5th Mathematical Contest of Friendship
in Honor and Memory of Grand Duchy of Lithuania

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Welcomed by Nazar AGAKHANOV,
Chairman of International Mathematical Olympiad Advisory Board

1. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be strictly increasing linear functions such that $f(x)$ is an integer if and only if $g(x)$ is an integer. Prove that $f(x) - g(x)$ is an integer for any $x \in \mathbb{R}$.

2. Let ABC be an isosceles triangle with $AB = AC$. The points D, E and F are taken on the sides BC, CA and AB, respectively, so that $\angle FDE = \angle ABC$ and FE is not parallel to BC. Prove that the line BC is tangent to the circumcircle of $\triangle DEF$ if and only if D is the midpoint of the side BC.

3. The number 1234567890 is written on the blackboard. Two players A and B play the following game taking alternate moves. In one move, a player erases the number which is written on the blackboard, say, m, subtracts from m any positive integer not exceeding the sum of the digits of m and writes the obtained result instead of m. The first player who reduces the number written on the blackboard to 0 wins. Determine which of the players has the winning strategy if the player A makes the first move.

4. A positive integer $n \geqslant 2$ is called *peculiar* if the number
$$\binom{n}{i} + \binom{n}{j} - i - j$$
is even for all integers i and j such that $0 \leqslant i \leqslant j \leqslant n$. Determine all *peculiar* numbers.