Organized by
VILNIUS UNIVERSITY

Sponsored by
INFORMACINĖS TECHNOLOGIJOS,
Publishing house ALMA LITTERA,
BALTIC AMADEUS,
LITHUANIAN SCHOOL OF YOUNG MATHEMATICIANS,
NATIONAL EXAM CENTRE,
Publishing house TEV,
Publishing house TYTO ALBA

22nd TEAM CONTEST OF LITHUANIA IN MATHEMATICS

Department of mathematics and informatics of Vilnius University
September the 29th 2007

1. Solve the system of equations

$$\begin{cases} 1-\dfrac{12}{y+3x}=\dfrac{2}{\sqrt{x}}, \\ 1+\dfrac{12}{y+3x}=\dfrac{6}{\sqrt{y}}. \end{cases}$$

2. Find all quadruplets a, b, c, d of real numbers (a; b; c; d), which satisfy the system of equations

$$\begin{cases} a+b=8, \\ ab+c+d=23, \\ ad+bc=28, \\ cd=12. \end{cases}$$

3. A polynomial $f(x)$ of degree three is such that it has three different real zeros and the coefficient of x^3 is positive. Show that $f'(a)+f'(b)+f'(c)>0$

4. Find the real solutions of $\dfrac{8}{\sqrt{x+6}-\sqrt{x-2}}\le 6-\sqrt{x+1}$

5. The real numbers a, b, c are such that they all three are either greater or all are less than 1. Prove that $\log_a bc+\log_b ca+\log_c ab\ge 4(\log_{ab} c+\log_{bc} a+\log_{ca} b)$.

6. Find all quadruplets (x, y, z, t) of positive integers x, y, z, t such that

$$x^2+y^2+z^2+t^2=3(x+y+z+t)$$

7. Let n be a positive integer and $S_n=1\cdot 2+2\cdot 3+...+n\cdot(n+1)$. Prove or disprove that there is always at least one perfect square between S_n and S_{n+1}.

8. Determine all positive integers n, which can be represented in the form $n=[a,b]+[b,c]+[c,a]$ where a, b, c are positive integers and $[p, q]$ is the lowest common multiple of the integers p and q.

9. We will try to choose the positive integers m and n in such a way that the number $\dfrac{m+1}{n}+\dfrac{n+1}{m}$ would be an integer.

(i) Select (at least) three such pairs (m, n) of positive integers m and n.
(ii) Find five such pairs (m, n) of a positive integers m and n;
(iii) Find seven such a pairs.
(iiii) Are there an infinitely many such pairs of positive integers?

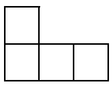
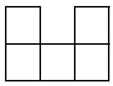
10. (A) Determine a natural number n such that $n > 2$ and the sum of squares of some n consecutive positive integers is a perfect square;
(B) Find at least 2 such natural numbers n;
(C) Is it possible to find 3 such positive integers n?

11. M is a finite set of points in a plane. Point O in the plane is called an "*almost centre of symmetry*" of set M, if it is possible to remove from M one point in such a way that among the remaining points O is the centre of symmetry in the usual sense.
(i) Find such an M, possessing such an *almost centre of symmetry* ;
(ii) Find such an M, possessing two *almost centres of symmetry*;
(iii) How many such *almost centres of symmetry* may a finite point set in the plane have?

12. For each sequence $S = \{a_1, a_2, ..., a_n\}$ of non-negative integers let the offspring of S be the sequence $T = \{b_1, b_2, ..., b_n\}$, where b_i is a number of integers in S to the right of a_i, that are less than a_i. For example, if $S = \{6, 1, 8, 0, 5, 7, 2, 2, 4, 0, 7, 7, 5\}$, then $T = \{8, 2, 10, 0, 4, 5, 1, 1, 1, 0, 1, 1, 0\}$. For a given sequence S_0, let S_1 be the offspring of S_0, S_2 the offspring of S_1 and so on. Is there always an integer j such that $S_j = S_{j+1}$?

13. A circle is divided into $2n$ congruent sectors, n of them coloured black and remaining n sectors coloured white. The white sectors are numbered clockwise from 1 to n, starting anywhere. Afterwards, the black sectors are numbered counter-clockwise from 1 to n, again starting anywhere. Prove that there exist n consequent sectors having all the numbers from 1 to n.

14. Let a_1, a_2, ..., a_n be an arbitrary arrangement of numbers 1, 2, ..., n on a circle. Find $\min \sum_{j=1}^{n} |a_j - a_{j+1}|$ and $\max \sum_{j=1}^{n} |a_j - a_{j+1}|$ where $a_{n+1} = a_1$ and the extreme are taken over all possible arrangements of 1, 2, ..., n.

15. Find the smallest possible integer n for which it is possible to cover an $n \times n$ chessboard using the same number of tiles  and  so that no two tiles overlap.

16. $ABCD$ is a convex quadrilateral inscribed in a circle with centre O, and with mutually perpendicular diagonals. The broken line AOC divides the quadrilateral into two parts. Find the possible ratio of areas of these parts.

17. Two touching circles S and T share a common tangent which meets S at A and T at B. Let AP be a diameter of S and let the tangent from P to T touch it at Q. Show that $AP = PQ$.

18. Consider triangles whose each side length squared is a rational number. Is it true that
(i) the square of the circumradius of every such triangle is rational?
(ii) the square of the inradius of every such triangle is rational?

19. Let w_a, w_b, w_c be the lengths of internal angle bisectors of a triangle ABC with the sides a, b, c. Let R be its circumradius. Prove that $\frac{b^2+c^2}{w_a} + \frac{c^2+a^2}{w_b} + \frac{a^2+b^2}{w_c} > 4R$.

20. Let $ABCD$ be a convex quadrilateral. Let O be the intersection of AC and BD. Let O and M be the intersections of the circumcircle of triangle OAD with the circumcircle of the triangle OBC. Let T and S be the intersections of OM with circumcircles of triangles OAB and triangle OCD respectively. Prove that M is the midpoint of TS.