and by T^1 the unit circle. A numerical invariant I is called a **continuous invariant** if for every $r \geq 0$ there exits a continuous function

$$f_r : \mathbb{R}^+ \times (\mathbb{R}^+ \times \mathbb{C}^+)^r \times T^1 \to \mathbb{C}$$

such that $I(F) = f_r(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$, where $(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$ are the data of $F \in \mathcal{S}^\sharp$ (remember that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are r-dimensional vectors). Examples of continuous numerical invariants are the H-invariants $H_F(n)$, the conductor q_F and the root number ω_F^*.

Theorem 4.6. ([42]) *Let I be a continuous invariant. Then the sets $\Re I(\mathcal{S})$, $\Im I(\mathcal{S})$, $\Re I(\mathcal{S}^\sharp)$ and $\Im I(\mathcal{S}^\sharp)$ are Lebesgue measurable.*

Roughly speaking, the proof of Theorem 4.6 is based on the fact that for a given continuous invariant I, the extended Selberg class $\mathcal{S}^\sharp$ can be endowed with a suitable metric, thus becoming a metric space with good properties.

4. Linear twists and structure theorems

The main tool for the results of Section 6 on the classification of the functions with degree $1 \leq d < 2$ are the **linear twists**

$$F(s, \alpha) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s} e(-n\alpha),$$

where $F \in \mathcal{S}^\sharp$, $\alpha \in \mathbb{R}$ and $e(x) = e^{2\pi i x}$. More precisely, the results of Section 6 require certain analytic properties of the linear twists. In order to get a first impression of the relevance of the linear twists, let us consider $F(s) = L(s, \chi)$ with a primitive Dirichlet character $\chi \pmod q$, let $\tau(\chi)$ be its associated Gauss sum and let $0 \le \alpha < 1$. By orthogonality we have

$$F(s,\alpha) = \frac{1}{\tau(\overline{\chi})} \sum_{(a,q)=1} \overline{\chi}(a)\zeta(s, \frac{a}{q} - \alpha), \tag{4.1}$$

where

$$\zeta(s,\lambda) = \sum_{n=1}^{\infty} e(n\lambda)n^{-s}$$

is the *Lerch zeta function*. It is well known that $\zeta(s,\lambda)$ has a simple pole at $s = 1$ if $\lambda \in \mathbb{Z}$, otherwise it is an entire function. Therefore, $F(s,\alpha)$ has a pole at $s = 1$ if and only if $\alpha = a/q$ with $(a,q) = 1$. Thus, for example, information on the modulus q of the character χ can be obtained from the polar structure of the linear twists of $L(s,\chi)$.

In order to study the analytic properties of the linear twists, Kaczorowski-Perelli [33], [37] start with

$$F_N(s,\alpha) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s} e(-n\alpha)e^{-n/N},$$

where $N > 0$, which is absolutely convergent over $\mathbb{C}$ and has the integral expression

$$F_N(s,\alpha) = \frac{1}{2\pi i} \int_{(2)} F(s+w)\Gamma(w)z_N^{-w}dw,$$

where $z_N = \frac{1}{N} + 2\pi i\alpha$ and the integration is on the line from $2 - i\infty$ to $2 + i\infty$. Shifting the line of integration to $\sigma = -K - \frac{1}{2}$, where K is a suitably large positive integer, and using the functional equation of $F(s)$ we obtain

$$F_N(s,\alpha) = R_N(s,\alpha) + \omega Q^{1-2s} \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{n^{1-s}} H_K(\frac{n}{Q^2 z_N}, s), \tag{4.2}$$

where $R_N(s,\alpha)$ is a term arising from the residues, and the functions

$$H_K(z,s) = \frac{1}{2\pi i} \int_{(-K-\frac{1}{2})} \prod_{j=1}^{r} \frac{\Gamma(\lambda_j(1-s) + \overline{\mu_j} - \lambda_j w)}{\Gamma(\lambda_j s + \mu_j + \lambda_j w)} \Gamma(w)z^w dw \tag{4.3}$$

are rather general cases of **Fox hypergeometric functions**. Since we will eventually let $N \to \infty$, we require some information on $H_K(-iy, s)$, especially when $y = \frac{n}{2\pi Q^2 \alpha}$.

An instance of such an approach to the study of the linear twists, in the case $F(s) = \zeta(s)$, can be found in Linnik [43]. In fact, starting from (4.1) with $\alpha = 0$, Linnik [43] obtained a new proof of the functional equation of the Dirichlet L-functions using the functional equation of $\zeta(s)$ in a direct way. In Linnik's special case, the hypergeometric functions (4.3) reduce to simple well known functions, and hence the right hand side of (4.2) becomes rather explicit. This is, unfortunately, not the case in more general situations.

For s *fixed*, the general Fox hypergeometric functions have been studied by Braaksma [7]. Roughly speaking, their behaviour depends on the value of the *main parameter* μ defined by

$$\mu = 2\sum_{j=1}^{r} \lambda_j - 1 = d_F - 1.$$

In the case $\mu = 0$, which corresponds to the degree 1 functions, the behaviour is simpler since only the "algebraic part" comes into play, while for $\mu > 0$ (*i.e.* $d_F > 1$) the behaviour is more complicated due to the presence of the "exponential part"; we refer to Braaksma [7] for the meaning of the algebraic and exponential parts. We remark that the case $\mu < 0$, although not directly related to the linear twists (see Theorem 3.1), has also some interest, and that in this case the situation is simpler thanks to nice convergence properties.

In order to study the linear twists, one has to develop a (z,s)-*variables* theory of the hypergeometric functions (4.3). We present here only the result for $d_F = 1$, where a clean statement can be given. Let

$$\beta = \prod_{j=1}^{r} \lambda_j^{2\lambda_j},$$

θ_F be the *shift* of $F \in \mathcal{S}^\sharp$ (see Section 4) and, given $R > 1$, let $K = K(R)$ be a suitably large positive integer.

Theorem 5.1. ([33], [40]) *Let $d_F = 1$, $y > 0$ and $\sigma < R$. If $y \neq \beta$ then $H_K(-iy, s)$ is holomorphic, while $H_K(-i\beta, s)$ has at most simple poles at the points $s_k = 1 - k - i\theta_F$ for $k = 0, 1, \ldots,$ with non-vanishing residue at $s = s_0$.*

In addition, suitable bounds for $H_K(-iy, s)$ as $y \to \infty$, required in (4.2), can be obtained; see Kaczorowski-Perelli [33], [40]. Moreover, it is in principle possible to check the vanishing or non-vanishing of the residue at each point s_k with $k \geq 1$, but there are non-trivial complications in details. We will say more on this later on, see Problem 5.3.

We refer to Kaczorowski-Perelli [37] and Kaczorowski [29] for the analytic properties of the hypergeometric functions when $1 < d_F < 2$, since in this case the statement is more involved due to the appearance of the above mentioned exponential part. We remark here that such an exponential part is reflected by the exponential factor in the Dirichlet series $D_F(s, \alpha)$ in Theorem 5.3 below.

The analytic properties of the linear twists of the functions $F \in \mathcal{S}_1^\sharp$ follow now from (4.2) and Theorem 5.1. Let $\alpha > 0$ and, in view of (4.2) and Theorem 5.1, define the **critical value** n_α (of course arising from the equation $\frac{n}{2\pi Q^2 \alpha} = \beta$) by

$$n_\alpha = q_F \alpha,$$

where q_F is the *conductor* of $F(s)$ defined in Section 4. Moreover, define $a_F(n_\alpha) = 0$ if $n_\alpha \notin \mathbb{N}$. We have

Theorem 5.2. ([33], [40]) *Let $F \in \mathcal{S}_1^\sharp$ and $\alpha > 0$. Then $F(s, \alpha)$ is entire if $a_F(n_\alpha) = 0$, while if $a_F(n_\alpha) \neq 0$ then $F(s, \alpha)$ has at most simple poles at the points $s_k = 1 - k - i\theta_F$ for $k = 0, 1, \ldots,$ with residue at s_0 equal to $c(F)\frac{\overline{a_F(n_\alpha)}}{n_\alpha^{i\theta_F}}$ and $c(F) \neq 0$.*

Clearly, the vanishing or non-vanishing of the residue at the points s_k with $k \geq 1$ is closely related to the analogous problem in Theorem 5.1.

In order to state the properties of the linear twists of functions $F \in \mathcal{S}_d^\sharp$ with $1 < d < 2$ we need a few more definitions. Let

$$\kappa = \frac{1}{d_F - 1}, \qquad A = (d_F - 1)q_F^{-\kappa}, \qquad s^* = \kappa\left(s + \frac{d_F}{2} - 1 + i\theta_F\right)$$

and

$$D_F(s, \alpha) = \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{n^s} e\Big(A\big(\frac{n}{\alpha}\big)^\kappa\Big).$$

Moreover, let σ^* denote the real part of s^* and $\sigma_a(F)$ be the abscissa of absolute convergence of $F(s)$. We have

Theorem 5.3. ([37]) *Let $1 < d < 2$, $F \in \mathcal{S}_d^\sharp$, $\alpha > 0$ and $J \geq 1$ be an integer. Then there exist a constant $c_0 \neq 0$ and polynomials $P_j(s)$, with $0 \leq j \leq J-1$ and $P_0(s) = c_0$ identically, such that for $\sigma^* > \sigma_a(F)$*

$$F(s,\alpha) = q_F^{\kappa s} \sum_{j=0}^{J-1} \alpha^{\kappa(d_F s - d_F/2 + i\theta_F + j)} P_j(s) D_F(s^* + j\kappa, \alpha) + G_J(s,\alpha) \tag{4.4}$$

where $G_J(s,\alpha)$ is holomorphic for s in the half-plane $\sigma^* > \sigma_a(F) - \kappa J$ and continuous for $\alpha > 0$.

Note that $\sigma^* > \sigma$ for $\sigma > \frac{1}{2}$ and $1 < d_F < 2$, hence (4.4) shows a kind of *overconvergence phenomenon* for $F(s,\alpha)$, which will be exploited in Section 6. Observe also that, contrary to Theorem 5.2 where a clean description of the analytic properties of the linear twists of degree 1 functions is given, in this case the properties of the linear twists are related to those of certain *non-linear* twists. However, due to the above overconvergence phenomenon, Theorem 5.3 provides a non-trivial continuation of $F(s,\alpha)$ to a strip to the left of $\sigma = 1$. This will be important in Section 6, where Theorems 5.2 and 5.3 will be applied to obtain a complete classification of the functions $F \in \mathcal{S}_d^\sharp$ with $1 \leq d < 5/3$.

The relation in (4.4) between the linear twists and suitable non-linear twists is just a special case of a more general theory, based on the properties of the Fox hypergeometric functions as in Kaczorowski-Perelli [37], where a *general non-linear twist* of $F \in \mathcal{S}^\sharp$ is related to its *conjugate non-linear twist*. We do not enter such a general theory in this survey.

We remark here that by axiom *(i)* we have $\sigma_a(F) \leq 1$ for every $F \in \mathcal{S}^\sharp$, but the exact value of $\sigma_a(F)$ is not known in general. Assuming the Selberg orthonormality conjecture, in Section 3 we saw that $\sigma_a(F) = 1$ for every $F \in \mathcal{S} \setminus \{1\}$. In the general case we raise the following

Problem 5.1. *Is it true that $\sigma_a(F) = 1$ for every $F \in \mathcal{S}_d^\sharp$ with $d > 0$?*

At the beginning of Section 2 we raised two questions. We gave an answer to the first question but not yet to the second one, asking if *all* L-functions are already known. An answer to such question is given by the following impressive conjecture.

Conjecture 6.1. (main conjecture) *The Selberg class $\mathcal{S}$ coincides with the class of the* GL(n) *automorphic L-functions.*

Conjecture 6.1, if true, lies very deep. In fact, on the one hand it *morally* implies the truth of the Langlands program, since the L-functions of arithmetic, algebraic and geometric nature (morally in $\mathcal{S}$) would become special cases of automorphic L-functions. On the other hand, if one accepts that $\mathcal{S}$ is the class of all L-functions, then Conjecture 6.1 implies that *all* L-functions are already known. Moreover, Conjecture 6.1 immediately implies *almost all* the other conjectures in this survey, since most such conjectures are known in the case of the automorphic L-functions (but *not all*, for example GRH).

Since the automorphic L-functions have integer degree, we can split Conjecture 6.1 into two parts as follows.

Conjecture 6.2. (general converse theorem) *For $d \in \mathbb{N}$*

$$\mathcal{S}_d = \{\text{automorphic } L\text{-functions of degree } d\}.$$

Conjecture 6.3. (degree conjecture) *For $d \notin \mathbb{N}$*

$$\mathcal{S}_d = \emptyset.$$

Since the standard γ-factors of the automorphic L-functions have all the λ-coefficients equal to $\frac{1}{2}$, it is clear that Conjectures 6.2 and 6.3 imply and motivate Conjectures 4.1 and 4.2 (restricted to $\mathcal{S}$).

Although we are mainly concerned with the Selberg class $\mathcal{S}$, it is interesting to raise similar problems for $\mathcal{S}^\sharp$ as well.

Problem 6.1. *What does $\mathcal{S}_d^\sharp$ contain for $d \in \mathbb{N}$?*

We remark here that $\mathcal{S}^\sharp$ is a model, introduced in Kaczorowski-Perelli [33], for the class of the "L-functions without Euler product". However, there are well known classes of L-functions which in general do not fall into $\mathcal{S}^\sharp$, see for instance the vector spaces of L-functions associated with holomorphic modular forms (where the functional equation has no conjugation on the right hand side). Possibly, a better definition of $\mathcal{S}^\sharp$ is obtained by allowing a slightly more general type of functional equation, for example relating $F(s)$ to $G(1-s)$ instead of $\overline{F}(1-s)$, where $G(s)$ satisfies the same properties of $F(s)$. At any rate, we do not expect substantial differences between the properties of such a class and $\mathcal{S}^\sharp$.

Coming back to the description of $\mathcal{S}^\sharp$, we expect that the degree conjecture holds for $\mathcal{S}^\sharp$ as well.

Conjecture 6.4. (strong degree conjecture) $\mathcal{S}_d^\sharp = \emptyset$ *for* $d \notin \mathbb{N}$.

Let $V^\sharp(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$ denote the real vector space of the functions of $\mathcal{S}^\sharp \cup \{0\}$ satisfying a given functional equation. A much weaker version of Conjecture 6.4 is

Problem 6.2. *Prove that* $\dim V^\sharp(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega) < \infty$ *if* $d \notin \mathbb{N}$.

Probably, $\dim V^\sharp(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega) = \infty$ for certain degree 2 functional equations (see Chapter II of Hecke [22]), thus the condition $d \notin \mathbb{N}$ in Problem 6.2 appears to be crucial.

Somehow, the degree conjecture reflects the *arithmetical nature* of the Selberg class. Although $\mathcal{S}^\sharp$ is obtained dropping the two arithmetical axioms of $\mathcal{S}$, Conjecture 6.4 suggests that $\mathcal{S}^\sharp$ still has some arithmetical content. Indeed, by axiom *(i)* every $F \in \mathcal{S}^\sharp$ is an *ordinary* Dirichlet series, *i.e.* the "frequences" are *integers*. We may therefore ask if the analog of the degree conjecture fails once axiom *(i)* is weakened to allow *general* Dirichlet series (see Section 2 for the definition). It turns out that this is essentially the case, as the following simple result shows. For *any* choice of $(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$ as in axiom *(iii)*, let $\mathcal{D}(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$ denote the real vector space of the *somewhere* absolutely convergent general Dirichlet series satisfying axioms *(ii)* and *(iii)*. We have

Theorem 6.1. ([41]) $\mathcal{D}(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$ *has an uncountable basis.*

Therefore, the degree conjecture definitely fails in this case. The proof is based on Hecke's theory of modular forms associated with the groups $G(\lambda)$, see Chapter II of Hecke [22], which provides examples of suitable L-functions with arbitrary non-negative μ-coefficient. Theorem 6.1 is slightly unsatisfactory due to the "somewhere absolutely convergent" general Dirichlet series in the definition of $\mathcal{D}(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$. Therefore we raise

Problem 6.3. *Is the analog of Theorem* 6.1 *true with "somewhere absolutely convergent" replaced by "absolutely convergent for $\sigma > 1$" in the definition of* $\mathcal{D}(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$?

In Section 3 we presented the results for degree $0 \leq d < 1$ (Theorems 3.1, 3.2 and 3.3), which confirm Conjectures 6.2 and 6.3 in that range. We remark here that a very simple proof of Theorem 3.1 can be obtained as a corollary of Theorem 5.4, see Kaczorowski-Perelli [40]. In fact, let $0 < d < 1$, $F \in \mathcal{S}_d^\sharp$, m with $a_F(m) \neq 0$ and let α be such that $n_\alpha = m$. Hence the non-linear twist $F_d(s,\alpha)$ has a pole on the line $\sigma = \frac{d+1}{2d} > 1$, a contradiction.

Now we turn to the next case, *i.e.* the classification of the L-functions of $\mathcal{S}_1$. By Conjecture 6.2 one expects that these are the Dirichlet L-functions with primitive characters and their shifts (see Section 3), and this is in fact the case as Theorem 6.3 below shows. We start with a complete description of the functions $F \in \mathcal{S}_1^\sharp$. For a positive integer q and complex numbers $\xi = \eta + i\theta$ and ω^* with $|\omega^*| = 1$, we denote by $\mathcal{S}_1^\sharp(q,\xi,\omega^*)$ the set of $F \in \mathcal{S}_1^\sharp$ such that (see Section 4)

$$q_F = q, \qquad \xi_F = \xi, \qquad \omega_F^* = \omega^*.$$

Since q_F, $\xi_F = \eta_F + i\theta_F$ and ω_F^* are invariants, $\mathcal{S}_1^\sharp$ is a disjoint union of these classes. Moreover, we write

$$V_1^\sharp(q,\xi,\omega^*) = \mathcal{S}_1^\sharp(q,\xi,\omega^*) \cup \{0\}.$$

If χ is a Dirichlet character we denote by f_χ its conductor and by χ^* the primitive character inducing χ. We also denote by ω_{χ^*} the ω-factor in the standard functional equation of $L(s,\chi^*)$ and, for $\eta \in \{-1, 0\}$, we write

$$\mathfrak{X}(q,\xi) = \begin{cases} \{\chi \pmod{q} \text{ with } \chi(-1) = 1\} & \text{if } \eta = -1 \\ \{\chi \pmod{q} \text{ with } \chi(-1) = -1\} & \text{if } \eta = 0. \end{cases}$$

Further, χ_0 denotes the principal character (mod q).

Theorem 6.2. ([33]) *i) If $F \in \mathcal{S}_1^\sharp$, then $q_F \in \mathbb{N}$, the sequence $a_F(n)n^{i\theta_F}$ is periodic of period q_F and $\eta_F \in \{-1, 0\}$.*

ii) Every $F \in \mathcal{S}_1^\sharp(q,\xi,\omega^*)$, with $q \in \mathbb{N}$, $\eta \in \{-1,0\}$ and $|\omega^*| = 1$, can be uniquely written as

$$F(s) = \sum_{\chi \in \mathfrak{X}(q,\xi)} P_\chi(s+i\theta) L(s+i\theta,\chi^*)$$

where $P_\chi \in \mathcal{S}_0^\sharp(\frac{q}{f_{\chi^*}}, \omega^*\overline{\omega_{\chi^*}})$. Moreover, $P_{\chi_0}(1) = 0$ if $\theta \neq 0$.

iii) For q, ξ and ω^* as above, $V_1^\sharp(q,\xi,\omega^*)$ is a real vector space with

$$\dim_{\mathbb{R}} V_1^\sharp(q,\xi,\omega^*) = \begin{cases} [\frac{q}{2}] + 1 & \textit{if } \xi = -1 \\ [\frac{q-1-\eta}{2}] & \textit{otherwise.} \end{cases}$$

Note that by Theorem 6.2 the functions in $\mathcal{S}_1^\sharp$ satisfy the Ramanujan conjecture. Adding the Euler product axiom, from Theorem 6.2 we obtain

Theorem 6.3. ([33]) *Let $F \in \mathcal{S}_1$. If $q_F = 1$, then $F(s) = \zeta(s)$. If $q_F \geq 2$, then there exists a primitive Dirichlet character χ (mod q_F) with $\chi(-1) = -(2\eta_F + 1)$ such that $F(s) = L(s + i\theta_F, \chi)$.*

We give a sketch of the first steps of the proof of Theorem 6.2, thus showing the relevance of the polar structure of $F(s,\alpha)$ in Theorem 5.2. Choose m with $a_F(m) \neq 0$ and let $\alpha = \frac{m}{q_F}$. Then the linear twist $F(s,\alpha)$ has a simple pole at $s = 1 - i\theta_F$, and hence the same holds for $F(s,\alpha+1)$ by the α-periodicity of linear twists. Therefore $n_{\alpha+1} = q_F(\frac{m}{q_F} + 1) \in \mathbb{N}$, thus $q_F \in \mathbb{N}$. Similarly, to show the periodicity of the coefficients we choose $\alpha = \frac{n}{q_F}$. Then $F(s,\alpha)$ has residue equal to $c(F)\overline{\frac{a_F(n)}{n^{i\theta_F}}}$ at $s = 1 - i\theta_F$, and hence $F(s,\alpha+1)$ has residue $c(F)\overline{\frac{a_F(n+q_F)}{(n+q_F)^{i\theta_F}}}$ at the same point, and the periodicity follows. Once the periodicity of the coefficients is established, Dirichlet characters enter the game, and $F(s)$ is expressed as a linear combination of Dirichlet

L-functions over the Dirichlet polynomials of $\mathcal{S}_0^\sharp$. The full description of the functions in $\mathcal{S}_1^\sharp$ follows then from a careful analysis of the functional equations satisfied by $F(s)$ and by the involved Dirichlet L-functions.

We refer to Soundararajan [71] for a different proof of Theorem 6.3. For previous partial results towards Theorems 6.2 and 6.3, and for related results, we refer to Bochner [1], Vignéras [77], Gérardin-Li [20], Conrey-Ghosh [12] and Funakura [18], and to the literature quoted there.

We remark that Theorem 6.2 confirms *(ii)* of Theorem 4.5 in the case of degree 1 functions: the triplet (q_F, ω_F^*, ξ_F) determines the functional equation of $F \in \mathcal{S}_1^\sharp$. Moreover, Theorem 6.3 clarifies the name and the meaning of the invariant η_F.

A simple consequence of Theorem 6.3 is

Corollary 6.1. ([52]) *The nomalized L-functions $L_f(s)$ associated with holomorphic newforms $f(z)$ on congruence subgroups of $SL(2,\mathbb{Z})$ are primitive.*

This is proved by contradiction, assuming that $L_f(s) = L(s+i\theta_1,\chi_1)L(s+i\theta_2,\chi_2)$ with χ_j primitive Dirichlet characters and $\theta_j \in \mathbb{R}$. Taking the Rankin-Selberg convolution of both sides (or twisting by a suitable character), the order of pole at $s = 1$ leads to a contradiction.

We remark that the classical converse theorem of Weil [79] characterizes the GL(2) L-functions by means of their twists by Dirichlet characters. Roughly speaking, a similar philosophy applies in general to the GL(n) converse theorems, see Cogdell and Piatetski-Shapiro [10]. In fact, the GL(n) L-functions are characterized in terms of suitable Rankin-Selberg convolutions, and the twists are of course special instances of such convolutions. Since Theorems 6.2 and 6.3 can be regarded as general converse theorems for degree 1 functions, it is clear that the use of the linear twists in their proofs fits well into the above philosophy. Note that the axioms of $\mathcal{S}$ do not explicitly include any property of the Rankin-Selberg convolutions; this increases the difficulties in proving general converse theorems in $\mathcal{S}$. We refer to Conrey-Farmer [11] for an interesting alternative to Weil's converse theorem, where the twists are replaced by the Euler product, although at present this approach produces much less general results.

Another instance showing the relevance of the twists in the problems of this section is the following. Given $F \in \mathcal{S}^\sharp$ and a Dirichlet character χ, define the **twist** of $F(s)$ as

$$F^\chi(s) = \sum_{n=1}^\infty \frac{a_F(n)\chi(n)}{n^s}.$$

The following two natural conjectures about twists are given in Kaczorowski-Perelli [35]; see also Selberg [69] for other conjectures on twists.

Conjecture 6.5. (twist conjecture) *Let $F \in \mathcal{S}$ with $q_F \in \mathbb{N}$, $m \in \mathbb{N}$ with $(m, q_F) = 1$ and let χ (mod m) be a primitive Dirichlet character. Then $F^\chi \in \mathcal{S}$.*

In addition, in [35] it is also conjectured that $F^\chi(s)$ *is primitive if and only if* $F(s)$ *is primitive.*

Conjecture 6.6. (twisted conductor conjecture) *Assume the twist conjecture. Then*

$$q_{F^\chi} = q_F m^{d_F}.$$

It is easy to see that *the conductor conjecture* (see Section 4) *and Conjectures* 6.5 *and* 6.6 *imply the degree conjecture*, since $q_F m^{d_F} \in \mathbb{N}$ for all $(m, q_F) = 1$ implies that $d_F \in \mathbb{N}$. Although the twists (by Dirichlet characters) and the linear twists are closely related, a problem of some interest is to give a proof of Theorem 6.3 closer to the spirit of Weil's converse theorem.

Problem 6.4. *Give a proof of Theorem* 6.3 *using the twists (by Dirichlet characters) instead of the linear twists.*

Turning to the range $1 < d < 2$ we have

Theorem 6.4. ([37]) $\mathcal{S}_d^\sharp = \emptyset$ *for* $1 < d < 5/3$. *Moreover, for* $1 < d < 2$ *there exist no* $F \in \mathcal{S}_d^\sharp$ *with a pole at* $s = 1$.

The second part of Theorem 6.4 follows immediately from Theorem 5.3 with $J = 1$. Indeed, assuming that $F(s)$ has a pole at $s = 1$, choosing $\alpha = 1$ and exploiting the α-periodicity of the linear twists, thanks to the overconvergence phenomenon the right hand side of (4.4) is holomorphic at $s = 1$, a contradiction. The holomorphic case is definitely more involved. By a Fourier transform argument, equation (4.4) is transformed into an identity of type

$$\Sigma_1(x) = x^{-\kappa/2}\Sigma_2(x) + O(x^{\sigma_a(F)-\kappa+\varepsilon}), \tag{4.5}$$

where $\Sigma_1(x)$ and $\Sigma_2(x)$ are certain exponential sums. Moreover, $\Sigma_1(x)$ is concentrated at the integers, while the same phenomenon is not visible in $\Sigma_2(x)$. This fact is exploited by computing the L^2-norm, weighted by the function $e(x)$, of both sides of (4.5). The weight is clearly irrelevant on the left hand side, but produces some cancellation on the right hand side when $1 < d < 5/3$, thus getting a contradiction in that range.

We remark that in order to avoid the use of the Ramanujan conjecture (axiom *(iv)*), in the proof of Theorem 6.4, the following lemma of some independent interest concerning a form of *Rankin-Selberg convolution* in $\mathcal{S}^\sharp$ is used. For $F \in \mathcal{S}^\sharp$ and $\sigma > 2\sigma_a(F)$ define

$$F \times \overline{F}(s) = \sum_{n=1}^{\infty} |a_F(n)|^2 n^{-s}. \tag{4.6}$$

Lemma 6.1. ([37]) *Let* $1 < d < 2$ *and* $F \in \mathcal{S}_d^\sharp$. *Then* $F \times \overline{F}(s)$ *is holomorphic for* $\sigma > \sigma_a(F) - \kappa$ *apart from a simple pole at* $= 1$.

Note that the simple pole at $s = 1$ of $F \times \overline{F}(s)$ is in agreement with the fact that all $F \in \mathcal{S}_d$ with $1 < d < 2$ (if any !) are primitive, and the Rankin-Selberg convolution (4.6) of a primitive function is expected to have simple pole at $s = 1$ (in agreement with the Selberg orthonormality conjecture).

J.Kaczorowski observed that the same range $1 < d < 5/3$ in Theorem 6.4 would follow from a direct application of the well known conjecture that $(\varepsilon, \frac{1}{2} + \varepsilon)$ is an exponent pair (see Chapter 3 of Montgomery [49]) to a certain exponential sum closely linked to the exponential sums in (4.5). Therefore, Theorem 6.4 appears to be the limit of the method in that respect. However, the whole range $1 < d < 2$ would follow from a natural multidimensional analog of such a conjecture. In fact, using a more refined Fourier transform argument giving an error $O(x^{\sigma_a(F)-\frac{3}{2}\kappa+\varepsilon})$ in (4.5), by a k-fold iteration of (4.5) one can get an expression for the coefficients $a_F(n)$ involving a certain k-dimensional exponential sum. The full range $1 < d < 2$ would then follow from a square-root cancellation bound for such a sum. We refer to Kaczorowski-Perelli [37] for further remarks on this subject.

From Theorem 6.4 we easily obtain still another proof of Theorem 3.1, see [37]. Indeed, assuming that there exists a function $F \in \mathcal{S}_d^\sharp$ with $0 < d < 1$, we may clearly assume (shifting if necessary) that $F(1) \neq 0$. Therefore $\zeta(s)F(s)$ is a polar function in $\mathcal{S}_d^\sharp$ with $1 < d < 2$, a contradiction. The same argument shows that *the degree conjecture restricted to the polar L-functions implies the degree conjecture* (and similarly for the strong degree conjecture). We conclude by stressing the importance of the *α-periodicity* of the linear twists, which plays a fundamental role in the proof of the structure theorems for $1 \le d < 2$.

Now we come back to the measure theoretic approach to invariants outlined in Section 4. We recall that a numerical invariant I is *additive* if $I(FG) = I(F)+I(G)$, and that the H-invariants are additive; hence in particular so is the degree. We refer to Section 4 for the definition of continuous invariants, and recall that in this case $I(\mathcal{S})$ and $I(\mathcal{S}^\sharp)$ are Lebesgue measurable by Theorem 4.6. We also recall that, given $\mathcal{A} \subset \mathbb{R}$, $\mathcal{A} - \mathcal{A}$ denotes the set of all real numbers of the form $a - a'$ with $a, a' \in \mathcal{A}$.

The next result is a 0–1 **law for additive invariants** which, in view of the degree conjectures, is particularly interesting in the case of the degree.

Theorem 6.5. ([42]) *Let I be a continuous, additive and real-valued invariant. Then either the set $I(\mathcal{S})$ has Lebesgue measure 0, or $I(\mathcal{S}) - I(\mathcal{S}) = \mathbb{R}$. The same holds for $I(\mathcal{S}^\sharp)$ as well.*

A similar result holds in the case of the root number, in the sense that the sets of values $\omega^*(\mathcal{S})$ and $\omega^*(\mathcal{S}^\sharp)$ taken by the root number ω_F^* over $\mathcal{S}$ and $\mathcal{S}^\sharp$ are either of measure zero or coincide with the unit circle T^1; see Kaczorowski-Perelli [42]. Further, similar results hold for certain subclasses of $\mathcal{S}^\sharp$, for example $\mathcal{S}_d$ and $\mathcal{S}_d^\sharp$ with a fixed d. Note that we already met related examples, involving the invariants ω_F^* and ξ_F (ω_F^* is not additive), where both alternatives happen. Indeed, from Theorems 6.2 and 6.3 we see that

$$\omega^*(\mathcal{S}_1^\sharp) = T^1 \quad \text{and} \quad \omega^*(\mathcal{S}_1) \text{ is countable}$$

$$\eta(\mathcal{S}_1^\sharp) = \eta(\mathcal{S}_1) = \{-1, 0\} \quad \text{and} \quad \theta(\mathcal{S}_1^\sharp) = \theta(\mathcal{S}_1) = \mathbb{R}.$$

In the case of the Selberg class $\mathcal{S}$, a stronger form of Theorem 6.5 is suggested by the following conjecture in Kaczorowski-Perelli [42].

Conjecture 6.7. (0–1 law conjecture) *Let I be a continuous, additive and real-valued invariant. Then either the set $I(\mathcal{S})$ is countable or it contains a half-line.*

We conclude the discussion of the measure theoretic results with the following simple conditional result related to the degree conjectures. Let

$$E = \{d > 0 : d \notin \mathbb{N} \text{ and } \mathcal{S}_d^\sharp \neq \emptyset\}$$

be the "exceptional set" for the degree conjecture. Clearly, E is measurable by Theorem 4.6, and let $\mu(E)$ denote its Lebesgue measure.

Theorem 6.6. ([16]) *Assume that every $F \in \mathcal{S}_d^\sharp$ with $d \in \mathbb{N}$ has a γ-factor with all $\lambda_j = \frac{1}{2}$. Then $E \cap \mathbb{Q} = \emptyset$ and $\mu(E) = 0$.*

The same result holds for $\mathcal{S}$ as well, under the same assumption restricted to the integer degrees of $\mathcal{S}$. Since Theorem 6.6 is an unpublished result, we give a detailed proof.

Proof of Theorem 6.6. Suppose first that there exists a function $F(s)$ of degree $d_F = \frac{d}{n} \in \mathbb{Q}\backslash\mathbb{N}$, with $(d, n) = 1$ and $n > 1$. Then $F^n \in \mathcal{S}_d^\sharp$ with $d \in \mathbb{N}$, hence by the assumption and Theorem 4.1, the n-th power of the γ-factor of $F(s)$ satisfies an identity of type

$$\gamma(s)^n = c_0 Q^s \prod_{j=1}^{l} \Gamma(\frac{s}{2} + \mu_j)^{n_j}, \tag{4.7}$$

where the μ_j are distinct. Our first aim is to show that n divides each n_j. To this end consider a row of μ_j's with equal imaginary part, and let μ_{j_0} be the one with smallest real part. Since the poles of the γ-factor of $F(s)^n$ are at the points $s_{j,k} = -2(\mu_j + k)$ with $k = 0, 1, \ldots$ and $j = 1, \ldots, l$, the multiplicity of $s_{j_0,0}$ is n_{j_0}. Hence $n \mid n_{j_0}$. Let now μ_{j_1} be the next one in the same row. If $\Re(\mu_{j_1} - \mu_{j_0})$ is not an integer, then the same argument shows that $n \mid n_{j_1}$. If $\Re(\mu_{j_1} - \mu_{j_0})$ is an integer, then the multiplicity of $s_{j_1,0}$ is $n_{j_0} + n_{j_1}$, thus $n \mid (n_{j_0} + n_{j_1})$ and hence $n \mid n_{j_1}$ in this case as well. Arguing similarly with the other μ_j's we see that $n \mid n_j$ for $j = 1, \ldots, l$ as required.

But from (4.7) we have

$$d_{F^n} = 2\sum_{j=1}^{l} \frac{n_j}{2} = d,$$

hence $n \mid d$, a contradiction. Therefore $E \cap \mathbb{Q} = \emptyset$. Moreover, $\mathcal{S}^\sharp$ is a multiplicative semigroup, hence for every $d \in E$ and $d' \in \mathbb{R}^+$ such that $d + d' \in \mathbb{Q} \setminus \mathbb{N}$ we have $\mathcal{S}^\sharp_{d'} = \emptyset$. Therefore

$$((\mathbb{Q} \setminus \mathbb{N}) - E) \cap E = \emptyset, \tag{4.8}$$

and hence $\mu(E) = 0$. In fact, given A and B with $\mu(A), \mu(B) > 0$, the set $A + B$ contains an open segment, see ex. 19 of Ch. 9 of Rudin [64]. Hence, if $\mu(E) > 0$ there exist $r \in \mathbb{Q} \setminus \mathbb{N}$ and $d_1, d_2 \in E$ such that $r = d_1 + d_2$, thus $r - d_1 = d_2$ contradicting (4.8). $\square$

Note that the first part of the proof of Theorem 6.6 is a special case of the following result (see Molteni [45]): if the multiplicity of all poles of a γ-factor $\gamma_1(s)$ is divisible by n, then there exists another γ-factor $\gamma_2(s)$ such that $\gamma_1(s) = \gamma_2(s)^n$. Note also that the crucial point that $E \cap \mathbb{Q} = \emptyset$ is a direct consequence of the assumption on the shape of the functional equation for integer degrees. A related problem is

Problem 6.5. *Replace the assumption in Theorem* 6.6 *by other function-theoretic properties of the functions with integer degree.*

5. Independence

In Section 3 we presented the simplest independence result in the Selberg class, namely the linear independence of the functions in $\mathcal{S}$ (see Theorem 3.8). We also remarked that the unique factorization (UF) and the Selberg orthonormality (SOC) conjectures (see Conjectures 3.2 and 3.3) are stronger forms of independence. In this section we present other independence results, concerning the functions in $\mathcal{S}$ and their zeros. Most such results are conditional, but nevertheless they form a very interesting part of the Selberg class theory.

We start with the following simple consequence of Theorem 3.8, concerning the algebraic independence in $\mathcal{S}$.

Corollary 7.1. ([30]) *$\mathcal{S}$ has unique factorization if and only if distinct primitive functions are algebraically independent.*

Clearly, we can't expect that distinct functions in $\mathcal{S}$ are algebraically independent, since $\mathcal{S}$ is a semigroup. However, we have

Theorem 7.1. ([45]) *Let $F, G \in \mathcal{S}$ satisfy $F(s)^a = G(s)^b$ with $a, b \in \mathbb{N}$. Then $F(s) = H(s)^b$ and $G(s) = H(s)^a$ for some $H \in \mathcal{S}$.*

The proof of Theorem 7.1 is based on the characterization of the solubility in $\mathcal{S}$ of the equation $X^k = F(s)$ in terms of the multiplicities of the zeros of $F(s)$ and of its Euler factors $F_p(s)$. Note that Theorem 7.1 and the linear independence imply that *pairs of distinct primitive functions in $\mathcal{S}$ are algebraically independent.*

In the 1940's, Selberg [67], [68] initiated the study of the moments, and hence of the statistical distribution, of $\log\zeta(\frac{1}{2}+it)$. Later, see [69], he outlined the extension of such investigations to the functions in $\mathcal{S}$, obtaining their statistical independence as well. Selberg's arguments have been streamlined by Bombieri-Hejhal [3], with the emphasis just on the probabilistic convergence of the relevant measures, and the goal of applications to the distribution of zeros of linear combinations of certain Euler products. Here we present Bombieri-Hejhal's version of Selberg's results on the normal distribution and statistical independence of the values of L-functions. Note that Bombieri-Hejhal [3] work in a moderately general setting, but their results easily carry over to the Selberg class (see Zanello [82]). Therefore, although Theorems 7.2, 7.3, 7.4 and 7.7 below were originally proved in Bombieri-Hejhal's setting, we state them in the setting of the Selberg class, and refer to [82] for the needed changes in the proofs. Moreover, we refer to the survey paper by Bombieri-Perelli [4] for a discussion of these matters and for a sketch of the proofs of Theorems 7.3, 7.4 and 7.7 below.

We start with two hypotheses needed in the results which follow. The first is a variant of the Selberg orthonormality conjecture (see Conjecture 3.3).

Hypothesis S**.** *The coefficients of $F_1,\dots,F_N\in\mathcal{S}$ satisfy,* for $x\to\infty$,

$$\sum_{p\le x}\frac{a_i(p)\overline{a_j(p)}}{p}=\delta_{ij}n_j\log\log x+c_{ij}+O(\frac{1}{\log x}),$$

where $n_j>0$, $c_{ij}\in\mathbb{C}$, and $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ otherwise.

Note that Hypothesis S is quantitatively stronger than SOC. However, it does not assume that the functions are primitive (in practice, it requires that $F_i(s)$ and $F_j(s)$ are *coprime* if $i\neq j$), and does not require $n_j=1$ for primitive functions. Note, however, that for most applications of SOC the reqirement $n_j>0$ for primitive functions would also suffice.

Writing

$$N_j(\sigma,T)=\sharp\{\rho=\beta+i\gamma:F_j(\rho)=0,\beta\ge\sigma,|\gamma|\le T\},$$

the second hypothesis is the following density estimate of Selberg [68] type, which acts as a substitute of GRH in the results below.

Hypothesis D**.** *The zeros of $F_1,\dots,F_N\in\mathcal{S}$ satisfy*

$$N_j(\sigma,T)\ll T^{1-a(\sigma-\frac{1}{2})}\log T$$

for some $0<a<1$, uniformly for $\sigma\ge\frac{1}{2}$ and $j=1,\dots,N$.

Note that Hypothesis D is known for the Riemann zeta function, for the Dirichlet L-functions and for certain GL(2) L-functions.

We need further notation. Given $F_1,\dots,F_N\in\mathcal{S}$ we write

$$V_j(t)=\frac{\log F_j(\frac{1}{2}+it)}{\sqrt{\pi n_j\log\log t}}\qquad\qquad j=1,\dots,N$$

and let

$$\mu_T(\Omega)=\frac{1}{T}|\{t\in[T,2T]:(V_1(t),\dots,V_N(t))\in\Omega\}|$$

be the associated probability measure on $\mathbb{C}^N$, where $\Omega\subset\mathbb{C}^N$ is an open set and $|\mathcal{A}|$ denotes here the Lebesgue measure of $\mathcal{A}$. Moreover, let $e^{-\pi\|\boldsymbol{z}\|^2}$ denote the gaussian measure on $\mathbb{C}^N$ and $d\omega$ be the euclidean density on $\mathbb{C}^N$.

Theorem 7.2. ([69], [3]) *Let $F_1, \dots, F_N \in \mathcal{S}$ satisfy Hypotheses S and D. Then, as $T \to \infty$, the measure μ_T converges weakly to the gaussian measure with associated density $e^{-\pi\|\boldsymbol{z}\|^2} d\omega$.*

By separating real and imaginary parts of the $V_j(t)$, Theorem 7.2 may be expressed as follows. If $F_1, \dots, F_N \in \mathcal{S}$ satisfy Hypotheses S and D, then the functions

$$\frac{\log|F_1(\frac{1}{2}+it)|}{\sqrt{\pi n_1 \log\log t}}, \quad \frac{\arg F_1(\frac{1}{2}+it)}{\sqrt{\pi n_1 \log\log t}}, \quad \dots, \quad \frac{\log|F_N(\frac{1}{2}+it)|}{\sqrt{\pi n_N \log\log t}}, \quad \frac{\arg F_N(\frac{1}{2}+it)}{\sqrt{\pi n_N \log\log t}}$$

become distributed, in the limit of large t, like independent random variables, each having gaussian density $e^{-\pi u^2} du$. It is interesting to observe how an independence hypothesis of SOC type (*i.e.* Hypothesis S) implies the *normal distribution* and *statistical independence* of the values of $\log F_J(s)$ on the critical line (Hypothesis D is a technical hypothesis in this context).

Theorem 7.2 is a deep result, and here we just mention the main steps in Bombieri-Hejhal's [3] proof. The starting point is the following approximation formula for $\log F_j(\frac{1}{2}+it)$, obtained by Mellin transform techniques: for $j = 1, \dots, N$ and $10 \le X \le |t|^2$

$$\log F_j(\frac{1}{2}+it) = \Sigma_j(t,X) + R_j(t,X), \tag{5.1}$$

where $\Sigma_j(t,X)$ is a suitable Dirichlet polynomial approximation and $R_j(t,X)$ is a certain sum over zeros of $F_j(s)$. In order to derive Theorem 7.2, in [3] the mixed moments of the $\Sigma_j(t,X)$ are evaluated, and an L^1-norm bound for the error term $R_j(t,X)$ is obtained:

$$\int_T^{2T} |R_j(t,X)| dt \ll T \frac{\log T}{\log X} \tag{5.2}$$

for $10 \le X \le T^{a/2}$, and

$$\int_T^{2T} \prod_{j=1}^N \Sigma_j(t,X)^{k_j} \overline{\Sigma_j(t,X)}^{l_j} dt = \delta_{\boldsymbol{kl}} \boldsymbol{k}! T \prod_{j=1}^N (n_j \log\log X)^{k_j} + O\Big((T(\log\log X)^{\frac{K+L-1}{2}}\Big) \tag{5.3}$$

for $10 \le X \le T^{1/(K+L+1)}$, where k_j, l_j are non-negative integers, $\boldsymbol{k} = (k_1, \dots, k_N)$ and $K = k_1 + \cdots + k_N$ (and similarly for $\boldsymbol{l}$ and L), $\boldsymbol{k}! = \prod_{j=1}^N k_j!$ and $\delta_{\boldsymbol{kl}} = 1$ if $\boldsymbol{k} = \boldsymbol{l}$, $\delta_{\boldsymbol{kl}} = 0$ otherwise. Theorem 7.2 follows then from (5.1), (5.2) and (5.3) by a probabilistic argument.

For future reference, we state here a *short intervals* version of Theorem 7.2. Let $M \ge 10$, $h = M/\log T$,

$$\tilde{V}_j(t) = \frac{\log F_j(\frac{1}{2}+i(t+h)) - \log F_j(\frac{1}{2}+it)}{\sqrt{2\pi n_j \log M}} \qquad j = 1, \dots, N$$

and let $\tilde{\mu}_T$ be the associated probability measure on $\mathbb{C}^N$ (like μ_T above). We have

Theorem 7.3. ([5]) *Let $F_1, \dots, F_N \in \mathcal{S}$ satisfy Hypotheses S and D and let $M = M(T) \to \infty$ with $M \le \frac{\log T}{\log\log T}$. Then, as $T \to \infty$, the measure $\tilde{\mu}_T$ converges weakly to the gaussian measure with associated density $e^{-\pi\|\boldsymbol{z}\|^2} d\omega$.*

The proof of Theorem 7.3 is a short intervals version of the argument in Theorem 7.2.

It is generally accepted that functional equation and Euler product are crucial ingredients for the validity of the Riemann Hypothesis. In the opposite direction, examples of L-functions without Euler product having zeros *off* the critical line are well known; see *e.g.* Section 10.25 of Titchmarsh [76] and Chapter 9 of Davenport [15]. Usually, such L-functions are linear combinations of Euler products satisfying a common functional equation, and fairly general methods often show that the number of zeros *on* the critical line up to T is $\gg T$ (for small degrees). In some cases, lower bounds of type $\gg Tf(T)$ with concrete functions $f(T) \to \infty$ have been obtained, see Section 1 of [3]. Using Theorem 7.2, Bombieri-Hejhal [3] obtained a very sharp and general result about the zeros on the critical line of linear combinations of Euler products. In order to state such a result we need the following mild condition of well-spacing for zeros.

Hypothesis H_0. *Let $F_1, \dots, F_N \in \mathcal{S}$ satisfy GRH and, moreover, let the zeros of each $F_j(s)$ satisfy*

$$\lim_{\varepsilon \to 0^+} \Big\{ \overline{\lim_{T\to\infty}} \frac{|\{T \le \gamma \le 2T : \gamma' - \gamma \le \varepsilon/\log T\}|}{T\log T} \Big\} = 0,$$

where $\frac{1}{2} + i\gamma'$ is the successor of $\frac{1}{2} + i\gamma$.

We will return on Hypothesis H_0 later on in this section. Given $F_1, \dots, F_N \in \mathcal{S}$ satisfying the *same* functional equation, let

$$F(s) = \sum_{j=1}^{N} c_j F_j(s), \tag{5.4}$$

where the coefficients c_j are such that $c_j\gamma(\frac{1}{2}+it)F_j(\frac{1}{2}+it) \in \mathbb{R}$ for $t \in \mathbb{R}$ and $j = 1, \dots, N$, where $\gamma(s)$ is the common γ-factor of the $F_j(s)$'s.

Theorem 7.4. ([2], [3]) *Let $F_1, \dots, F_N \in \mathcal{S}$ satisfy the same functional equation, GRH and Hypothesis H_0, and let $F(s)$ be defined as in* (5.4)*. Then all but $o(T\log T)$ zeros of $F(s)$ up to T are simple and lie on the critical line.*

Roughly speaking, the proof of Theorem 7.4 rests on the fact that on a given interval of the critical line of length roughly $1/\log T$, *one* of the functions $F_j(s)$ dominates with oscillations larger than the others, thus in that interval the function $F(s)$ follows fairly closely the behaviour of the $F_j(s)$ with largest oscillation. The proof is rather complicated because of, among other things, the weak measure theoretic setting in Theorem 7.2.

In view of Theorem 7.4, one may ask about the number of *off-line* zeros up to T of the function $F(s)$ in (5.4). We refer to Hejhal [24], [25] for several results in this direction. Here we only recall that there is some expectation that the order of magnitude of the number of such zeros is about

$$\frac{T\log T}{\sqrt{\log\log T}}.$$

By Hadamard's theory, the L-functions are essentially determined by their zeros, thus one expects that "independent" L-functions have "independent" zeros. One of the simplest forms of independence of the zeros is asking for **distinct zeros**, in the sense defined below; other forms of independence will be presented later on in this section. In view of SOC and of Theorem 7.2, distinct primitive functions are expected to be independent in several ways, and hence we may expect that they have few common zeros. Therefore, by factorization, distinct functions in $\mathcal{S}$ should have many distinct zeros. In order to make rigorous and quantitative such heuristic observations, we need some notation.

Given $F, G \in \mathcal{S}$, we define two counting functions of the distinct zeros, *with multiplicity*, as follows. The **asymmetric difference** of the zeros of $F(s)$ and $G(s)$, *i.e.* the number of zeros of $F(s)$ which are not zeros of $G(s)$, is defined by

$$D(T; F, G) = \sum_{|\gamma| \le T} \max(m_F(\rho) - m_G(\rho), 0),$$

where $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $F(s)$ and is counted without multiplicity, and $m_F(\rho)$ (resp. $m_G(\rho)$) denotes the multiplicitiy of ρ as zero of $F(s)$ (resp. of $G(s)$). The **symmetric difference** is then defined as

$$D_{F,G}(T) = D(T;F,G) + D(T;G,F) = \sum_{|\gamma|\leq T} |m_F(\rho) - m_G(\rho)|,$$

where ρ runs over the non-trivial zeros of $F(s)G(s)$ and is counted without multiplicity, and counts the number of zeros and poles of $F(s)/G(s)$ in the critical strip (excluding the contribution of possible trivial zeros, and of poles at $s = 1$, of $F(s)$ and $G(s)$).

Of course, the asymmetric difference is more difficult to study than the symmetric difference since, in general, we cannot expect a positive lower bound for $D(T;F,G)$, as the example $G(s) = F(s)^2$ shows. In the symmetric case we expect that $D_{F,G}(T) \gg T\log T$ as soon as $F(s) \neq G(s)$; for example, this is trivially the case when $d_F \neq d_G$ (see (5.5) below). Recalling that $f(x) = \Omega(g(x))$ is the negation of $f(x) = o(g(x))$, in the general case we have

Theorem 7.5. ([53]) *Let $F, G \in \mathcal{S}$ be distinct. Then $D_{F,G}(T) = \Omega(T)$.*

The proof follows by a comparison of Landau's formula, expressing the von Mangoldt function in terms of an exponential sum over the zeros, for $F(s)$ and $G(s)$. In particular, the Euler products of $F(s)$ and $G(s)$ are used. We remark that the same Ω-estimate has been obtained by Bombieri-Perelli [6] for the number of zeros and poles of a class of exponential sums $f(s)$ assuming only certain *function theoretic properties*, disregarding their arithmetic aspects. In this case the proof is more involved and uses Nevanlinna's theory. Choosing $f(s) = F(s)/G(s)$, Theorem 2 of [6] yields $\Omega(T)$ distinct zeros in the case of two Dirichlet series $F(s)$ and $G(s)$ satisfying the *same* functional equation, without any assumption on the Euler product.

Clearly, in order to get non-trivial lower bounds for the asymmetric difference $D(T;F,G)$ we have to assure at least that $F(s) \nmid G(s)$. A convenient way to do this is to assume that $d_F \geq d_G$. Since clearly (see Section 2)

$$D(T;F,G) \geq N_F(T) - N_G(T) \gg T\log T \qquad \text{if } \ d_F > d_G, \tag{5.5}$$

as far as we are not concerned with *asymptotic formulae* for $D(T;F,G)$ we may simply assume that $d_F = d_G$. An unconditional result in this direction has been obtained by Srinivas [72].

Theorem 7.6. ([72]) *Let $F, G \in \mathcal{S}$ be distinct and $d_F = d_G$. Then for T sufficiently large there exists a zero ρ of $F(s)$ with $m_F(\rho) > m_G(\rho)$ such that $|T - \rho| \ll \log\log T$. Hence, in particular,*

$$D(T;F,G) \gg \frac{T}{\log\log T}.$$

Of course, the same holds switching the roles of $F(s)$ and $G(s)$. The proof is based on a contour integration argument applied to the quotient $G(s)/F(s)$. The order $\log\log T$ of the short intervals in Theorem 7.6 should be compared with the order $1/\log\log\log T$ of the short intervals containing a zero of a given $F \in \mathcal{S}$ (extension to $\mathcal{S}$ of Littlewood's theorem, see Theorem 9.12 of [76], obtained by Anirban-Srinivas, unpublished).

Problem 7.1. *Replace the estimates $\Omega(T)$ in Theorem 7.5 and $\gg T/\log\log T$ in Theorem 7.6 by the lower bound $\gg T$.*

Theorem 7.3 allows to obtain the best possible lower bound for $D(T;F,G)$, under Hypothesis S and the technical Hypothesis D.

Theorem 7.7. ([5]) *Let $F, G \in \mathcal{S}$ be distinct and satisfy Hypotheses S and D, and let $d_F = d_G$. Then*

$$D(T; F, G) \gg T \log T.$$

Again, the same result holds switching $F(s)$ and $G(s)$. The proof of Theorem 7.7 rests on a similar phenomenon as in Theorem 7.4, *i.e.* usually one of the functions is dominating on short intervals of length roughly $1/\log T$. Also, the weak measure theoretic setting of Theorem 7.3 gives rise to some complications in the proof. A consequence is that the argument is by contradiction, thus the constant in the $\gg$-symbol is not effectively computable. However, as remarked in Section 4 of Bombieri-Perelli [4], an effective constant can be obtained at the cost of substantial additional complications in the proof. Further, we already remarked that Hypothesis D is known for many classical L-functions of degree ≤ 2. Since such L-functions satisfy SOC as well, Theorem 7.7 is unconditional in those cases. Again, we refer to the survey paper [4] for a discussion of the distinct zeros problem.

We remark at this point that the first result on distinct zeros has been obtained by Fujii [17]. He proved, by Selberg's moments method, that $D(T; F, G) \gg T \log T$ in the case of two Dirichlet L-functions. Moreover, Raghunathan [61], [62] obtained that $D(T; F, G) \to \infty$ for certain classical L-functions $F(s)$ and $G(s)$, by a method based on converse theorems of Hecke type.

The problem of the **strongly-distinct zeros**, *i.e.* the zeros placed at different points, appears to be more difficult. Conrey-Ghosh-Gonek [13], [14] dealt with the case of two Dirichlet L-functions, by considering the more difficult problem of getting simple zeros of $L(s, \chi_1)L(s, \chi_2)$. They obtained $\gg T^{6/11}$ such zeros up to T and, assuming the Riemann Hypothesis for one of the two functions, got a positive proportion of strongly distinct zeros. However, apparently the analytic techniques in [13] and [14] do not extend to higher degree L-functions.

Problem 7.2. *Deal with the strongly-distinct zeros problem for the functions in $\mathcal{S}$.*

We conclude the discussion of distinct zeros with the following conjectures.

Conjecture 7.1. (simple zeros conjecture) *Let $F \in \mathcal{S}$ be primitive. Then all but $o(T \log T)$ non-trivial zeros of $F(s)$ up to T are simple.*

Conjecture 7.2. (distinct zeros conjecture) *Let $F, G \in \mathcal{S}$ be distinct primitive functions. Then all but $o(T \log T)$ non-trivial zeros of $F(s)$ and $G(s)$ are strongly-distinct.*

Although apparently Conjecture 7.1 cannot be regarded as an independence statement, we included it here in view of its relevance in the problem of the strongly-distinct zeros. Conjectures 7.1 and 7.2 are not at all the strongest conjectures of this type. In fact, it is generally expected that a non-trivial zero with multiplicity greater than 1 of a primitive function can occur only at the point $s = \frac{1}{2}$. The same phenomenon is expected to hold for the distinct zeros as well, but one has to more careful here. In fact, the *shifts* of the primitive functions are expected to be primitive (see Section 3) and GRH is expected to hold, hence it is likely that distinct primitive function have common zeros other than $s = \frac{1}{2}$. However, if $F, G \in \mathcal{S}$ are primitive and **normalized**, *i.e.* the shifts θ_F and θ_G vanish (as for the classical L-functions), then it is expected that the only common zero can occur at $s = \frac{1}{2}$. The above expectations take into account the Birch and Swinnerton-Dyer conjecture.

Another form of independence is the **functional independence of the zeros**. By this we mean, roughly speaking, the following problem: given $F_1, \ldots, F_N \in \mathcal{S}$ and a holomorphic function $H(\boldsymbol{z}, s)$, $\boldsymbol{z} = (z_1, \dots, z_N)$, can

$$H(\log F_1(s), \dots, \log F_N(s), s) \quad \text{or} \quad H(\frac{F_1'}{F_1}(s), \dots, \frac{F_N'}{F_N}(s), s)$$

have only finitely many singularities in the half-plane $\sigma \geq \frac{1}{2}$?

We consider first the problem with the logarithms. It is clear that one needs to impose some natural restrictions on the functions $F_1, \dots, F_N \in \mathcal{S}$ and on $H(\boldsymbol{z}, s)$ in order to get infinitely many singularities. To this end, we consider a region $\mathcal{D}$ containing the half-plane $\sigma \geq \frac{1}{2}$ and holomorphic functions $H(\boldsymbol{z}, s)$ on $\mathbb{C}^N \times \mathcal{D}$ such that for every $s_0 \in \mathcal{D}$ and $\varepsilon > 0$

$$H(\boldsymbol{z}, s_0) \ll e^{\varepsilon \|\boldsymbol{z}\|} \qquad\qquad \|\boldsymbol{z}\| \to \infty;$$

we denote by $\mathcal{H}$ the ring of such functions. Moreover, we say that $\deg H = 0$ if for every $s_0 \in \mathcal{D}$, $H(\boldsymbol{z}, s_0)$ is constant as a function of $\boldsymbol{z}$. Further, for $H \in \mathcal{H}$ we write

$$h(s) = H(\log F_1(s), \dots, \log F_N(s), s),$$

which is holomorphic in the region obtained by suitably cutting $\mathcal{D}$ at the singularities of the $\log F_j(s)$'s. Note that the growth condition on $H(\boldsymbol{z}, s)$ cannot be significantly relaxed, as shown by the example with $N = 1$, $F(s) = \zeta(s)^k$, $H(z, s) = e^{z/k}$ and $k \in \mathbb{N}$, where $h(s)$ has only the singularity at $s = 1$. Moreover, the $\log F_1(s), \dots, \log F_N(s)$ must be linearly independent over $\mathbb{Q}$, otherwise there are simple examples of $H \in \mathcal{H}$ such that $h(s)$ has no singularities at all. The following result shows that $h(s)$ has always infinitely many singularities in the half-plane $\sigma \geq \frac{1}{2}$, unless there are obvious reasons for the cancellation of the singularities.

Theorem 7.8. ([32]) *Let $F_1, \dots, F_N \in \mathcal{S}$ be such that $\log F_1(s), ..., \log F_N(s)$ are linearly independent over $\mathbb{Q}$, and let $H \in \mathcal{H}$ with $\deg H \neq 0$. Then $h(s)$ has infinitely many singularities in the half-plane $\sigma \geq \frac{1}{2}$.*

Since the polynomials belong to $\mathcal{H}$, Theorem 7.8 provides, in particular, a kind of *algebraic independence* of the zeros, in the sense explained above. Moreover, if the unique factorization conjecture holds (see Conjecture 3.2), then $h(s)$ has infinitely many singularities in $\sigma \geq \frac{1}{2}$ for any distinct primitive functions $F_1, \dots, F_N \in \mathcal{S}$ and $H \in \mathcal{H}$ with $\deg H \neq 0$. We also remark that Theorem 7.8 allows to describe the structure of the functions $H \in \mathcal{H}$ such that $h(s)$ is holomorphic on $\mathcal{D}$, depending on the number of $F_1, \dots, F_N \in \mathcal{S}$ with linearly independent logarithms, see [32]. As a consequence, one obtains the following corollary of Theorem 7.8 (see [32]):

if $F_1, \dots, F_N \in \mathcal{S}$ and $H \in \mathcal{H}$, then $h(s)$ is either holomorphic on $\mathcal{D}$ or has infinitely many singularities on $\sigma \geq \frac{1}{2}$.

Moreover, non-trivial examples of vanishing $\mathbb{Q}$-linear forms of logarithms, and hence of holomorphic $h(s)$, can be obtained in the framework of Artin and Hecke L-functions (see Kaczorowski-Perelli [32]).

The proof of Theorem 7.8 is based on the following lemma, which is of some independent interest. Given $\rho \in \mathbb{C}$ we write $\boldsymbol{m}(\rho) = (m_1(\rho), \dots, m_N(\rho))$, where $m_j(\rho)$ denotes, as usual, the multiplicity of ρ as zero of $F_j(s)$. Moreover, ρ_j denotes a non-trivial zero of $F_j(s)$ in the half-plane $\sigma \geq \frac{1}{2}$.

Lemma 7.1. ([32]) *Let $F_1, \dots, F_N \in \mathcal{S}$ be as in Theorem 7.8. Then there exist infinitely many N-tuples $(\rho_1, \dots, \rho_N)$ such that the vectors $\boldsymbol{m}(\rho_1), \dots, \boldsymbol{m}(\rho_N)$ form a basis of $\mathbb{R}^N$.*

Very likely, Lemma 7.1 can be made quantitative in the sense that many such N-tuples $(\rho_1, \dots, \rho_N)$ up to T can be obtained. Therefore, quantitative versions of Theorem 7.8 are within reach. Accordingly, we raise the following

Problem 7.3. *Get a quantitative version of Theorem* 7.8*, with at least* $\Omega(T)$ *singularities.*

The analogous problem with the logarithmic derivatives in place of the logarithms is more delicate, since poles are easier to cancel than logarithmic singularities. In this case we cannot even expect as general results as before. In fact, let $F_1,\dots,F_N\in\mathcal{S}$, m_{F_j} be the polar order of $F_j(s)$, $P(\boldsymbol{z})$ be a polynomial of degree $k\geq 1$, and let

$$h(s)=\Big(\prod_{j=1}^{N}(s-1)^{m_{F_j}+1}F_j(s)\Big)^kP\Big(\frac{F_1'}{F_1}(s),\dots,\frac{F_N'}{F_N}(s)\Big);$$

clearly, $h(s)$ in an entire function. We have

Theorem 7.9. ([32]) *Let* $P(\boldsymbol{z})$ *be a polynomial with degree* >0*,* $F_1,\dots,F_N\in\mathcal{S}$*,* $k_j,\alpha_j\in\mathbb{N}$ *and* $\eta_j\in\mathbb{C}$ *for* $j=1,\dots,N$*. Then*

$$h(s)=P\Big(\frac{F_1'}{F_1}(\alpha_1s+\eta_1)^{(k_1)},\dots,\frac{F_N'}{F_N}(\alpha_Ns+\eta_N)^{(k_N)}\Big)$$

is either constant or has infinitely many poles.

Note that Theorem 7.9 is weaker than the above stated corollary of Theorem 7.8, in the sense that the function $h(s)$ is of less general type and there is no non-trivial lower bound for the real part of the singularities. The proof of Theorem 7.9 is based on a Mellin transform argument. Similarly to Problem 7.3, one may ask for explicit lower bounds for the number of poles up to T in Theorem 7.9.

Clearly, the functional independence of the zeros is closely related, at least *morally*, to the distinct zeros; we therefore raise the following

Problem 7.4. *Are there direct implications between the results on the functional independence and on the distinct zeros ?*

We finally turn to a very strong form of independence of the zeros, namely the **pair correlation**. Following Montgomery [48], given $F,G\in\mathcal{S}$ and $\alpha\in\mathbb{R}$, Murty-Perelli [54] defined the (*asymmetric* and *normalized*) *pair correlation function*

$$\mathcal{F}(\alpha;F,\overline{G})=\frac{\pi}{d_FT\log T}\sum_{|\gamma_F|,|\gamma_G|\leq T}T^{i\alpha d_F(\gamma_F-\gamma_G)}w(\gamma_F-\gamma_G),$$

where γ_F,γ_G are the imaginary parts of the non-trivial zeros of $F(s)$, $G(s)$ and $w(u)=4/(4+u^2)$, and studied the behaviour of $\mathcal{F}(\alpha;F,\overline{G})$ under GRH. Writing

$$\Lambda_F(n,x)=\begin{cases}\Lambda_F(n)(\frac{n}{x})^{1/2} & \text{if } n\leq x\\ \Lambda_F(n)(\frac{x}{n})^{3/2} & \text{if } n>x\end{cases}\quad\text{and}\quad \Psi_{F\times\overline{G}}(x)=\sum_{n=1}^{\infty}\Lambda_F(n,x)\overline{\Lambda_G(n,x)}$$

they obtained

Proposition 7.1. ([54]) *Assume GRH and let* $F,G\in\mathcal{S}$*,* $\varepsilon>0$ *and* $X=T^{\alpha d_F}$*. Then, uniformly for* $0\leq\alpha d_F\leq 1-\varepsilon$*, as* $T\to\infty$ *we have*

$$\mathcal{F}(\alpha;F,\overline{G})=\frac{1}{d_FX\log T}\Psi_{F\times\overline{G}}(X)+(1+o(1))d_GT^{-2\alpha d_F}\log T+o(1).$$

We may consider only $\alpha\geq 0$ since $\mathcal{F}(-\alpha;F,\overline{G})=\overline{\mathcal{F}(\alpha;F,\overline{G})}$. Note that $\Psi_{F\times\overline{G}}(x)$ is related by partial summation to

$$\psi_{F\times\overline{G}}(x)=\sum_{n\leq x}\Lambda_F(n)\overline{\Lambda_G(n)},$$

for which SOC suggests the asymptotic formula

$$\psi_{F\times\overline{G}}(x) = (\delta_{F,G} + o(1))x\log x, \tag{5.6}$$

provided $F(s)$ and $G(s)$ are primitive. Therefore, assuming GRH and (5.6), Proposition 7.1 yields

$$\mathcal{F}(\alpha; F, \overline{G}) = \delta_{F,G}\alpha + (1 + o(1))d_G T^{-2\alpha d_F}\log T + o(1)$$

uniformly for $0 \leq \alpha d_F \leq 1 - \varepsilon$. This is the analog of the Theorem in Montgomery [48], and still in analogy with [48], the following conjecture has been stated in [54].

Conjecture 7.3. (pair correlation conjecture, PC) *Let $F, G \in \mathcal{S}$ be primitive. Then*

$$\mathcal{F}(\alpha; F, \overline{G}) = \begin{cases} \delta_{F,G}|\alpha| + (1 + o(1))d_G T^{-2|\alpha|d_F}\log T + o(1) & \text{if } |\alpha| \leq 1 \\ \delta_{F,G} + o(1) & \text{if } |\alpha| \geq 1 \end{cases}$$

as $T \to \infty$, uniformly for α in any bounded interval.

Clearly, Conjecture 7.3 can be suitably modified if $F, G \in \mathcal{S}$ are not primitive, see (3.5) of [54]. As already remarked above, Conjecture 7.3 is a very strong independence statement. For example, by convolution with suitable kernels (see [48]) PC yields

Theorem 7.10. ([54]) *PC implies Conjectures 7.1 and 7.2. Moreover, if the asymptotic formula of PC holds for some $\alpha_0 > 0$, then the unique factorization conjecture follows.*

We remark that the UF conjecture is deduced in Theorem 7.10 using the properties of the pair correlation function $\mathcal{F}(\alpha; F, \overline{G})$ in a direct way. However, it is quite clear from Proposition 7.1 that PC has implications on the following version of SOC

$$\Psi_{F\times\overline{G}}(x) = (\delta_{F,G} + o(1))x\log x. \tag{5.7}$$

Precisely, (5.7) *follows from GRH and*

$$\mathcal{F}(\alpha_0; F, \overline{G}) = \delta_{F,G}\alpha_0 + o(1) \tag{5.8}$$

for a suitably small $0 < \alpha_0 = \alpha_0(F, G)$. But (5.7) can be used as a substitute of SOC in results like Theorems 3.6 and 3.7. Hence, in turn, GRH and (5.8) imply results of such type, in particular the Artin conjecture.

The pair correlation conjecture may also be formulated, more explicitly, in terms of the differences of the imaginary parts of the zeros, as in (12) of [48]. In that way one sees that Hypothesis H_0 is in fact a weak version of PC, dealing only with the behaviour at $\alpha = 0$.

The above treatment of PC follows Montgomery's [48] approach, thus assumes GRH. However, Rudnik-Sarnak [65], [66] investigated suitably weighted versions of the n-level correlation of the zeros of automorphic L-functions (assuming a mild form of the Ramanujan conjecture when the degree is > 4) without assuming the Riemann Hypothesis for such L-functions, thus getting interesting general results. This point of view has been carried over by Murty-Zaharescu [55] to the framework of the Selberg class.

To this end, Murty-Zaharescu [55] defined the pair correlation function in the following slightly modified way (here $w(z) = \frac{4}{4-z^2}$)

$$\tilde{\mathcal{F}}(\alpha; F, G) = \frac{\pi}{d_F T\log T}\sum_{|\gamma_F|,|\gamma_G|\leq T} T^{\alpha d_F(\rho_F+\rho_G-1)}w(\rho_F + \rho_G - 1),$$

so that $\tilde{\mathcal{F}}(\alpha; F, \overline{G}) = \mathcal{F}(\alpha; F, \overline{G})$ under GRH. Observing that an off-line zero $\rho = \beta + i\gamma$ of a primitive $F \in \mathcal{S}$ gives rise to a term of order $T^{\alpha d_F(2\beta-1)}/T\log T$ in $\tilde{\mathcal{F}}(\alpha; F, \overline{F})$, and that such a term tends to infinity as $T \to \infty$ provided α is suitably large, it is clear that Conjecture 7.3, with $\tilde{\mathcal{F}}(\alpha; F, \overline{G})$ in place of $\mathcal{F}(\alpha; F, \overline{G})$, *morally* (at least) assumes GRH. In order to have a GRH-free form of PC, in [55] the following conjecture has been formulated.

Conjecture 7.4. (weak pair correlation conjecture, WPC) *Let $F, G \in \mathcal{S}$ be primitive. Then there exists a constant $c_{F,G} > 0$ such that for any $0 < \alpha < c_{F,G}$*

$$\tilde{\mathcal{F}}(\alpha; F, \overline{G}) = \delta_{F,G}\alpha + o(1) \qquad\qquad \text{as } T \to \infty.$$

As pointed out in [55], results of this type are indeed *proved* in Rudnik-Sarnak [66] in the framework of automorphic L-functions. Moreover, it is clear from the second statement of Theorem 7.10 and from the discussion after it that WPC has interesting consequences. In fact, the following unconditional version of Proposition 7.1 holds (recall that ϑ_F appears in axiom *(v)* of the Selberg class).

Proposition 7.2. ([55]) *Let $F, G \in \mathcal{S}$, $\varepsilon > 0$ and $X = T^{\alpha d_F}$. Then for $\varepsilon \le \alpha d_F \le 1 - \varepsilon$, as $T \to \infty$ we have*

$$\tilde{\mathcal{F}}(\alpha; F, \overline{G}) = \frac{1}{d_F X \log T} \Psi_{F\times\overline{G}}(X) + O(T^{-\delta}),$$

where $\delta = \varepsilon \min(\frac{1}{2}, 1 - \vartheta_F - \vartheta_G)$.

In view of the discussion after Theorem 7.10, WPC implies (5.7) and hence

Corollary 7.2. ([55]) *WPC implies UF, the Artin conjecture and the Langlands reciprocity conjecture for solvable extensions of $\mathbb{Q}$.*

It is natural to raise at this point the following

Problem 7.5. *Prove that some form of PC implies SOC.*

The proof of Proposition 7.2 is based on a suitable version of Landau's formula, see Proposition 1 of [55]. Such a technique allows to introduce weight functions, thus giving more general explicit formulae for the pair correlation of the zeros of functions in $\mathcal{S}$, of type

$$\sum_{|\gamma_F|,|\gamma_G| \le T} f(\rho_F + \rho_G) = \sum_{n=1}^{\infty} \Lambda_F(n)\Lambda_G(n) g(n) \ + \ \text{error}, \tag{5.9}$$

where $g(u)$ is a suitable weight function and $f(s)$ is its Mellin transform (see Theorems 3 and 4 and Corollaries 1 and 2 in Murty-Zaharescu [55]).

In Section 6 we already met an extension to $\mathcal{S}^\sharp$ of the classical Rankin-Selberg convolution, namely $F \times \overline{F}(s)$. Of course, the same type of convolution can be defined more generally for two functions $F, G \in \mathcal{S}^\sharp$ as

$$F \times G(s) = \sum_{n=1}^{\infty} a_F(n) a_G(n) n^{-s}. \tag{5.10}$$

Another type of extension to $\mathcal{S}$ of the Rankin-Selberg convolution appears in Narayanan [56], and is defined as follows (see also [55]). For $F, G \in \mathcal{S}$ define

$$F \otimes G(s) = \prod_p (F \otimes G)_p(s), \tag{5.11}$$

where for all but finitely many primes p

$$\log(F \otimes G)_p(s) = \sum_{m=1}^{\infty} \frac{m b_F(p^m) b_G(p^m)}{p^{ms}}.$$

Note that in view of the multiplicity one property of $\mathcal{S}$ (see Theorem 3.4) the definition of the Euler factors at a finite number of primes is not critical. Note also that the convolutions (5.10) and (5.11) are closely related, and that (5.11) is expected to have better analytic properties (see Chapter II of Moroz [50]).

Assuming that $F\otimes G \in \mathcal{S}$, from (5.11) we see that (apart from $n = p^m$ with p in a finite set)

$$\Lambda_F(n)\Lambda_G(n) = \Lambda(n)\Lambda_{F\otimes G}(n), \tag{5.12}$$

where $\Lambda(n)$ denotes the classical von Mangoldt function associated with $\zeta(s)$. Hence the explicit formulae of type (5.9) and Proposition 7.2 imply

Corollary 7.3. ([55]) *Let $F, G \in \mathcal{S}$ be such that $F\otimes G \in \mathcal{S}$. Then for $f(s)$ as in* (5.9)

$$\sum_{|\gamma_F|,|\gamma_G|\le T} f(\rho_F + \rho_G) = \sum_{|\gamma|,|\gamma_{F\otimes G}|\le T} f(\rho + \rho_{F\otimes G}) \ + \ \textit{error},$$

where ρ and $\rho_{F\otimes G}$ denote the non-trivial zeros of $\zeta(s)$ and $F\otimes G(s)$. Moreover, for $\varepsilon > 0$ and $\varepsilon \le \alpha d_F \le 1-\varepsilon$, as $T \to \infty$

$$\tilde{\mathcal{F}}(\alpha; F, \overline{G}) = \tilde{\mathcal{F}}(\alpha; \zeta, F\otimes\overline{G}) + O(T^{-\delta}),$$

where $\delta = \varepsilon \min(\frac{1}{2}, 1 - \vartheta_F - \vartheta_G)$.

Corollary 7.3 reflects once again the *universality* of the pair correlation of the L-functions, as well as the expectation that $F, G \in \mathcal{S}$ *are coprime if and only if the Rankin-Selberg convolution $F\otimes\overline{G}(s)$ is holomorphic at $s = 1$* (or, equivalently, *if and only if $F\otimes\overline{G}(s)$ is coprime with $\zeta(s)$*). Another consequence of Proposition 7.2 pointed out in [55] is that, in analogy to Rudnik-Sarnak [66], *if $F, G \in \mathcal{S}$ are primitive and $F\otimes\overline{G}(s)$ has nice analytic properties, then WPC holds for $\tilde{\mathcal{F}}(\alpha; F, \overline{G})$*. In fact, by standard arguments in prime number theory one gets, *via* (5.12), the expected asymptotic formula (5.7) from mild information on the analytic continuation, polar structure at $s = 1$ and zero-free regions of $F\otimes\overline{G}(s)$.

We finally state a natural problem on the extension of the pair correlation problems (see Section 7 of [55]).

Problem 7.6. *Investigate the N-level correlation of the zeros of functions in $\mathcal{S}$ (or the correlation of the zeros of N-tuples of functions in $\mathcal{S}$).*

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