Second Order and Resampling Approximation of Finite Population U-Statistics Based on Stratified Samples

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Abstract. We construct one-term Edgeworth expansions to distributions of U statistics and Studentized U statistics, based on stratified samples drawn without replacement. Replacing the cumulants defining the expansions by consistent jackknife estimators we obtain Empirical Edgeworth expansions. The expansions provide second order approximations that improve upon the normal approximation. Theoretical results are illustrated by a simulation study where we compare various approximations to the distribution of the commonly used Gini's mean difference estimator.

Key Words: Edgeworth expansion, jackknife, sampling without replacement, U-statistic, normal approximation.

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1 INTRODUCTION

Stratified simple random sampling without replacement (STSRS for short) is widely used in surveys. It generalizes sampling with replacement and without replacement. In order to assess the precision of survey estimates it is important to develop approximations to distributions of various classes of estimators based on STSRS samples.

Here we study estimators which are U statistics. It is well known that, generally, non-degenerate U-statistics are asymptotically normal, see [18]. An improvement upon the normal approximation is provided by an Edgeworth expansion, see [14]. In this paper we construct the one-term Edgeworth expansions for asymptotically linear U statistics and Studentized U statistics.

In the case where the population parameters (cumulants) defining the expansion are unknown it is convenient to use Empirical Edgeworth expansions where unknown parameters are replaced by their estimators ([3], [16], [17], [22]). We define the jackknife estimators and show their consistency.

Since general symmetric statistics can be approximated by U statistics up to second order (e.g., via Hoeffding's decomposition, see [2], [9]) our results extend to general asymptotically linear symmetric statistic.

In a simulation study we compare the accuracy of the normal approximation, Edgeworth expansion and empirical Edgeworth expansion for the commonly used STSRS estimator of Gini's mean difference. The simulation demonstrates that the Empirical Edgeworth expansion outperforms the normal approximation, but tends to be less accurate than the true Edgeworth expansion.

The outline of the paper is as follows. Results on Edgeworth expansions and Empirical Edgeworth expansions are given in Sections 2 and 3. Results of a simulation study are referred to in Section 4. Proofs and technical details are given in the Appendix.

2 Edgeworth expansion

Consider the population $\mathcal{X} = \{x_1, \dots, x_N\}$ and assume that we want to estimate the population parameter $u = \sum_{1 \le i < j \le N} t(x_i, x_j)$, where t is a symmetric function (i.e., $t(x, y) = t(y, x)$). Suppose that the population is divided in h non-intersecting strata $\mathcal{X} = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_h$. Here $\mathcal{X}_k = \{x_{k.1}, \dots, x_{k.N_k}\}$ and $N_1 + \ldots + N_k = N$. From every $\mathcal{X}_k$ we draw (without replacement) the simple random sample $\mathbb{X}_k = \{X_{k.1}, \dots, X_{k.n_k}\}$, $k = 1, \dots, h$, so that the samples $\mathbb{X}_1, \dots, \mathbb{X}_h$ are independent. The statistic, based on the stratified sample without replacement $\mathbb{X} = (\mathbb{X}_1, \dots, \mathbb{X}_h)$,

$$\hat{u} = \hat{u}(\mathbb{X}) = \sum_{1 \le j \le h} \sum_{\{x,y\} \subset \mathbb{X}_j} w_j t(x,y) + \sum_{1 \le j < r \le h} \sum_{x \in \mathbb{X}_j} \sum_{y \in \mathbb{X}_r} w_{jr} t(x,y), \qquad (1)$$

is an unbiased estimator of the parameter u. Here

$$w_j = w_j(\mathbb{X}) = \binom{N_j}{2} \binom{n_j}{2}^{-1}, \qquad w_{jr} = w_{jr}(\mathbb{X}) = N_j N_r (n_j n_r)^{-1}. \qquad (2)$$

Note that in the case where $t(x,y) = |x-y|$ and x_i, $i = 1, \dots, N$, denote real valued measurements, we obtain the estimator of Gini's mean difference.

Our analysis of the distribution of $\hat{u}$ uses linearization by means of Hoeffding's ([18]) decomposition. We approximate $\hat{u}$ by a linear statistic L of the form

$$L = \sum_{k=1}^{h} L_k, \qquad L_k = \sum_{x \in \mathbb{X}_k} g_k(x).$$

Choosing (first order influence) functions g_k that minimize the expectation $\mathbf{E}|\hat{u} - u - L|^2$ and denoting the remaining quadratic part Q we obtain Hoeffding's decomposition

$$\hat{u} = u + L + Q. \qquad (3)$$

Note that the linear part L and quadratic part Q in (3) are uncorrelated. Explicit formulas for L and Q are given in (20) below.

We shall assume that the variance σ_L^2 of the linear part L is positive, $\sigma_L^2 > 0$. By the central limit theorem, for large $n = n_1 + \ldots + n_h$ and N, the distribution of L/σ_L can be approximated by the standard normal distribution, see [4], [15]. In the case where the linear part dominates the statistic and we have $\sigma^2/\sigma_L^2 \approx 1$, the normal approximation applies to $(\hat{u} - u)/\sigma$ as well. Here σ^2 denotes the variance of $\hat{u}$.

Assuming that sample sizes n_k are large for every k, we construct the one-term Edgeworth expansion for the distribution function $F_u(x) = \mathbf{P}\{\hat{u} - u \le \sigma x\}$ and for the distribution function $F_S(x) = \mathbf{P}\{\hat{u} - u \le xS\}$ of Studentized $U-$statistic. S^2 denotes the classical (delete-one) jackknife estimator of the variance σ^2,

$$S^2 = \sum_{k=1}^{h} q_k \frac{n_k - 1}{n_k} v_k^2, \qquad v_k^2 = \sum_{i=1}^{n_k} (\hat{u}(\mathbb{X}_{k|i}) - \overline{u}_k)^2, \qquad \overline{u}_k = \frac{1}{n_k} \sum_{k=1}^{n_k} \hat{u}(\mathbb{X}_{k|i}). \qquad (4)$$

Here $\mathbb{X}_{k|i}$ denotes the STSRS sample obtained from $\mathbb{X}$ by removing the observation $X_{k.i}$.

Theorem 2.1 *The one-term Edgeworth expansions of* $F_u(x)$ *and* $F_S(x)$ *are functions* $G_u(x)$ *and* $G_S(x)$,

$$G_u(x) = \Phi(x) - \frac{\alpha + 3\kappa}{6\sigma^3}\Phi'(x)(x^2-1), \tag{5}$$

$$G_S(x) = \Phi(x) + \frac{\alpha + \alpha' x^2 + 3\kappa(x^2+1)}{6\sigma^3}\Phi'(x). \tag{6}$$

Here Φ *denotes the standard normal distribution function and* Φ' *denotes its derivative. The moments* α, α' *and* κ *are given in (8) below.*

Denote

$$\tau_k^2 = N_k p_k q_k, \qquad p_k = n_k/N_k, \qquad q_k = (N_k - n_k)/N_k \tag{7}$$

and introduce the moments

$$\alpha = \sum_{r=1}^{h}(q_r - p_r)\tau_r^2\alpha_r, \qquad \alpha' = \sum_{r=1}^{h}(1+q_r)\tau_r^2\alpha_r, \tag{8}$$

$$\kappa = \sum_{r=1}^{h}\tau_r^4\kappa_{rr} + 2\sum_{1\le k<r\le h}\tau_k^2\tau_r^2\kappa_{kr}.$$

Here

$$\alpha_k = \mathbf{E}g_k^3(X_{k.1}) = \frac{1}{N_k}\sum_{x\in\mathcal{X}_k} g_k^3(x),$$

$$\kappa_{kk} = \mathbf{E}\psi_k(X_{k.1}, X_{k.2})g_k(X_{k.1})g_k(X_{k.2}) = \frac{1}{\binom{N_k}{2}}\sum_{\{x,y\}\in\mathcal{X}_k}\psi_k(x,y)g_k(x)g_k(y),$$

$$\kappa_{kr} = \mathbf{E}\psi_{kr}(X_{k.1}, X_{r.1})g_k(X_{k.1})g_r(X_{r.1}) = \frac{1}{N_k N_r}\sum_{x\in\mathcal{X}_k,\, y\in\mathcal{X}_r}\psi_{kr}(x,y)g_k(x)g_r(y).$$

Functions $\psi_k(X_{k.i}, X_{k.j})$ and $\psi_{kr}(X_{k.i}, X_{r.j})$ (sometimes called the second order influence functions) reflect the contribution of pairs of observations $\{X_{k.i}, X_{k.j}\}$ and $\{X_{k.i}, X_{r.j}\}$. Explicit expressions of g_k and ψ_k, ψ_{kr} are given in (21) and (22) below.

Remark 2.1. Theorem 2.1 provides *formal* Edgeworth expansions in the asymptotic framework where the number of strata remains bounded and, for every k, *sample sizes* n_k *diverge to infinity* and $\sigma_L^2/\sigma^2 \to 1$. To be more precise assume that we have a sequence of stratified populations

$\mathcal{X}^{(\nu)} = \mathcal{X}_1^{(\nu)} \cup \ldots \cup \mathcal{X}_h^{(\nu)}$ of sizes $N^{(\nu)} = N_1^{(\nu)} + \ldots + N_h^{(\nu)}$, and sequence of STSRS samples $\mathbb{X}^{(\nu)} = (\mathbb{X}_1^{(\nu)}, \ldots, \mathbb{X}_h^{(\nu)})$ of sizes $n^{(\nu)} = n_1^{(\nu)} + \ldots + n_h^{(\nu)}$. Assume that for every $k = 1, \ldots, h$ the numbers $n_{*k} = \min\{n_k^{(\nu)}, N_k^{(\nu)} - n_k^{(\nu)}\}$ diverge to infinity and $\sigma_L^2/\sigma^2 \to 1$ as $\nu \to \infty$. In this case one can prove that the expansions (5), (6) provide approximations to the distribution functions F_u and F_S that improve upon the normal approximation if, in addition, the linear part L satisfies a Cramér type condition and the kernel t satisfies appropriate moment conditions. Pushing harder on the methods used in this paper it is possible to establish $O(n_*^{-1})$ bound for the remainder $\sup_x |F_u(x) - G_u(x)|$ (under $4+\delta$ moment conditions) and $o(n_*^{-1/2})$ bound for the remainder $\sup_x |F_S(x) - G_S(x)|$ (under $6+\delta$ moment conditions) in the particular case where $n_{*k} \approx n_{*r}$, $1 \le k < r \le h$. Here $n_* = n_{*1} + \ldots + n_{*h}$ denotes the "actual sample size". Rigorous proofs of such bounds in the case of simple random sampling ($h = 1$) are given in [7], [10].

Remark 2.2. Normal approximation of $U-$statistics based on samples without replacement was studied in [8], [21], [25]. One term Edgeworth expansion for finite population $U-$ statistics based on simple random samples without replacement was established in [20], see also [10]. A similar result for Studentized $U-$statistic was shown in [7]. Theorem 2.1 extends these results to *stratified* samples. The extension is non-trivial: a more complex structure of Hoeffding's decomposition for STSRS samples yields new formulae for cumulants. In particular, the cumulant $\alpha + 3\kappa$ defining the Edgeworth correction term of (5) includes the sum $\sum_{r<k} \tau_r^2 \tau_k^2 \kappa_{kr}$, see (8), reflecting the contribution of cross strata interactions.

One term Edgeworth expansions for a linear STSRS statistic were obtained in [11], [12]. Theorem 2.1 extends these results to a more general class of U statistics.

Remark 2.3. Edgeworth expansions also provide a theoretical tool for analysis and design of resampling approximations like bootstrap and empirical Edegworth expansions, see [1], [5], [12], [16], [17], [22], [24]. In the next section we construct empirical Edgeworth expansions. The bootstrap approximation of STSRS $U-$statistics will be studied elsewhere.

3 Empirical Edgeworth expansion

Empirical Edgeworth expansions of statistics of *independent observations* were studied in [3], [5], [16], [17], [22]. Here we construct empirical Edgeworth expansions that are suited to $U-$statistics based on STSRS samples.

Let us define the jackknife estimators $\hat{\alpha}_k$ and $\hat{\kappa}_{kr}$ of the moments α_k and κ_{kr}. Substitution of these estimators in (8) give jackknife estimators $\hat{\alpha}, \hat{\alpha}', \hat{\kappa}$ of α, α', κ. Put

$$\hat{\alpha}_k = \frac{1}{n_k} \sum_{i=1}^{n_k} V_{k|i}^3, \quad \hat{\kappa}_{kk} = \frac{1}{\binom{n_k}{2}} \sum_{1 \le i < j \le n_k} W_{k|ij} V_{k|i} V_{k|j}, \quad 1 \le k \le h, \tag{9}$$

$$\hat{\kappa}_{kr} = \frac{1}{n_k n_r} \sum_{i=1}^{n_k} \sum_{j=1}^{n_r} W_{kr|ij} V_{k|i} V_{r|j}, \qquad 1 \le k < r \le h. \tag{10}$$

Here we denote

$$V_{k|i} = \overline{u}_k - \hat{u}(\mathbb{X}_{k|i}), \tag{11}$$

$$W_{k|ij} = \tilde{u}_k - \frac{n_k - 1}{n_k} \overline{u}_{k(k)|i} - \frac{n_k - 1}{n_k} \overline{u}_{k(k)|j} + \frac{n_k - 2}{n_k} \hat{u}(\mathbb{X}_{k|ij}), \tag{12}$$

$$W_{kr|ij} = \tilde{u}_{kr} - \overline{u}_{k(r)|i} - \overline{u}_{r(k)|j} + \hat{u}(\mathbb{X}_{kr|ij}) \tag{13}$$

and, for $1 \le k, r \le h$, $k \ne r$ we write

$$\tilde{u}_k = \frac{1}{\binom{n_k}{2}} \sum_{1 \le i < j \le n_k} \hat{u}(\mathbb{X}_{k|ij}), \qquad \tilde{u}_{kr} = \frac{1}{n_k n_r} \sum_{i=1}^{n_k} \sum_{1=j}^{n_r} \hat{u}(\mathbb{X}_{kr|ij}),$$

$$\overline{u}_{k(k)|i} = \frac{1}{n_k - 1} \sum_{1 \le j \le n_k, j \ne i} \hat{u}(\mathbb{X}_{k|ij}), \qquad \overline{u}_{k(r)|i} = \frac{1}{n_r} \sum_{j=1}^{n_r} \hat{u}(\mathbb{X}_{kr|ij}). \tag{14}$$

Here $\mathbb{X}_{k|ij}$ denotes the STSRS sample obtained from $\mathbb{X}$ by removing the observations $X_{k.i}$ and $X_{k.j}$ from $\mathbb{X}_k$; $\mathbb{X}_{kr|ij}$ denotes the STSRS sample obtained from $\mathbb{X}$ by removing the observation $X_{k.i}$ from $\mathbb{X}_k$ and $X_{r.j}$ from $\mathbb{X}_r$.

Define $\hat{\alpha}$, $\hat{\alpha}'$ and $\hat{\kappa}$ by formulas (8), where α_k and κ_{kr} are replaced by their estimators $\hat{\alpha}_k$ and $\hat{\kappa}_{kr}$. One can show under appropriate moment conditions, see Appendix below, that as $n_k \to \infty$

$$\frac{\hat{\alpha}}{S^3} - \frac{\alpha}{\sigma^3} = o_P(1), \qquad \frac{\hat{\alpha}'}{S^3} - \frac{\alpha'}{\sigma^3} = o_P(1), \qquad \frac{\hat{\kappa}}{S^3} - \frac{\kappa}{\sigma^3} = o_P(1). \tag{15}$$

Therefore the empirical Edgeworth expansions

$$H_u(x) = \Phi(x) - \frac{\hat{\alpha} + 3\hat{\kappa}}{6S^3} \Phi'(x)(x^2 - 1), \tag{16}$$

$$H_S(x) = \Phi(x) + \frac{\hat{\alpha} + \hat{\alpha}' x^2 + 3\hat{\kappa}(x^2 + 1)}{6S^3} \Phi'(x). \tag{17}$$

match G_u and G_S (in probability). In this case the empirical Edgeworth expansions provide approximations which are second-order correct in probability.

4 Simulation study

In this section, we examine the performance of the Edgeworth expansion (6) and the empirical Edgeworth expansion (17) by simulation.

Two stratified populations $\mathcal{X}^{(r)} \subset \mathbb{R}$, $r = 1, 2$, are generated, each consisting of three strata of sizes $N_1 = N_2 = 500$ and $N_3 = 200$. For Gini's mean difference estimator $\hat{u}$ (defined in (1) and below) based on a stratified sample without replacement with sizes $n_1 = n_2 = n_3 = 50$, we evaluate approximation errors $\Delta_{\Phi}^{(r)}(x) = F_S(x) - \Phi(x)$, $\Delta_E^{(r)}(x) = F_S(x) - \tilde{G}_S(x)$, and $\Delta_{EE}^{(r)}(x) = F_S(x) - \tilde{H}_S(x)$. Here F_S denotes the distribution function of the Studentized statistic $(\hat{u} - u)/S$ obtained by a Monte-Carlo simulation and $\tilde{G}_S$, respectively, $\tilde{H}_S$ denote the distribution functions which are obtained from G_S and H_S via the transformation $\tilde{f}(x) := \sup_{y \le x}((0 \vee f(y)) \wedge 1)$.

Table 1. Approximation accuracy

$x =$	-2.33	-1.65	-1.29	-0.68	0.0	0.68	1.29	1.65	2.33
$\Phi(x) \approx$	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
$10^3\,\Delta_{\Phi}^{(1)}(x) \approx$	7.76	12.9	13.8	9.63	3.68	7.87	10.7	8.97	3.60
$10^3\Delta_E^{(1)}(x) \approx$	2.29	2.05	2.0	1.69	1.08	0.41	-0.91	-1.96	-1.86
$10^3\Delta_{EE}^{(1)}(x) \approx$	2.94	3.41	3.57	3.01	1.99	1.72	0.65	-0.59	-1.22
$10^3\,\Delta_{\Phi}^{(2)}(x) \approx$	19.5	32.9	36.2	24.9	10.2	17.9	25.3	20.5	7.22
$10^3\Delta_E^{(2)}(x) \approx$	6.38	6.19	6.93	4.78	0.92	-2.26	-3.94	-6.23	-2.7
$10^3\Delta_{EE}^{(2)}(x) \approx$	13.3	20.6	23.1	17.2	8.24	10.1	12.25	8.18	1.01

Table 1 demonstrates that the Edgeworth expansion improves upon the normal approximation (as expected). The empirical Edgeworth expansion outperforms the normal approximation, but tends to be less accurate than the true Edgeworth expansion.

Table 2 gives values of the ratios α/σ^3, α'/σ^3 and κ/σ^3 which define the Edgeworth expansion G_S. It also provides estimated values of the mean square errors of their estimators $\alpha_* = \hat{\alpha}/S^3$, $\alpha'_* = \hat{\alpha}'/S^3$ and $\kappa_* = \hat{\kappa}/S^3$, cf. (15). The latter define the empirical Edgeworth expansion H_S.

Table 2. Moments and estimated MSE of jackknife estimators

	α/σ^3	α'/σ^3	κ/σ^3	MSE(α_*)	MSE(α'_*)	MSE(κ_*)
$r = 1$	0.11	0.29	-0.023	0.0004	0.0028	0.00001
$r = 2$	0.21	0.59	-0.023	0.0028	0.0223	0.00004

We conclude this section with a short description of the study populations $\mathcal{X}^{(r)} = \mathcal{X}_1^{(r)} \cup \mathcal{X}_2^{(r)} \cup \mathcal{X}_3^{(r)}$, $r = 1, 2$. The elements of the first population are realized values of independent normal variables: $\mathcal{X}_1^{(1)} \sim \mathcal{N}(2,1)$; $\mathcal{X}_2^{(1)} \sim \mathcal{N}(3,4)$; $\mathcal{X}_3^{(1)} \sim \mathcal{N}(6,9)$. The second population combines symmetric and skewed strata: $\mathcal{X}_1^{(2)} = \{1 + 0.004(i-1), 1 \le i \le N_1\}$ (uniform distribution on $[1,3]$); $\mathcal{X}_2^{(2)} = 2 - \ln(1 - 0,002(i-1)), 1 \le i \le N_2\}$ (exponential distribution); $\mathcal{X}_3^{(2)} \sim \chi^2 + 3$.

5 Proof of (5)

Before the proof we collect some useful facts about Hoeffding's decomposition for STSRS samples, see [8], [9] for proofs and details.

5.1. Hoeffding decomposition. Let us consider a general $U-$statistic

$$U = U(\mathbb{X}) = \sum_{1\le k\le h} \sum_{\{x_i,x_j\}\subset\mathbb{X}_k} t_k(x_i,x_j) + \sum_{1\le k<r\le h} \sum_{x_i\in\mathbb{X}_k} \sum_{y_j\in\mathbb{X}_r} t_{kr}(x_i,y_j), \quad (18)$$

where t_k are symmetric functions. Hoeffding's decomposition

$$U = \mathbf{E}U + L + Q, \quad (19)$$

decomposes $U - \mathbf{E}U$ into the sum of the linear part L and quadratic part Q which are centered and uncorrelated. Here

$$L = \sum_{1\le k\le h} L_k, \qquad Q = \sum_{1\le k\le r\le h} Q_{kr}, \quad (20)$$
$$L_k = \sum_{x\in\mathbb{X}_k} g_k(x), \qquad Q_{kk} = \sum_{\{x,y\}\subset\mathbb{X}_k} \psi_k(x,y), \qquad Q_{kr} = \sum_{x\in\mathbb{X}_k} \sum_{y\in\mathbb{X}_r} \psi_{kr}(x,y).$$

The functions g_k, ψ_k and ψ_{kr}, for $k<r$, are defined as follows. Denote $\tilde{t}_k(x,y) = t_k(x,y) - \mathbf{E}t_k(X_{k.1},X_{k.2})$ and $\tilde{t}_{kr}(x,y) = t_{kr}(x,y) - \mathbf{E}t_{kr}(X_{k.1},X_{r.1})$. We have

$$g_k(x) = (n_k - 1)t_k^*(x) + \sum_{1\le j\le h,\, j\ne k} n_j t_{k|j}^*(x), \quad (21)$$

where

$$t_k^*(x) = \frac{N_k - 1}{N_k - 2}\mathbf{E}(\tilde{t}_k(X_{k.1},X_{k.2})|X_{k.1} = x),$$

and, for $k<r$,

$$t_{k|r}^*(x) = \mathbf{E}(\tilde{t}_{kr}(X_{k.1},X_{r.1})|X_{k.1} = x), \quad t_{r|k}^*(x) = \mathbf{E}(\tilde{t}_{kr}(X_{k.1},X_{r.1})|X_{r.1} = x).$$

Furthermore, we have for $k < r$

$$\psi_k(x,y) = \tilde{t}_k(x,y) - t_k^*(x) - t_k^*(y), \quad \psi_{kr}(x,y) = \tilde{t}_{kr}(x,y) - t_{k|r}^*(x) - t_{r|k}^*(y). \tag{22}$$

For $r > k$ we denote $\psi_{rk}(y,x) := \psi_{kr}(x,y)$.

Note that for every k and r we have

$$\mathbf{E}g_k(X_{k.1}) = 0, \qquad \mathbf{E}\psi_k(X_{k.1}, X_{k.2}) = 0, \qquad \mathbf{E}\psi_{kr}(X_{k.1}, X_{r.1}) = 0. \tag{23}$$

Moreover, the following identities hold

$$\mathbf{E}(\psi_k(X_{k.i}, X_{k.j})|X_{k.j}) = 0, \qquad i \neq j, \tag{24}$$

$$\mathbf{E}(\psi_{kr}(X_{k.i}, X_{r.j})|X_{r.j}) = 0, \qquad \mathbf{E}(\psi_{kr}(X_{k.i}, X_{r.j})|X_{k.i}) = 0. \tag{25}$$

It follows from (24) and (25) that the parts L and Q are uncorrelated and the variances σ_U^2, σ_L^2 and σ_Q^2 of U, L and Q satisfy

$$\sigma_U^2 = \sigma_L^2 + \sigma_Q^2. \tag{26}$$

Let $\sigma^2(L_k)$, $\sigma^2(Q_{kk})$, and $\sigma^2(Q_{kr})$ denote the variance of L_k, Q_{kk}, and Q_{kr}. Using (24) and (25) one can show (see Bloznelis (2003b)) that

$$\sigma_L^2 = \sum_{1 \le k \le h} \sigma^2(L_k), \qquad \sigma_Q^2 = \sum_{1 \le k \le r \le h} \sigma^2(Q_{kr}), \tag{27}$$

$$\sigma^2(L_k) = \frac{N_k}{N_k - 1}\tau_k^2 \sigma_k^2, \qquad \sigma^2(Q_{kk}) = \frac{\binom{N_k - n_k}{2}\binom{n_k}{2}}{\binom{N_k - 2}{2}}\sigma_{kk}^2,$$

$$\sigma^2(Q_{kr}) = \frac{N_k}{N_k - 1}\frac{N_r}{N_r - 1}\tau_k^2 \tau_r^2 \sigma_{kr}^2,$$

where we denote

$$\sigma_k^2 = \mathbf{E}g_k^2(X_{k.1}), \quad \sigma_{kk}^2 = \mathbf{E}\psi_{kk}^2(X_{k.1}, X_{k.2}), \quad \sigma_{kr}^2 = \mathbf{E}\psi_{kr}^2(X_{k.1}, X_{r.1}). \tag{28}$$

Remark 5.1. Note that every $U-$statistic defined by (18) can be written in the form (1). Just denote $t(x,u) := t_j(x,y)w_j^{-1}$, for $\{x,y\} \subset \mathcal{X}_j$ and $t(x,y) = t_{jr}(x,y)w_{jr}^{-1}$ for $x \in \mathcal{X}_j$, $y \in \mathcal{X}_r$, $j < r$.

5.2. Proof of (5).

We shall show that

$$G_U(x) = \Phi(x) - \frac{\alpha + 3\kappa}{6\sigma_U^3}\Phi'(x)(x^2 - 1), \tag{29}$$

is the one-term Edgeworth expansion of the probability distribution function $F_U = \mathbf{P}\{U - \mathbf{E}U \le \sigma_U x\}$ of a general $U-$statistics given by (18). The quantities α, κ are defined by formula (8) applied to the kernels (21) and (22).

We assume without loss of generality that $\mathbf{E}U = 0$ and $\sigma_U^2 = 1$. In order to construct an Edgeworth expansion of F_U we write an asymptotic expansion to the characteristic function

$$f_U(t) = \mathbf{E}\exp\{itU\}$$

and apply Fourier inversion formula. We shall show that the function

$$g(t) = g_L(t) + g_Q(t), \tag{30}$$

where

$$g_L(t) = \exp\{-\frac{t^2}{2}\}\Big(1 + \frac{(it)^3}{6}\alpha\Big), \qquad g_Q(t) = \exp\{-\frac{t^2}{2}\}\frac{(it)^3}{2}\kappa \tag{31}$$

is an one-term expansion of the function $f_U(t)$. An application of the inversion formula (see, e.g. Chung (1974), page 159) gives the one-term asymptotic expansion

$$G_u(x) = \frac{1}{2} + \frac{i}{2\pi}\lim_{M\to\infty}\mathbf{V}.\mathbf{P}.\int_{|t|\le M} e^{-itx} g(t) t^{-1} dt.$$

Here $\mathbf{V}.\mathbf{P}.\int_R = \lim_{h\downarrow 0}(\int_{-\infty}^{-h} + \int_h^{+\infty})$ denotes Cauchy's Principal Value.

Let us prove that $g(t)$ is the one-term asymptotic expansion of $f_U(t)$. For simplicity we consider the case of two strata only. It is convenient to represent the characteristic function $f_U(t)$ in the Edrős-Rényi (1959) form, see (35) below. Using this representation we replace $f_U(t)$ by $f_L(t) + 2^{-1}\kappa(it)^3 f_L(t)$, see Lemma 8.1 below. Here $f_L(t) = \mathbf{E}\exp\{itL\}$ denotes the characteristic function of the random variable L. Finally we replace $f_L(t)$ by its asymptotic expansion $g_L(t)$. To see that $g_L(t)$ is indeed an asymptotic expansion of $f_L(t)$ we split $f_L(t) = f_{L_1}(t) f_{L_2}(t)$ and, for every $k = 1, 2$, we replace $f_{L_k}(t) = \mathbf{E}\exp\{itL_k\}$ by its one-term asymptotic expansion (see, Robinson (1978), formula (13))

$$f_{L_k}(t) \approx \exp\{-t^2\frac{\sigma^2(L_k)}{2}\}(1 + \frac{(it)^3}{6}(q_k - p_k)\tau_k^2\alpha_k).$$

Collecting the main terms and using the approximation $\sigma_U^2 \approx \sigma_L^2$ we obtain

$$f_L(t) \approx \exp\{-t^2\sigma_L^2/2\}(1 + \frac{(it)^3}{6}\alpha) \approx g_L(t),$$

thus completing the proof of the fact that $g(t)$ is the one-term expansion of $f_U(t)$.

In the remaining part of the proof we construct the Edrős-Rényi representation for $f_U(t)$ and prove Lemma 8.1. The proof of the lemma is given in Section 8.

Write $\mathcal{U} = \mathcal{X} \cup \mathcal{Y}$, where $\mathcal{X} = \{x_1, \ldots, x_{N_1}\}$ and $\mathcal{Y} = \{y_1, \ldots, y_{N_2}\}$. Let $X^* = (X_1, \ldots, X_{N_1})$ and $Y^* = (Y_1, \ldots, Y_{N_2})$ denote random permutations of the sequences $\{x_i\}$ and $\{y_j\}$. We assume that X^* and Y^* are independent. The random vector $(\mathbb{X}, \mathbb{Y})$, where $\mathbb{X} = \{X_1, \ldots, X_{n_1}\}$ and $\mathbb{Y} = \{Y_1, \ldots, Y_{n_2}\}$, represents the STSI sample drawn from the population $\mathcal{U}$.

We shall write $g_x, g_y, \psi_x, \psi_y, \psi_{xy}$ and $\sigma_x, \sigma_y, \sigma_{xx}, \sigma_{yy}, \sigma_{xy}$, $\kappa_x, \kappa_y, \kappa_{xy}$ instead of g_1, g_2, ψ_1, ψ_2, ψ_{12} and $\sigma_1, \sigma_2, \sigma_{11}, \sigma_{22}, \sigma_{12}$, κ_1, κ_2, κ_{12}. Similarly, we write $L_x, L_y, Q_x, Q_y, Q_{xy}$ instead of L_1, L_2, Q_{11}, Q_{22}, Q_{12} and p_x, q_x, p_y, q_y instead of p_1, q_1, p_2, q_2.

Let

$$\nu = \{\nu_1, \ldots, \nu_{N_1}\} \qquad \text{and} \qquad \eta = \{\eta_1, \ldots, \eta_{N_2}\}$$

be independent sequences of independent Bernoulli random variables with probabilities

$$\mathbf{P}\{\nu_i = 1\} = 1 - \mathbf{P}\{\nu_i = 0\} = p_x, \qquad \mathbf{P}\{\eta_j = 1\} = 1 - \mathbf{P}\{\eta_j = 0\} = p_y,$$

$1 \le i \le N_1$ and $1 \le j \le N_2$. After we write U in the form (19), i.e., $U = L_x + L_y + Q_x + Q_y + Q_{xy}$, it is easy to see that the distribution of U coincides with the conditional distribution of the sum

$$U^* = L_x^* + L_y^* + Q_x^* + Q_y^* + Q_{xy}^*, \tag{32}$$

given the events $\{\nu_1 + \ldots + \nu_{N_1} = n_1\}$ and $\{\eta_1 + \ldots + \eta_{N_2} = n_2\}$. Here

$$L_x^* = \sum_{i=1}^{N_1} g_x(X_i)\nu_i, \quad L_y^* = \sum_{j=1}^{N_2} g_y(Y_j)\eta_j, \quad Q_{xy}^* = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \psi_{xy}(X_i, Y_j)\nu_i\eta_j,$$

$$Q_x^* = \sum_{1 \le i < j \le N_1} \psi_x(X_i, X_j)\nu_i\nu_j, \qquad Q_y^* = \sum_{1 \le i < j \le N_2} \psi_y(Y_i, Y_j)\eta_i\eta_j.$$

Denote $\omega_i = \nu_i - p_x$ and $\xi_j = \eta_j - p_y$ and let $\mathbb{L}_x, \mathbb{L}_y, \mathbb{Q}_x, \mathbb{Q}_y, \mathbb{Q}_{xy}$ be defined in the same way as $L_x^*, L_y^*, Q_x^*, Q_y^*, Q_{xy}^*$ but with ν_i and η_j replaced by ω_i and ξ_j respectively. It follows from (24), (25) that almost surely

$$\mathbb{L}_x = L_x^*, \quad \mathbb{L}_y = L_y^*, \quad \mathbb{Q}_x = Q_x^*, \quad \mathbb{Q}_y = Q_y^*, \quad \mathbb{Q}_{xy} = Q_{xy}^*.$$

In particular, from (32) we have $U^* = \mathbb{U}$, where

$$\mathbb{U} = \mathbb{L} + \mathbb{Q}, \qquad \text{where} \quad \mathbb{L} = \mathbb{L}_x + \mathbb{L}_y, \qquad \mathbb{Q} = \mathbb{Q}_x + \mathbb{Q}_y + \mathbb{Q}_{xy}. \tag{33}$$

Denote $S_x = \sum_{i=1}^{N_1} w_i$ and $S_y = \sum_{j=1}^{N_2} \xi_j$ and write

$$H = t\mathbb{U} + (s_1/\tau_1)S_x + (s_2/\tau_2)S_y, \qquad \tilde{\mathbb{L}} = t\mathbb{L} + (s_1/\tau_1)S_x + (s_2/\tau_2)S_y. \tag{34}$$

Note that the distribution of U coincides with the conditional distribution of $\mathbb{U}$ given the events $\{S_x = 0\}$ and $\{S_y = 0\}$. Using this fact we can write the characteristic function $f_U(t)$ in the following way, see Erdős and Rényi (1959),

$$f_U(t) = \lambda_1\lambda_2 \int_{-\pi\tau_1}^{\pi\tau_1} ds_1 \int_{-\pi\tau_1}^{\pi\tau_1} ds_2 \mathbf{E} \exp\{iH\}. \tag{35}$$

Here

$$\lambda_1^{-1} = 2\pi\tau_1 \mathbf{P}\{S_x = 0\}, \qquad \lambda_2^{-1} = 2\pi\tau_2 \mathbf{P}\{S_y = 0\}.$$

6 Proof of (6)

The proof goes along the same lines as that of Theorem 1 of Bloznelis (2003a), who considered the case of the simple random sample drawn without replacement.

We assume without loss of generality that $\sigma^2 = 1$, where σ^2 denotes the variance of $\hat{u}(\mathbb{X})$.

In order to fix notation let g_k, ψ_k and ψ_{kr} denote the kernels of the decomposition $\hat{u} = u + L + Q$, see (3), defined by (21) and (22).

The proof consists of two steps. Firstly we replace S by the short stochastic expansion $1 + 2^{-1}L^*$, where

$$L^* = \sum_{k=1}^{h} L_k^*, \qquad L_k^* = \sum_{i=1}^{n_k} f_k(X_{k.i}). \tag{36}$$

Here $f_k = f_{k.1} + f_{k.2} + f_{k.3}$ with

$$\begin{aligned}
f_{k.1}(x) &= q_k(g_k^2(x) - \sigma_k^2), \qquad \sigma_k^2 = \mathbf{E} g_k^2(X_{k.1}),\\
f_{k.2}(x) &= 2q_k n_k \frac{n_k - 1}{n_k - 2} \frac{N_k - 1}{N_k - 2} \mathbf{E}(\psi_k(X_{k.1}, X_{k.2}) g_k(X_{k.2}) | X_{k.1} = x),\\
f_{k.3}(x) &= 2 \sum_{r=1}^{h} \mathbb{I}_{\{r \neq k\}} q_r \frac{n_r^2}{n_r - 1} E(\psi_{kr}(X_{k.1}, X_{r.1}) g_r(X_{r.1}) | X_{k.1} = x).
\end{aligned}$$

Secondly, we apply (29) to the probability

$$P\{\hat{u} - u \le x(1 + 2^{-1}L^*)\} = P\{\tilde{U} \le x\},$$

where

$$\tilde{U} = \hat{u} - u - 2^{-1}xL^*$$

is a $U-$statistic of the form (18).

6.1. Step.1. Here we replace S by $1+2^{-1}L^*$. It is convenient to represent the difference $\overline{u}_k - \hat{u}(\mathbb{X}_{k|i})$ in the following way

$$\begin{aligned}
&\overline{u}_k - \hat{u}(\mathbb{X}_{k|i}) = Z_k(i) + W_k(i) + M_k(i),\\
&Z_k(i) = \frac{n_k}{n_k - 1} \sum_{1\le j\le n_k} g_k(X_{k.j})(\mathbb{I}_{\{i=j\}} - \frac{1}{n_k}),\\
&W_k(i) = \frac{n_k}{n_k - 2} \sum_{1\le j<r\le n_k} \psi_k(X_{k.j}, X_{k.r})(\mathbb{I}_{\{i\in\{j,r\}\}} - \frac{2}{n_k}),\\
&M_k(i) = \frac{n_k}{n_k - 1} \sum_{r=1}^{h} \mathbb{I}_{\{r\ne k\}} \sum_{s=1}^{n_r}\sum_{j=1}^{n_k} \psi_{kr}(X_{k.j}, X_{r.s})(\mathbb{I}_{\{i=j\}} - \frac{1}{n_k}).
\end{aligned}$$

Write $v_k^2 = \tilde{v}_k + r_k$, where

$$\tilde{v}_k = \sum_{i=1}^{n_k} Z_k^2(i) + 2\sum_{i=1}^{n_k} Z_k(i)(W_k(i) + M_k(i)), \qquad r_k = \sum_{k=1}^{n_k}(W_k(i) + M_k(i))^2. \tag{37}$$

A calculation shows that

$$\begin{aligned}
&\sum_{i=1}^{n_k} Z_k^2(i) = \frac{n_k}{n_k - 1}\sum_{i=1}^{n_k} g_k^2(X_{k.i}) - \frac{2n_k}{(n_k-1)^2} H_{k.1},\\
&\sum_{i=1}^{n_k} Z_k(i)W_k(i) = \frac{n_k^2}{(n_k-1)(n_k-2)}(H_{k.4} - \frac{2}{n_k}H_{k.2}H_{k.3}),\\
&\sum_{i=1}^{n_k} Z_k(i)M_k(i) = \frac{n_k^2}{(n_k-1)^2}(H_{k.5} - \frac{1}{n_k}H_{k.2}H_{k.6}).
\end{aligned}$$

Here we denote

$$\begin{aligned}
&H_{k.1} = \sum_{1\le i<j\le n_k} g_k(X_{k.i})g_k(X_{k.j}),\\
&H_{k.2} = \sum_{i=1}^{n_k} g_k(X_{k.i}), \quad H_{k.3} = \sum_{1\le i<j\le n_k} \psi_k(X_{k.i}, X_{k.j}),
\end{aligned}$$

$$H_{k.4} = \sum_{1 \le i < j \le n_k} \psi_k(X_{k.i}, X_{k.j})(g_k(X_{k.i}) + g_k(X_{k.j})),$$

$$H_{k.5} = \sum_{r=1}^{h} \mathbb{I}_{\{r \neq k\}} \sum_{s=1}^{n_r} \sum_{i=1}^{n_k} \psi_{kr}(X_{k.i}, X_{r.s}) g_k(X_{k.i}),$$

$$H_{k.6} = \sum_{r=1}^{h} \mathbb{I}_{\{r \neq k\}} \sum_{s=1}^{n_r} \sum_{i=1}^{n_k} \psi_{kr}(X_{k.i}, X_{r.s}).$$

Collecting these formulas and (37) in (4) we obtain

$$S^2 = 1 + \sum_{k=1}^{h} \sum_{i=1}^{n_k} f_{k.1}(X_{k.i}) + Q^* + Q^\star + R, \tag{38}$$

$$Q^* = \sum_{k=1}^{h} Q_k^*, \qquad Q_k^* = 2q_k \frac{n_k}{n_k - 2} H_{k.4},$$

$$Q^\star = \sum_{k=1}^{h} Q_k^\star, \qquad Q_k^\star = 2q_k \frac{n_k}{n_k - 1} H_{k.5}.$$

Here $R = R_1 - R_2 - R_3 - R_4 - R_5$, where

$$R_1 = \sum_{k=1}^{h} q_k \frac{n_k - 1}{n_k} r_k, \qquad R_2 = 1 - \sum_{k=1}^{h} q_k n_k \sigma_k^2,$$

$$R_3 = 2 \sum_{k=1}^{h} q_k \frac{1}{n_k - 1} H_{k.1}, \quad R_4 = 4 \sum_{k=1}^{h} q_k \frac{1}{n_k - 2} H_{k.2} H_{k.3},$$

$$R_5 = 2 \sum_{k=1}^{h} q_k \frac{1}{n_k - 1} H_{k.2} H_{k.6}.$$

Furthermore, using Hoeffding's decomposition (see (19)) we approximate Q^* and $Q^\star$ by the linear statistics

$$Q^* = \sum_{k=1}^{h} \sum_{i=1}^{n_k} f_{k.2}(X_{k.i}) + R_7, \tag{39}$$

$$Q^\star = \sum_{k=1}^{h} \sum_{i=1}^{n_k} f_{k.3}(X_{k.i}) + R_8. \tag{40}$$

Under appropriate moment conditions one can show that the remainders R_i, $1 \le i \le 7$ are negligible, see Bloznelis (2003a). In order to show that the remainder R_8 is negligible it is convenient to write $Q^\star$ in the form

$$Q^\star = \sum_{k<r} Q^\star_{kr},$$
$$Q^\star_{kr} = \sum_{i=1}^{n_k}\sum_{j=1}^{n_r} 2\psi_{kr}(X_{k.i}, X_{r.j})\Big(q_k \frac{n_k}{n_k-1} g_k(X_{k.i}) + q_r \frac{n_r}{n_r-1} g_r(X_{r.j})\Big).$$

Then we approximate $Q^\star$ by the linear part of decomposition (see (19)) and obtain the approximation (40).

Finally, combining (38) and (39), (40) we obtain $S^2 \approx 1 + L^*$. Furthermore, for small L^* we can write $S \approx \sqrt{1+L^*} \approx 1 + 2^{-1}L^*$.

6.2. Step 2. Here we construct the one-term Edgeworth expansion to the distribution function $P\{\tilde{U} \le x\}$. We are going to apply (29). Firstly, we write Hoeffding's decomposition of $\tilde{U}$, $\tilde{U} = \tilde{L} + Q$. Here $\tilde{L} = L - 2^{-1}xL^*$ denotes the linear part and Q denotes the quadratic part, where L and Q are the same as in (3). Let $\tilde{\sigma}_U^2$ and $\tilde{\sigma}_L^2$ denote the variances of $\tilde{U}$ and $\tilde{L}$ and let $\tilde{\alpha}, \tilde{\alpha}_r, \tilde{\kappa}, \tilde{\kappa}_{rk}$ denote moments of various parts of $\tilde{U}$ defined by (8).

Substituting $\tilde{\alpha}$, $\tilde{\kappa}$, $\tilde{\sigma}_U^2$ in (29) we obtain the one-term Edgeworth expansion $\tilde{G}(x)$ of the probability $P\{\tilde{U} \le \tilde{\sigma}_U x\}$. A calculation shows that $\tilde{\sigma}_U = \sigma + R(\sigma)$, $\tilde{\alpha} = \alpha + R(\alpha)$, $\tilde{\alpha}_k = \alpha_k + R(\alpha_k)$, $\tilde{\kappa} = \kappa + R(\kappa)$, $\tilde{\kappa}_{kr} = \kappa_{kr} + R(\kappa_{kr})$, where $R(\sigma), \dots, R(\kappa_{kr})$ are negligible remainders. Therefore, $\tilde{G}(x)$ can be replaced by $G_u(x)$ (= the one term Edgeworth expansion of $F_u(x) = \mathbf{P}\{\hat{u} - \mathbf{E}\hat{u} \le x\sigma\}$, see (5)). It follows that $G_u(x/\tilde{\sigma}_U)$ is the one-term Edgeworth expansion of $P\{\tilde{U} \le x\}$. In the remaining part of the proof we shall show that $G_u(x/\tilde{\sigma}_U) = G_S(x) + R$, where G_S is given by (6) and where R is a negligible remainder.

Denote $\delta = \tilde{\sigma}_U^2 - 1$ and expand using Taylor's formula

$$\begin{aligned} G_u(\frac{x}{\tilde{\sigma}_U}) &= G_u(x + x(\frac{1}{\tilde{\sigma}_U} - 1)) \\ &= G_u(x) + G_u'(x)x(\frac{1}{\tilde{\sigma}_U} - 1) + R_1 \\ &= G_u(x) + \Phi'(x)x(\frac{1}{\tilde{\sigma}_U} - 1) + R_1 + R_2 \\ &= G_u(x) - 2^{-1}\delta\Phi'(x)x + R_1 + R_2 + R_3. \end{aligned}$$

In the last step we used the approximation (via Taylor's expansion) $\tilde{\sigma}_U^{-1} - 1 \approx -\delta/2$. Here R_i, $i = 1, 2, 3$ are negligible remainders. In order to evaluate the factor δ we apply variance decomposition formula (see, (26)),

$$\tilde{\sigma}_U^2 = \tilde{\sigma}_L^2 + \sigma_Q^2 = \sigma_U^2 + \tilde{\sigma}_L^2 - \sigma_L^2 = 1 + \tilde{\sigma}_L^2 - \sigma_L^2. \tag{41}$$

Therefore, $\delta = \tilde{\sigma}_L^2 - \sigma_L^2$. From (27) we have

$$\delta = \sum_{k=1}^{h} \tau_k^2 \frac{N_k}{N_k - 1} (\tilde{\sigma}_k^2 - \sigma_k^2), \qquad \text{where} \qquad \tilde{\sigma}_k^2 = \mathbf{E}(g_k(X_{k.1}) - 2^{-1} x f_k(X_{k.1}))^2.$$

A calculation shows that

$$\tilde{\sigma}_k^2 = \sigma_k^2 - x\mathbf{E} g_k(X_{k.1}) f_k(X_{k.1}) + r_k, \qquad r_k = x^2 \mathbf{E} f_k^2(X_{k.1}),$$

where

$$\mathbf{E} g_k(X_{k.1}) f_k(X_{k.1}) = \alpha_k q_k + 2 \sum_{r=1}^{h} \tau_r^2 \kappa_{kr}.$$

We obtain

$$\delta = -x \sum_{k=1}^{h} q_k \tau_k^2 \alpha_k - 2x \sum_{k=1}^{h} \tau_k^4 \kappa_{kk} - 4x \sum_{1 \le k < r \le h} \tau_k^2 \tau_r^2 \kappa_{kr} + R_4.$$

Here $R_4 = \sum r_k$ denotes the remainder. One can shown that R_4 is negligible. Substitution of this formula in (41) gives the desired result $G_u(x/\tilde{\sigma}_U) = G_S(x) + R$ with negligible remainder R. The proof of (6) is complete.

7 Consistency of jackknife estimators

We sketch the proof of (15). Assume without loss of generality that $\sigma^2 = 1$. Substitution of (3), (20) in (11), (12), (13) gives

$$\begin{aligned} V_{k|i} &= g_k(X_{k.i}) + R_{k|i}, \\ W_{k|ij} &= \psi_k(X_{k.i}, X_{k.j}) + R_{k|ij}, \\ W_{kr|ij} &= \psi_{kr}(X_{k.i}, X_{r.j}) + R_{kr|ij}, \end{aligned}$$

where

$$\begin{aligned} R_{k|i} &= -\frac{1}{n_k - 1} \sum_{x \in \mathbb{X}'_{k|i}} g_k(x) \\ &+ \sum_{x \in \mathbb{X}'_{k|i}} \psi_k(X_{k.i}, x) - \frac{2}{n_k - 2} \sum_{\{x,y\} \subset \mathbb{X}'_{k|i}} \psi_k(x, y) \\ &+ \sum_{1 \le s \le h,\, s \ne k} \sum_{y \in \mathbb{X}_s} (\psi_{ks}(X_{k.i}, y) - \frac{1}{n_k - 1} \sum_{x \in \mathbb{X}'_{k|i}} \psi_{ks}(x, y)), \end{aligned}$$

$$
\begin{aligned}
R_{k|ij} &= \frac{1}{\binom{n_k-2}{2}} \sum_{\{x,y\}\subset \mathbb{X}'_{k|ij}} \psi_k(x,y) \\
&- \frac{1}{n_k-2} \sum_{x\in \mathbb{X}'_{k|ij}} \psi_k(X_{k.i},x) - \frac{1}{n_k-2} \sum_{x\in \mathbb{X}'_{k|ij}} \psi_k(X_{k.j},x), \\
R_{kr|ij} &= \frac{1}{n_k-1}\frac{1}{n_r-1} \sum_{x\in \mathbb{X}'_{k|i}} \sum_{y\in \mathbb{X}'_{r|j}} \psi_{kr}(x,y) \\
&- \frac{1}{n_k-1} \sum_{x\in \mathbb{X}'_{k|i}} \psi_{kr}(x,X_{r.j}) - \frac{1}{n_r-1} \sum_{x\in \mathbb{X}'_{r|j}} \psi_{kr}(X_{k.i},x).
\end{aligned}
$$

Here we denote $\mathbb{X}'_{k|i} = \mathbb{X}_k \setminus \{X_{k.i}\}$ and $\mathbb{X}'_{k|ij} = \mathbb{X}_k \setminus \{X_{k.i}, X_{k.j}\}$. One can show (under appropriate moment conditions on ψ_k, ψ_{kr} and g_k) that the remainder terms $R_{k|i}$, $R_{k|ij}$, $R_{kr|ij}$ can be neglected. Therefore, we obtain from (9) and (10) that

$$
\begin{aligned}
&\hat\alpha_k \approx \frac{1}{n_k}\sum_{x\in\mathbb{X}_k} g_k^3(x), \qquad \hat\kappa_{kk} \approx \frac{1}{\binom{n_k}{2}} \sum_{\{x,y\}\subset\mathbb{X}_k} \psi_k(x,y) g_k(x) g_k(y), \\
&\hat\kappa_{kr} \approx \frac{1}{n_k}\frac{1}{n_r} \sum_{x\in\mathbb{X}_k}\sum_{y\in\mathbb{X}_r} \psi_{kr}(x,y) g_k(x) g_r(y).
\end{aligned}
$$

Finally, (15) follows by the law of large numbers. A rigorous proof in the case of a simple random sample ($h = 1$) is given in [6].

8 Expansions

Denote, for $r > 0$,

$$
\begin{aligned}
B_{xr} &= \sigma_x^{-r}\mathbf{E}|g_x(X_1)|^r, & B_{yr} &= \sigma_y^{-r}\mathbf{E}|g_y(Y_1)|^r, \\
\gamma_x &= \tau_1^6\tau_2^2\mathbf{E}\psi_x^2(X_1,X_2)\psi_{xy}^2(X_1,Y_1), & \gamma_y &= \tau_2^6\tau_1^2\mathbf{E}\psi_y^2(Y_1,Y_2)\psi_{xy}^2(X_1,Y_1).
\end{aligned}
$$

Lemma 8.1. *There exist (large) absolute constants $C, c, c_T > 0$ such that for $\tau_1, \tau_2, N_1^{1/2}B_{x3}^{-1}, N_2^{1/2}B_{y3}^{-1} > c$ and we have, for $|t| \le T$,*

$$
|f_U(t) - f_L(t) - 2^{-1}\kappa(it)^3 f_L(t)| \le C|t|(1+|t|^3)(\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3). \tag{42}
$$

Here

$$
T^{-1} = c_T \max\{B_{x3}\sigma_x; B_{y3}\sigma_y\} \tag{43}
$$

and

$$\begin{aligned}
\mathcal{R}_1 &= \sigma_Q^2\Big(1+\frac{\sigma^2(Q_x)+\sigma^2(Q_{xy})}{\tau_1\sigma_x^2}+\frac{\sigma^2(Q_y)+\sigma^2(Q_{xy})}{\tau_2\sigma_y^2}\Big),\\
\mathcal{R}_2 &= \Big(\frac{\gamma_x}{\tau_1^2}(1+\frac{1}{\sigma_y^2\tau_2})+\frac{\gamma_y}{\tau_2^2}(1+\frac{1}{\sigma_x^2\tau_1})\Big),\\
\mathcal{R}_3 &= \Big(B_{x4}^{1/2}\tau_1^{-1}(\sigma(Q_{xy})+\sigma(Q_x))+B_{y4}^{1/2}\tau_2^{-1}(\sigma(Q_{xy})+\sigma(Q_y))\Big).
\end{aligned}$$

For typical examples of statistics we have $\mathcal{R}_1+\mathcal{R}_2+\mathcal{R}_3=o(|\alpha|+|\kappa|)$ as $\alpha,\kappa\to 0$. Note that the bound (42) is not sufficient to establish the validity of the expansion (29). For this purpose one needs an improved version of (42) where the factor $|t|(1+|t|^3)$ in the right were replaced by a function which is integrable over the interval $|t|>0$ with respect to the measure $\frac{dt}{|t|}$.

Before the proof of Lemma 8.1 we introduce some notation

8.1. Notation. In what follows c and C with indices or without denote generic absolute constants. For complex numbers a,b we write $a\ll b$ if $|a|\le c|b|$. Given a complex valued function $z(t,s_1,s_2)$ write

$$\mathcal{J}(z)=\lambda_1\lambda_2\int_{-\pi\tau_1}^{\pi\tau_1}ds_1\int_{-\pi\tau_2}^{\pi\tau_2}ds_2 z(t,s_1,s_2).$$

For a number $A>0$ and complex valued functions $z_1(t,s_1,s_2)$ and $z_2(t,s_1,s_2)$ write $z_1\prec A$ if $|\mathcal{J}(z_1)|\ll A$ and write $z_1\sim z_2$ if $z_1-z_2\prec\mathcal{R}$, where $\mathcal{R}=|t|(1+|t|^3)(\mathcal{R}_1+\mathcal{R}_2+\mathcal{R}_3)$.

For $k=1,2,\dots$ denote $\Omega_k=\{1,\dots,k\}$ and $\Omega_{k|2}=\{(i,j):\,1\le i<j\le k\}$. For sums

$$S=\sum_{1\le i\le k}s_i,\qquad T=\sum_{1\le i\le k}\sum_{1\le j\le r}t_{ij},\qquad V=\sum_{1\le i<j\le k}v_{ij}$$

and sets $A\subset\Omega_k$, $B\subset\Omega_k\times\Omega_r$, $D\subset\Omega_{k|2}$ write for short

$$S(A)=\sum_{i\in A}s_i,\qquad T(B)=\sum_{(i,j)\in B}t_{ij},\qquad V(D)=\sum_{(i,j)\in D}v_{ij}.$$

For integers $1<m_1<N_1$ and $1<m_2<N_2$ put

$$\begin{aligned}
A_1^x &= \Omega_{m_1},\qquad A_2^x=\Omega_{N_1}\setminus\Omega_{m_1},\qquad A_1^y=\Omega_{m_2},\qquad A_2^y=\Omega_{N_2}\setminus\Omega_{m_2},\\
B_1 &= \Omega_{m_1}\times\Omega_{m_2},\qquad B_2=\Omega_{m_1}\times(\Omega_{N_2}\setminus\Omega_{m_2}),\\
B_3 &= (\Omega_{N_1}\setminus\Omega_{m_1})\times\Omega_{m_2},\qquad B_4=(\Omega_{N_1}\setminus\Omega_{m_1})\times(\Omega_{N_2}\setminus\Omega_{m_2}),\\
D_1^x &= \Omega_{m_1|2},\qquad D_2^x=\Omega_{m_1}\times(\Omega_{N_1}\setminus\Omega_{m_1}),\qquad D_3^x=\Omega_{N_1|2}\setminus(D_1^x\cup D_2^x),\\
D_1^y &= \Omega_{m_2|2},\qquad D_2^y=\Omega_{m_2}\times(\Omega_{N_2}\setminus\Omega_{m_2}),\qquad D_3^y=\Omega_{N_2|2}\setminus(D_1^y\cup D_2^y).
\end{aligned}$$

Split

$$
\begin{aligned}
\tilde{\mathbb{L}} &= \tilde{\mathbb{L}}_1 + \tilde{\mathbb{L}}_2, \quad \tilde{\mathbb{L}}_1 = \tilde{\mathbb{L}}_1^x + \tilde{\mathbb{L}}_1^y, \qquad \tilde{\mathbb{L}}_2 = \tilde{\mathbb{L}}_2^x + \tilde{\mathbb{L}}_2^y, \\
\tilde{\mathbb{L}}_k^x &= t\mathbb{L}_x(A_k^x) + \frac{s_1}{\tau_1} S_x(A_k^x), \quad \tilde{\mathbb{L}}_k^y = t\mathbb{L}_y(A_k^y) + \frac{s_2}{\tau_2} S_y(A_k^y), \qquad k = 1, 2, \\
\mathbb{L}_x &= \mathbb{L}_x(A_1^x) + \mathbb{L}_x(A_2^x), \qquad \mathbb{L}_y = \mathbb{L}_y(A_1^y) + \mathbb{L}_y(A_2^y), \\
\mathbb{Q}_{xy} &= \sum_{1\le i\le 4} \mathbb{Q}_{xy}(B_i), \qquad \mathbb{Q}_x = \sum_{1\le i\le 3} \mathbb{Q}_x(D_i^x), \qquad \mathbb{Q}_y = \sum_{1\le i\le 3} \mathbb{Q}_y(D_i^y)
\end{aligned}
$$

and write $H = L_1 + F_1 + F$, where $L_1 = L_{1x} + L_{1y}$,

$$
\begin{aligned}
L_{1x} &= \tilde{\mathbb{L}}_1^x + t\mathbb{Q}_x(D_2^x) + t\mathbb{Q}_{xy}(B_2), \\
L_{1y} &= \tilde{\mathbb{L}}_1^y + t\mathbb{Q}_y(D_2^y) + t\mathbb{Q}_{xy}(B_3), \\
F_1 &= t\mathbb{Q}_x(D_1^x) + t\mathbb{Q}_y(D_1^y) + t\mathbb{Q}_{xy}(B_1), \\
F &= \tilde{\mathbb{L}}_2 + F_2, \qquad F_2 = t\mathbb{Q}_x(D_3^x) + t\mathbb{Q}_y(D_3^y) + t\mathbb{Q}_{xy}(B_4).
\end{aligned}
$$

Note that the random variable F does not depend on ω_i, $i \le m_1$ and ξ_j, $j \le m_2$. Furthermore we have

$$
L_{1x} = \sum_{1\le i\le m_1} a_i\omega_i, \qquad L_{1y} = \sum_{1\le j\le m_2} b_j\xi_j, \tag{44}
$$

where

$$
\begin{aligned}
a_i &= tg_x(X_i) + s_1\tau_1^{-1} + tv_i^x + tu_i^x, \qquad b_j = tg_y(Y_j) + s_2\tau_2^{-1} + tv_j^y + tu_j^y, \\
v_i^x &= \sum_{m_1<j\le N_1} \psi_x(X_i, X_j)\omega_j, \qquad u_i^x = \sum_{m_2<j\le N_2} \psi_{xy}(X_i, Y_j)\xi_j, \\
v_j^y &= \sum_{m_2<i\le N_2} \psi_y(Y_j, Y_i)\xi_i, \qquad u_j^y = \sum_{m_1<i\le N_1} \psi_{xy}(X_i, Y_j)\omega_i.
\end{aligned}
$$

Note that conditionally given all the random variables but $\{\omega_i,\, i \le m_1\}$ and $\{\xi_j,\, j \le m_2\}$ the random variable L_{1x} (respectively L_{1y}) is a linear statistic of random variables $\omega_1, \dots, \omega_{m_1}$ (respectively $\xi_1, \dots, \xi_{m_2}$).

Given $B \subset \Omega_{m_1}$ denote

$$
\begin{aligned}
\tilde{J}_{xy}^x(B) &= \exp\{it\mathbb{L}_x(B) + it\mathbb{Q}_x(B \times (\Omega_{N_1} \setminus \Omega_{m_1})) \\
&+ it\mathbb{Q}_{xy}(B \times (\Omega_{N_2} \setminus \Omega_{m_2})) + i(s_1/\tau_1)S_x(B)\}, \\
\tilde{J}_y^x(B) &= \exp\{it\mathbb{L}_x(B) + it\mathbb{Q}_{xy}(B \times (\Omega_{N_2} \setminus \Omega_{m_2})) + i(s_1/\tau_1)S_x(B)\}, \\
\tilde{J}_x^x(B) &= \exp\{it\mathbb{L}_x(B) + it\mathbb{Q}_x(B \times (\Omega_{N_1} \setminus \Omega_{m_1})) + i(s_1/\tau_1)S_x(B)\} \\
\tilde{J}^x(B) &= \exp\{it\mathbb{L}_x(B) + i(s_1/\tau_1)S_x(B)\}.
\end{aligned}
$$

Let $J^x_{xy}(B)$ (respectively $J^x_y(B)$, $J^x_x(B)$ and $J^x(B)$) denote the absolute value of the conditional expectation of $\tilde J^x_{xy}(B)$ (respectively $\tilde J^x_y(B)$, $\tilde J^x_x(B)$ and $\tilde J^x(B)$) given all the random variables but $\{\omega_j,\, j\in B\}$. Write

$$\varkappa^x_{xy}(B)=\frac{1}{|B|}\sum_{j\in B}(v^x_j+u^x_j)^2,\qquad \varkappa^x_x(B)=\frac{1}{|B|}\sum_{j\in B}(v^x_j)^2,\qquad \varkappa^x_y(B)=\frac{1}{|B|}\sum_{j\in B}(u^x_j)^2.$$

Given $D\subset\Omega_{m_2}$ denote

$$\begin{aligned}
\tilde J^y_{xy}(D) &= \exp\{it\mathbb{L}_y(D)+it\mathbb{Q}_y(D\times(\Omega_{N_2}\setminus\Omega_{m_2}))\\
&+ it\mathbb{Q}_{xy}((\Omega_{N_1}\setminus\Omega_{m_1})\times D)+i(s_2/\tau_2)S_x(D)\},\\
\tilde J^y_x(D) &= \exp\{it\mathbb{L}_y(D)+it\mathbb{Q}_{xy}((\Omega_{N_1}\setminus\Omega_{m_1})\times D)+i(s_2/\tau_2)S_x(D)\},\\
\tilde J^y_y(D) &= \exp\{it\mathbb{L}_y(D)+it\mathbb{Q}_y(D\times(\Omega_{N_2}\setminus\Omega_{m_2}))+i(s_2/\tau_2)S_y(D)\}\\
\tilde J^y(D) &= \exp\{it\mathbb{L}_y(D)+i(s_2/\tau_2)S_x(D)\}.
\end{aligned}$$

Let $J^y_{xy}(D)$ (respectively $J^y_y(D)$, $J^y_x(D)$ and $J^y(D)$) denote the absolute value of the conditional expectation of $\tilde J^y_{xy}(D)$ (respectively $\tilde J^y_y(D)$, $\tilde J^y_x(D)$ and $\tilde J^y(D)$) given all the random variables but $\{\xi_j,\, j\in D\}$. Furthermore, write

$$\varkappa^y_{xy}(D)=\frac{1}{|D|}\sum_{j\in D}(v^y_j+u^y_j)^2,\qquad \varkappa^y_x(D)=\frac{1}{|D|}\sum_{j\in D}(u^y_j)^2,\qquad \varkappa^y_y(D)=\frac{1}{|D|}\sum_{j\in D}(v^y_j)^2.$$

8.2. Proof of Lemma 8.1. Note that

$$f_U(t)=\mathcal{J}(\mathbf{E}\exp\{iH\}),\qquad\text{and}\qquad f_L(t)=\mathcal{J}(\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}).$$

Therefore in order to prove the lemma we need to show that

$$\mathbf{E}\exp\{iH\}\sim(1+2^{-1}\kappa(it)^3)\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}.$$

We shall prove this relation in two steps

$$\mathbf{E}\exp\{iH\}\ \sim\ \mathbf{E}\exp\{i\tilde{\mathbb{L}}\}+it\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}\mathbb{Q},\tag{45}$$
$$it\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}\ \sim\ 2^{-1}\kappa(it)^3\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}.\tag{46}$$

8.2.1. Proof of (45). Introduce the integer valued functions $m_1=m_1(s_1)$ and $m_2=m_2(s_2)$,

$$m_1\approx C'N_1\frac{\ln(3+s_1^2)}{3+s_1^2},\qquad m_2\approx C'N_2\frac{\ln(3+s_2^2)}{3+s_2^2},\tag{47}$$

where $C' > 0$ is a sufficiently large constant. Denote

$$m_3 = \lfloor m_1/2 \rfloor, \qquad m_4 = \lfloor m_2/2 \rfloor.$$

Write

$$f_1 = \mathbf{E}\exp\{iH\}, \qquad f_2 = \mathbf{E}J, \qquad f_3 = i\mathbf{E}JF_1, \qquad J = \exp\{i(L_1 + F)\}.$$

In order to prove (45) we shall show that

$$\begin{aligned} f_1 &\sim f_2 + f_3, && (48)\\ f_2 &\sim \mathbf{E}\exp\{i\tilde{\mathbb{L}}\} + i\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}(t\mathbb{Q} - F_1), && (49)\\ f_3 &\sim i\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}F_1. && (50) \end{aligned}$$

Let us prove (48). Expanding the exponent in powers of $it\mathbb{Q}_{xy}(B_1)$ we obtain

$$\begin{aligned} f_1 &= f_4 + f_5 + R_1, && (51)\\ f_4 : &= \mathbf{E}J\exp\{i(t\mathbb{Q}_x(D_1^x) + t\mathbb{Q}_y(D_1^y))\}, &&\\ f_5 : &= it\mathbf{E}J\exp\{i(t\mathbb{Q}_x(D_1^x) + t\mathbb{Q}_y(D_1^y))\}\mathbb{Q}_{xy}(B_1), &&\\ |R_1| &\leq t^2\mathbf{E}\mathbb{Q}_{xy}^2(B_1). && \end{aligned}$$

A simple calculation shows that

$$f_5 = f_6 + R_2 + R_3 + R_4, \qquad f_6 := it\mathbf{E}J\mathbb{Q}_{xy}(B_1), \tag{52}$$

where we denote

$$R_2 = it\mathbf{E}J\mathbb{Q}_{xy}(B_1)(\exp\{itQ_y(D_1^y)\}-1), \qquad R_3 = it\mathbf{E}J\mathbb{Q}_{xy}(B_1)(\exp\{itQ_x(D_1^x)\}-1),$$

$$R_4 = it\mathbf{E}J\mathbb{Q}_{xy}(B_1)(\exp\{it\mathbb{Q}_y(D_1^y)\} - 1)(\exp\{it\mathbb{Q}_x(D_1^x)\} - 1).$$

Expanding the exponent in f_4 in powers of $it\mathbb{Q}_x(D_1^x)$ we obtain

$$f_4 = f_7 + f_8 + f_9, \tag{53}$$

where $f_7 := \mathbf{E}J\exp\{it\mathbb{Q}_x(D_1^x)\}$ and $f_8 := it\mathbf{E}J\exp\{it\mathbb{Q}_x(D_1^x)\}\mathbb{Q}_y(D_1^y)$ and

$$f_9 := (it)^2\mathbf{E}J\exp\{it\mathbb{Q}_x(D_1^x)\}\mathbb{Q}_y^2(D_1^y)\theta_y, \qquad |\theta_y| \leq 1.$$

The random variable θ_y is a function of $\mathbb{Q}_y(D_1^y)$. Furthermore, expanding in powers of $it\mathbb{Q}_x(D_1^x)$ we obtain

$$f_7 = f_2 + it\mathbf{E}J\mathbb{Q}_x(D_1^x) + R_5, \qquad R_5 := (it)^2\mathbf{E}J\mathbb{Q}_x^2(D_1^x)\theta_x, \tag{54}$$

where $|\theta_x| \le 1$ is a function of $\mathbb{Q}_x(D_1^x)$. Similarly, we get

$$f_8 = it\mathbf{E}J\mathbb{Q}_y(D_1^y) + R_6, \qquad |R_6| \le t^2|\mathbb{Q}_x(D_1^x)\mathbb{Q}_y(D_1^y)|. \tag{55}$$

Finally,

$$f_9 = R_7 + R_8, \qquad R_7 := (it)^2\mathbf{E}J\mathbb{Q}_y^2(D_1^y)\theta_y, \qquad |R_8| \le |t|^3|\mathbb{Q}_x(D_1^x)|\mathbb{Q}_y^2(D_1^y). \tag{56}$$

Combining (51)-(56) we obtain

$$f_1 = f_2 + f_3 + R, \qquad R := R_1 + \ldots + R_8. \tag{57}$$

In order to prove (48) we show that

$$R_k \prec \mathcal{R}, \qquad k = 1, \ldots, 8. \tag{58}$$

For $k = 1, 4, 6, 8$ the proof is simple. We have

$$\begin{aligned} |R_4| &\le t^2\mathbf{E}|\mathbb{Q}_x(D_1^x)|^{1/2}|\mathbb{Q}_y(D_1^y)|^{1/2}|\mathbb{Q}_{xy}(B_1)| \\ &\le t^2\mathbf{E}|\mathbb{Q}_x(D_1^x)\mathbb{Q}_y(D_1^y)| + t^2\mathbf{E}\mathbb{Q}_{xy}^2(B_1) \\ &\le t^2(\mathbf{E}\mathbb{Q}_x^2(D_1^x))^{1/2}(\mathbf{E}\mathbb{Q}_y^2(D_1^y))^{1/2} + t^2\mathbf{E}\mathbb{Q}_{xy}^2(B_1), \\ |R_8| &\le |t|^3(\mathbf{E}\mathbb{Q}_x^2(D_1^x))^{1/2}\mathbf{E}\mathbb{Q}_y^2(D_1^y), \\ |R_6| &\le t^2(\mathbf{E}\mathbb{Q}_x^2(D_1^x))^{1/2}(\mathbf{E}\mathbb{Q}_y^2(D_1^y))^{1/2}. \end{aligned}$$

Combining these inequalities, (47) and (120) we obtain (58) for $k = 1, 4, 6, 8$.

Let us prove (58) for $k = 2$. Write $B_1' = (\Omega_{m_1} \setminus \Omega_{m_3}) \times \Omega_{m_2}$. By symmetry, it suffices to show (58) for

$$R_2' := it\mathbf{E}J\mathbb{Q}_{xy}(B_1')(\exp\{it\mathbb{Q}_y(D_1^y)\} - 1) = (it)^2\mathbf{E}J\mathbb{Q}_{xy}(B_1')\mathbb{Q}_y(D_1^y)\theta_y^*,$$

where $|\theta_y^*| \le 1$ is a function of $\mathbb{Q}_y(D_1^y)$. We have

$$R_2' \le t^2\mathbf{E}J_{xy}^x(\Omega_{m_3})|\mathbb{Q}_{xy}(B_1')\mathbb{Q}_y(D_1^y)|.$$

Invoking inequalities $|J_{xy}^x(\Omega_3)| \le 1$ and $|ab| \le a^2 + b^2$ we write

$$R_2' \le t^2R_{21} + t^2R_{22}, \qquad R_{21} := \mathbf{E}\mathbb{Q}_{xy}^2(B_1'), \qquad R_{22} := \mathbf{E}J_{xy}^x(\Omega_{m_3})\mathbb{Q}_y^2(D_1^y). \tag{59}$$

Lemma 9.1 implies the bound $t^2R_{21} \prec \mathcal{R}$. Let us prove that $t^2R_{22} \prec \mathcal{R}$. Write

$$R_{22} = \mathbf{E}f_*, \qquad f_* := J_{xy}^x(\Omega_{m_3})\mathbb{Q}_y^2(D_1^y) \tag{60}$$

and split

$$f_* = f_*\mathbb{I}_x + f_*\bar{\mathbb{I}}_x, \qquad \bar{\mathbb{I}}_x := 1 - \mathbb{I}_x, \qquad \mathbb{I}_x := \mathbb{I}_{\{\alpha\varkappa^x_{xy}(\Omega_{m_3})<\sigma_x^2\}}. \tag{61}$$

The number $\alpha > 0$ is defined in (123) below. By Chebyshev's inequality and symmetry,

$$\begin{aligned}\mathbf{E}f_*\bar{\mathbb{I}}_x &\le (\alpha/\sigma_x^2)\mathbf{E}\varkappa^x_{xy}(\Omega_{m_3})\mathbb{Q}_y^2(D_1^y) = (\alpha/\sigma_x^2)\mathbf{E}(v_1^x + u_1^x)^2\mathbb{Q}_y^2(D_1^y)\\ &= (\alpha/\sigma_x^2)(V_1 + V_2).\end{aligned}$$

Here we denote

$$V_1 = \mathbf{E}(v_1^x)^2\mathbb{Q}_y^2(D_1^y), \qquad V_2 = \mathbf{E}(u_1^x)^2\mathbb{Q}_y^2(D_1^y).$$

By symmetry, (119) and (120) we have

$$V_1 = (N_1 - m_1)p_xq_x\sigma^2_{xx}\mathbf{E}\mathbb{Q}_y^2(D_1^y) \ll \tau_1^{-2}\sigma^2(Q_x)(m_2/N_2)^2\sigma^2(Q_y).$$

Similarly,

$$V_2 = (N_2 - m_2)p_yq_y\mathbf{E}\psi^2_{xy}(X_1, Y_{N_2})\mathbb{Q}_y^2(D_1^y) \le \tau_2^2m_2^2p_y^2q_y^2\mathbf{E}\psi^2_{xy}(X_1, Y_{N_2})\psi_y^2(Y_1, Y_2).$$

Invoking the simple bound $\mathbf{E}\psi^2_{xy}(X_1, Y_{N_2})\psi_y^2(Y_1, Y_2) \le \sigma^2_{xy}\sigma^2_{yy}$ and (119) we obtain

$$V_2 \ll \tau_1^{-2}(m_2/N_2)^2\sigma^2(Q_{xy})\sigma^2(Q_y).$$

Combining bounds for V_1 and V_2 we obtain,

$$\begin{aligned}\mathbf{E}f_*\bar{\mathbb{I}}_x &\ll (m_2/N_2)^2\tau_1^{-2}\sigma_x^{-2}\sigma^2(Q_y)(\sigma^2(Q_{xy}) + \sigma^2(Q_x)) \qquad (62)\\ &\prec \tau_1^{-1}\sigma_x^{-2}\sigma^2(Q_y)(\sigma^2(Q_{xy}) + \sigma^2(Q_x)).\end{aligned}$$

In the last step we used the simple bound $\int\mathbb{I}_{\{|s_1|\le\pi\tau_1\}}ds_1 \le \tau_1$.

Furthermore, combining (132) and (120) we obtain

$$\mathbf{E}f_*\mathbb{I}_x \le \mathbf{E}\mathbb{Q}_y^2(D_1^y)\mathbf{E}_XJ_x(\Omega_3)\mathbb{I}_x \prec \sigma^2(Q_y). \tag{63}$$

Finally, collecting (62) and (63) in (60) we obtain the bound $t^2R_{22} \prec \mathcal{R}$, thus completing the proof of (58) for R_2. The proof of (58) for R_3, R_5, R_7 is almost the same. We arrive to (48).

Let us prove (50). We shall show that

$$\begin{aligned}it\mathbf{E}J\mathbb{Q}_{xy}(B_1) &\sim it\mathbf{E}exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_{xy}(B_1), &(64)\\ it\mathbf{E}J\mathbb{Q}_x(D_1^x) &\sim it\mathbf{E}exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_x(D_1^x), &(65)\\ it\mathbf{E}J\mathbb{Q}_y(D_1^y) &\sim it\mathbf{E}exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_y(D_1^y). &(66)\end{aligned}$$

Write

$$J = J_0 \exp\{iF_3\}, \qquad J_0 = \exp\{i(\tilde{\mathbb{L}} + F_2)\}, \tag{67}$$
$$F_3 = t\mathbb{Q}_x(D_2^x) + t\mathbb{Q}_y(D_2^y) + t\mathbb{Q}_{xy}(B_2) + t\mathbb{Q}_{xy}(B_3).$$

In the proof of (64), (65), (66) we replace J by J_0 and then replace J_0 by $\exp\{i\tilde{\mathbb{L}}\}$.

Let us prove (64). Denote $B_{11} = (\Omega_{m_1} \setminus \Omega_{m_3}) \times (\Omega_{m_2} \setminus \Omega_{m_4})$. By symmetry it suffices to show that

$$it\mathbf{E}J\mathbb{Q}_{xy}(B_{11}) \sim it\mathbf{E}exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_{xy}(B_{11}). \tag{68}$$

We shall prove that

$$it\mathbf{E}J\mathbb{Q}_{xy}(B_{11}) \sim it\mathbf{E}J_0\mathbb{Q}_{xy}(B_{11}), \qquad it\mathbf{E}J_0\mathbb{Q}_{xy}(B_{11}) \sim it\mathbf{E}exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_{xy}(B_{11}). \tag{69}$$

Write

$$F_3 = H_{31} + t\mathbb{Q}_{xy}(B_2), \qquad H_{31} = H_{32} + t\mathbb{Q}_{xy}(B_3),$$
$$H_{32} = H_{33} + t\mathbb{Q}_y(D_2^y), \qquad H_{33} = t\mathbb{Q}_x(D_2^x).$$

Expanding in powers of $it\mathbb{Q}_{xy}(B_2)$ we obtain

$$\mathbf{E}J\mathbb{Q}_{xy}(B_{11}) = \mathbf{E}J_0 \exp\{iH_{31}\}\mathbb{Q}_{xy}(B_{11}) + itR_1,$$
$$R_1 : = \mathbf{E}J_0 \exp\{iH_{31}\}\mathbb{Q}_{xy}(B_{11})\mathbb{Q}_{xy}(B_2)\theta_1,$$

where the random variable $|\theta_1| \le 1$ is a function of $\mathbb{Q}_{xy}(B_2)$. Similarly, expanding in powers of $it\mathbb{Q}_{xy}(B_3)$, $it\mathbb{Q}_x(D_1^x)$ and $it\mathbb{Q}_y(D_1^y)$ we obtain

$$\mathbf{E}J_0 \exp\{iH_{31}\}\mathbb{Q}_{xy}(B_{11}) = \mathbf{E}J_0 \exp\{iH_{32}\}\mathbb{Q}_{xy}(B_{11}) + itR_2,$$
$$R_2 : = \mathbf{E}J_0 \exp\{iH_{32}\}\mathbb{Q}_{xy}(B_{11})\mathbb{Q}_{xy}(B_3)\theta_2,$$
$$\mathbf{E}J_0 \exp\{iH_{32}\}\mathbb{Q}_{xy}(B_{11}) = \mathbf{E}J_0 \exp\{iH_{33}\}\mathbb{Q}_{xy}(B_{11}) + itR_3,$$
$$R_3 : = \mathbf{E}J_0 \exp\{iH_{33}\}\mathbb{Q}_{xy}(B_{11})\mathbb{Q}_y(D_2^y)\theta_3,$$
$$\mathbf{E}J_0 \exp\{iH_{33}\}\mathbb{Q}_{xy}(B_{11}) = \mathbf{E}J_0\mathbb{Q}_{xy}(B_{11}) + itR_4,$$
$$R_4 : = \mathbf{E}J_0\mathbb{Q}_{xy}(B_{11})\mathbb{Q}_x(D_2^x)\theta_4.$$

In order to prove the first part of (69) we shall show that

$$t^2 R_k \prec \mathcal{R}, \qquad k = 1, 2, 3, 4. \tag{70}$$

Let us prove (70) for $k = 1$. We have

$$\begin{aligned} R_1 &\leq \mathbf{E}J^y_{xy}(\Omega_{m_4})|\mathbb{Q}_{xy}(B_{11})\mathbb{Q}_{xy}(B_2)| = R_{11} + R_{12}, \qquad (71)\\ R_{11} : &= \mathbf{E}J^y_{xy}(\Omega_{m_4})|\mathbb{Q}_{xy}(B_{11})\mathbb{Q}_{xy}(B_2)|\mathbb{I}_y,\\ R_{12} : &= \mathbf{E}J^y_{xy}(\Omega_{m_4})|\mathbb{Q}_{xy}(B_{11})\mathbb{Q}_{xy}(B_2)|\bar{\mathbb{I}}_y. \end{aligned}$$

Here

$$\bar{\mathbb{I}}_y := 1 - \mathbb{I}_y, \qquad \mathbb{I}_y := \mathbb{I}_{\{\alpha \varkappa^y_{xy}(\Omega_{m_4}) < \sigma^2_y\}}.$$

Random variables v^y_i and u^y_i are defined in (44) and the constant α is defined in (123). We shall show that

$$t^2 R_{11} \prec \mathcal{R}, \qquad t^2 R_{12} \prec \mathcal{R}. \qquad (72)$$

Let us prove the second bound. Using the simple inequality $J^y_{xy}(\Omega_{m_4}) \leq 1$ and Cauchy-Schwartz we obtain

$$R_{12} \leq R_{13}^{1/2} R_{14}^{1/2}, \qquad R_{13} := \mathbf{E}\mathbb{Q}^2_{xy}(B_{11})\bar{\mathbb{I}}_y, \quad R_{14} := \mathbf{E}\mathbb{Q}^2_{xy}(B_2). \qquad (73)$$

Lemma 9.1 implies $R_{14} \ll (m_1/N_1)\sigma^2(Q_{xy})$. Furthermore, by Chebyshev's inequality and symmetry

$$\begin{aligned} R_{13} &\ll \sigma_y^{-2}\mathbf{E}\mathbb{Q}^2_{xy}(B_{11})\varkappa^y_{xy}(\Omega_{m_4}) = \sigma_y^{-2}\mathbf{E}\mathbb{Q}^2_{xy}(B_{11})((v^y_1)^2 + (u^y_1)^2)\\ &= \sigma_y^{-2}|B_{11}|p_x q_x p_y q_y (V_1 + V_2). \qquad (74) \end{aligned}$$

Here $|B_{11}| < m_1 m_2$ denotes the number of elements of the set B_{11} and

$$\begin{aligned} V_1 : &= \mathbf{E}\psi^2_{xy}(X_{m_1}, Y_{m_2})(v^y_1)^2 = (N_2 - m_2)p_y q_y \mathbf{E}\psi^2_{xy}(X_{m_1}, Y_{m_2})\psi^2_y(Y_1, Y_{N_2})\\ &\ll \tau_2^2\sigma^2_{xy}\sigma^2_{yy} \ll \tau_1^{-2}\tau_2^{-4}\sigma^2(Q_{xy})\sigma^2(Q_y),\\ V_2 : &= \mathbf{E}\psi^2_{xy}(X_{m_1}, Y_{m_2})(u^y_1)^2 = (N_1 - m_1)p_x q_x \mathbf{E}\psi^2_{xy}(X_{m_1}, Y_{m_2})\psi^2_{xy}(X_{N_1}, Y_1)\\ &\ll \tau_1^2\sigma^4_{xy} \ll \tau_1^{-2}\tau_2^{-4}\sigma^4(Q_{xy}). \end{aligned}$$

We obtain

$$R_{13} \ll (m_1 m_2/N_1 N_2)\tau_2^{-2}\sigma_y^{-2}\sigma^2(Q_{xy})(\sigma^2(Q_{xy}) + \sigma^2(Q_y)).$$

Collecting the bounds for R_{13} and R_{14} in (73) we obtain $t^2 R_{12} \prec \mathcal{R}$.

Let us prove the first bound of (72). We have

$$R_{11} \leq R_{15} + R_{16}, \qquad R_{15} := \mathbf{E}\mathbb{Q}^2_{xy}(B_{11}), \quad R_{16} := \mathbf{E}J^y_{xy}(\Omega_{m_4})\mathbb{I}_y\mathbb{Q}^2_{xy}(B_2),$$

where $t^2 R_{15} \ll t^2 (m_1 m_2/N_1 N_2)\sigma^2(Q_{xy}) \prec \mathcal{R}$. Furthermore, by symmetry,

$$R_{16} = m_1 p_x q_x \mathbf{E} J^y_{xy}(\Omega_{m_4})\mathbb{I}_y D^2, \qquad D := \sum_{j=m_2+1}^{N_2} \psi_{xy}(X_1, Y_j)\xi_j. \tag{75}$$

We have, see (134),

$$J^y_{xy}(\Omega_{m_4})\mathbb{I}_y \le W_1 W_2 \mathbb{I}_y, \tag{76}$$

where

$$W_1^2 := \prod_{k=1}^{m_4} u_{\{1,y\}}(t g_y(Y_k) + s_2/\tau_2), \qquad W_2^2 := \prod_{k=1}^{m_4} v_{\{1,y\}}(t v^y_k + t u^y_k).$$

Invoking the bound, which can be shown in the same way as (135),

$$W_2 \mathbb{I}_y \le r_1, \qquad r_1 := \exp\{16^{-1}\Theta(1)(m_4/N_2)\sigma_y^2 \tau_2^2 t^2\}, \tag{77}$$

we obtain, by symmetry,

$$\mathbf{E} J^y_{xy}(\Omega_{m_4})\mathbb{I}_y D^2 \le r_1 \mathbf{E} W_1 D^2 = r_1 (N_2 - m_2) p_y q_y \mathbf{E} W_1 \psi^2_{xy}(X_1, Y_{N_2}). \tag{78}$$

The conditional expectation of $W_1 r_1$ given Y_{N_2} is not greater than $(3+s_2^2)^{-10}$, see (139). Therefore, we obtain

$$\mathbf{E} J^y_{xy}(\Omega_{m_4})\mathbb{I}_y D^2 \le (3 + s_2^2)^{-10} \tau_2^2 \sigma^2_{xy}.$$

Substituting this bound in (75) we obtain

$$R_{16} \le (3 + s_2^2)^{-10}(m_1/N_1)\tau_1^2 \tau_2^2 \sigma^2_{xy}. \tag{79}$$

This implies $t^2 R_{16} \prec \mathcal{R}$ thus completing the proof of (72). We have shown (70) for $k = 1$. For $k = 2, 3, 4$ the proof is much the same.

Let us prove the second part of (69). Expanding the exponent of J_0 in powers of iF_2 we obtain

$$it \mathbf{E} J_0 \mathbb{Q}_{xy}(B_{11}) = it \mathbf{E} exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_{xy}(B_{11}) + (it)^2 R, \quad R := \mathbf{E} exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_{xy}(B_{11}) F_2 \theta_F,$$

where the random variable $|\theta_F| \le 1$ is a function of F_2. We shall show that $t^2 R \prec \mathcal{R}$.

We have $|R| \le R_1 + R_2 + R_3$, where

$$\begin{aligned} R_1 :&= \mathbf{E} \tilde{J}^x(\Omega_{m_3}) \tilde{J}^y(\Omega_{m_4}) |\mathbb{Q}_{xy}(B_{11})\mathbb{Q}_{xy}(B_4)|, \\ R_2 :&= \mathbf{E} \tilde{J}^x(\Omega_{m_3}) \tilde{J}^y(\Omega_{m_4}) |\mathbb{Q}_{xy}(B_{11})\mathbb{Q}_{x}(D_3^x)|, \\ R_3 :&= \mathbf{E} \tilde{J}^x(\Omega_{m_3}) \tilde{J}^y(\Omega_{m_4}) |\mathbb{Q}_{xy}(B_{11})\mathbb{Q}_{y}(D_3^y)|. \end{aligned}$$

We shall show that
$$t^2 R_k \prec \mathcal{R}, \qquad k=1,2,3. \tag{80}$$
For $k=1$ we write
$$R_1 \le R_{11}+R_{12}, \qquad R_{11} := \mathbf{E}\mathbb{Q}_{xy}^2(B_{11}), \qquad R_{12} := \mathbf{E}\tilde{J}^x(\Omega_{m_3})\tilde{J}^y(\Omega_{m_4})\mathbb{Q}_{xy}^2(B_4).$$
Here we use simple inequalities $\tilde{J}^x(\Omega_{m_3}) \le 1$ and $\tilde{J}^y(\Omega_{m_4}) \le 1$. Clearly, $t^2 R_{11} \prec \mathcal{R}$. Furthermore, by symmetry,
$$R_{12} = (N_1-m_1)(N_2-m_2)p_x q_x p_y q_y \mathbf{E}\tilde{J}^x(\Omega_{m_3})\tilde{J}^y(\Omega_{m_4})\psi_{xy}^2(X_{N_1},Y_{N_2}).$$
Invoking (132), (133) we obtain
$$R_{12} \ll \tau_1^2\tau_2^2\sigma_{xy}^2(3+s_1^2)^{-10}(3+s_2^2)^{-10}.$$
This implies $t^2R_{12} \prec \mathcal{R}$ thus proving (80) for $k=1$. The proof for $k=2,3$ is much the same. The proof of (69) is complete. We arrived at (68).

Let us prove (65). We shall show that
$$it\mathbf{E}J\mathbb{Q}_x(D_1^x) \sim it\mathbf{E}J_0\mathbb{Q}_x(D_1^x), \qquad it\mathbf{E}J_0\mathbb{Q}_x(D_1^x) \sim it\mathbf{E}exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_x(D_1^x). \tag{81}$$
Write
$$\begin{aligned} F_3 &= H_{31}^* + t\mathbb{Q}_y(D_2^y), \qquad H_{31}^* := H_{32}^* + t\mathbb{Q}_{xy}(B_3), \\ H_{32}^* : &= H_{33}^* + t\mathbb{Q}_{xy}(B_2), \qquad H_{33}^* := t\mathbb{Q}_x(D_2^x). \end{aligned}$$

Let us prove that
$$it\mathbf{E}J\mathbb{Q}_x(D_1^x) \sim it\mathbf{E}J_0\exp\{iH_{31}^*\}\mathbb{Q}_x(D_1^x). \tag{82}$$
Expanding the exponent in J in powers of $it\mathbb{Q}_y(D_2^y)$ we obtain
$$\begin{aligned} \mathbf{E}J\mathbb{Q}_x(D_1^x) &= \mathbf{E}J_0\exp\{iH_{31}^*\}\mathbb{Q}_x(D_1^x) + itR_1 + t^2R_2, \\ R_1 : &= \mathbf{E}J_0\exp\{iH_{31}^*\}\mathbb{Q}_x(D_1^x)\mathbb{Q}_y(D_2^y), \qquad |R_2| \le \mathbf{E}|\mathbb{Q}_x(D_1^x)|\mathbb{Q}_y^2(D_2^y). \end{aligned}$$
We have, by symmetry and independence,
$$\begin{aligned} |R_2| &\le \mathbf{E}|\mathbb{Q}_x(D_1^x)|\mathbf{E}\mathbb{Q}_y^2(D_2^y) \le (\mathbf{E}\mathbb{Q}_x^2(D_1^x))^{1/2}\mathbf{E}\mathbb{Q}_y^2(D_2^y) \\ &= (\binom{m_1}{2}p_x^2q_x^2\sigma_{xx}^2)^{1/2}m_2(N_2-m_2)p_y^2q_y^2\sigma_{yy}^2 \\ &\ll (m_1/N_1)(m_2/N_2)\sigma(Q_x)\sigma^2(Q_y). \end{aligned}$$

Therefore, $t^3 R_2 \prec \mathcal{R}$.

Let us prove that $t^2 R_1 \prec \mathcal{R}$. By symmetry, it suffices to show that $t^2 R_1^* \prec \mathcal{R}$, where we denote

$$R_1^* = \mathbf{E} J_0 \exp\{iH_{31}^*\} Q_x(D_1^x) Q_y(D_y^*), \qquad D_y^* = (\Omega_{m_2} \setminus \Omega_{m_4}) \times (\Omega_{N_2} \setminus \Omega_{m_2}).$$

We have

$$|R_1^*| \le \mathbf{E} J_x^y(\Omega_{m_4}) |\mathbb{Q}_x(D_1^x) \mathbb{Q}_y(D_y^*)| \le (1+s_1^2) R_{11}^* + (1+s_1^2)^{-1} R_{12}^*,$$

where $R_{11}^* := \mathbf{E} J_x^y(\Omega_{m_4}) \mathbb{Q}_x^2(D_1^x)$ and $R_{12}^* := \mathbf{E} \mathbb{Q}_y^2(D_y^*)$. Here we applied the simple inequality $ab \le a^2c^2 + b^2c^{-2}$ and the inequality $J_x^y(\Omega_{m_4}) \le 1$. Since $R_{12}^* \ll (m_2/N_2)\sigma^2(Q_y)$ we obtain $t^2(1+s_1^2)^{-1} R_{12}^* \prec \mathcal{R}$. We complete the proof of (82) by showing that

$$t^2(1+s_1^2) R_{11}^* \prec \mathcal{R}. \tag{83}$$

Write $R_{11}^* = R_2^* + R_3^*$, where we denote

$$\begin{aligned} R_2^* &= \mathbf{E} J_x^y(\Omega_{m_4}) \mathbb{Q}_x^2(D_1^x) \mathbb{I}_y^*, \qquad R_3^* = \mathbf{E} J_x^y(\Omega_{m_4}) \mathbb{Q}_x^2(D_1^x) \overline{\mathbb{I}}_y^*, \\ \mathbb{I}_y^* &= \mathbb{I}_{\{\alpha \varkappa_x^y(\Omega_{m_4}) < \sigma_y^2\}}, \qquad \overline{\mathbb{I}}_y^* = 1 - \mathbb{I}_y^* \end{aligned}$$

and the constant α is defined in (123) below. Recall that the random variable $\varkappa_x^y(\Omega_{m_4})$ is introduced in the subsection **8.1**. We shall show that

$$t^2(1+s_1^2) R_k^* \prec \mathcal{R}, \qquad k = 2, 3. \tag{84}$$

We have, by Chebyshev's inequality and symmetry,

$$\begin{aligned} R_3^* &\ll \sigma_y^{-2} \mathbf{E} \mathbb{Q}_x^2(D_1^x) \varkappa_x^y(\Omega_{m_4}) = \sigma_y^{-2} \mathbf{E} \mathbb{Q}_x^2(D_1^x) (u_1^y)^2 \\ &= \sigma_y^{-2} \binom{m_1}{2} (N_1 - m_1) p_x^3 q_x^3 \mathbf{E} \psi_x^2(X_1, X_2) \psi_{xy}^2(X_{N_1}, Y_1) \\ &\ll \sigma_y^{-2} (m_1/N_1)^2 \tau_2^{-2} \sigma^2(Q_x) \sigma^2(Q_{xy}). \end{aligned}$$

This implies (84) for $k = 3$. In order to bound R_2^* we apply (133) and use symmetry,

$$R_2^* \ll \frac{1}{(3+s_2^2)^{10}} \mathbf{E} \mathbb{Q}_x^2(D_1^x) = \frac{\binom{m_1}{2}}{(3+s_2^2)^{10}} p_x^2 q_x^2 \sigma_{xx}^2 \ll \frac{m_1^2}{N_1^2} \frac{1}{(3+s_2^2)^{10}} \sigma^2(Q_x).$$

We obtain (84), thus completing the proof of (82).

Let us prove that

$$itE J_0 \exp\{iH_{31}^*\}\mathbb{Q}_x(D_1^x) \sim itE J_0 \exp\{iH_{32}^*\}\mathbb{Q}_x(D_1^x). \tag{85}$$

Expanding the exponent in powers of $it\mathbb{Q}_{xy}(B_3)$ we obtain

$$\begin{aligned} \mathbf{E}J_0 \exp\{iH_{31}^*\}\mathbb{Q}_x(D_1^x) &= \mathbf{E}J_0 \exp\{iH_{32}^*\}\mathbb{Q}_x(D_1^x) + itR_1 + t^2R_2, \\ R_1 : &= \mathbf{E}J_0 \exp\{iH_{32}^*\}\mathbb{Q}_x(D_1^x)\mathbb{Q}_{xy}(B_3), \\ |R_2| &\leq \mathbf{E}|\mathbb{Q}_x(D_1^x)|\mathbb{Q}_{xy}^2(B_3). \end{aligned}$$

Invoking the inequality $a \leq c^2 + a^2c^{-2}$ we obtain

$$|R_2| \leq (1+s_1^2)^{-1}\mathbf{E}\mathbb{Q}_{xy}^2(B_3) + (1+s_1^2)\mathbf{E}\mathbb{Q}_x^2(D_1^x)\mathbb{Q}_{xy}^2(B_3). \tag{86}$$

By symmetry,

$$\begin{aligned} \mathbf{E}\mathbb{Q}_{xy}^2(B_3) &= m_2(N_1-m_1)p_xq_xp_yq_y\sigma_{xy}^2 \ll (m_2/N_2)\sigma^2(Q_{xy}), \\ \mathbf{E}\mathbb{Q}_x^2(D_1^x)\mathbb{Q}_{xy}^2(B_3) &= \binom{m_1}{2}m_2(N_1-m_1)p_x^3q_x^3p_yq_y\mathbf{E}\psi_x^2(X_1,X_2)\psi_{xy}^2(X_{N_1},Y_1) \\ &\ll \frac{m_1^2}{N_1^2}\frac{m_2}{N_2}\tau_1^6\tau_2^2\sigma_{xx}^2\sigma_{xy}^2 \ll \frac{m_1^2}{N_1^2}\frac{m_2}{N_2}\sigma^2(Q_x)\sigma^2(Q_{xy}). \end{aligned}$$

Collecting these bounds in (86) we obtain that $t^3R_2 \prec \mathcal{R}$.

Let us show that $t^2R_1 \prec \mathcal{R}$. By symmetry, it suffices to show that $t^2R_1^* \prec \mathcal{R}$, where we denote

$$R_1^* = \mathbf{E}J_0 \exp\{iH_{32}^*\}\mathbb{Q}_x(D_1^x)\mathbb{Q}_{xy}(B_3'), \qquad B_3' = (\Omega_{N_1}\setminus\Omega_{m_1})\times(\Omega_{m_2}\setminus\Omega_{m_4}).$$

We have

$$\begin{aligned} |R_1^*| &\leq \mathbf{E}J^y(\Omega_{m_4})|\mathbb{Q}_x(D_1^x)\mathbb{Q}_{xy}(B_3')| \\ &\leq (1+s_1^2)\mathbf{E}J^y(\Omega_{m_4})\mathbb{Q}_x^2(D_1^x) + (1+s_1^2)^{-1}\mathbf{E}\mathbb{Q}_{xy}^2(B_3'). \end{aligned}$$

In the last step we used the simple inequality $ab \leq a^2c^2 + b^2c^{-2}$. Using the simple bound,

$$\mathbf{E}\mathbb{Q}_{xy}^2(B_3') \leq (m_2-m_4)(N_1-m_1)p_xq_xp_yq_y\sigma_{xy}^2 \leq (m_2/N_2)\sigma^2(Q_{xy})$$

we obtain

$$t^2(1+s_1^2)^{-1}\mathbf{E}\mathbb{Q}_{xy}^2(B_3') \prec \mathcal{R}. \tag{87}$$

Furthermore, invoking inequality (133) and using symmetry we obtain

$$\mathbf{E}J^y(\Omega_{m_4})\mathbb{Q}_x^2(D_1^x) \ll \frac{1}{(3+s_2^2)^{10}}\mathbf{E}\mathbb{Q}_x^2(D_1^x) \ll \frac{1}{(3+s_2^2)^{10}}\frac{m_1^2}{N_1^2}\sigma^2(Q_x).$$

This bound implies

$$t^2(1+s_1^2)\mathbf{E}J^y(\Omega_{m_4})\mathbb{Q}_x^2(D_1^x) \prec \mathcal{R}. \qquad (88)$$

Combining (87) and (88) we obtain $t^2 R_1^* \prec \mathcal{R}$ thus completing the proof of (85).

The proof of the formulas

$$\begin{aligned} it\mathbf{E}J_0\exp\{iH_{32}^*\}\mathbb{Q}_x(D_1^x) &\sim it\mathbf{E}J_0\exp\{iH_{33}^*\}\mathbb{Q}_x(D_1^x), \\ it\mathbf{E}J_0\exp\{iH_{33}^*\}\mathbb{Q}_x(D_1^x) &\sim it\mathbf{E}J_0\mathbb{Q}_x(D_1^x) \end{aligned}$$

is similar to that of (85) but simpler. Combining these bounds and (82), (85) we obtain the first formula of (81).

Let us prove the second formula of (81). Expanding the exponent in powers of iF_2 we obtain

$$\mathbf{E}J_0\mathbb{Q}_x(D_1^x) = \mathbf{E}\exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_x(D_1^x) + iR, \qquad R := \mathbf{E}\exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_x(D_1^x)F_2\theta_F,$$

where the random variable $|\theta_F| \le 1$ is a function of F_2. We have

$$|R| \le \mathbf{E}J^y(\Omega_{m_2})|\mathbb{Q}_x(D_1^x)F_2| \le (1+s_1^2)R_1 + (1+s_1^2)^{-1}R_2, \qquad (89)$$

where $R_1 := \mathbf{E}J^y(\Omega_{m_2})\mathbb{Q}_x^2(D_1^x)$ and $R_2 := \mathbf{E}J^y(\Omega_{m_2})F_2^2$. By symmetry and independence,

$$R_1 = \binom{m_1}{2} p_x^2 q_x^2 \sigma_{xx}^2 \mathbf{E}J^y(\Omega_{m_2}) \ll \frac{m_1^2}{N_1^2}\sigma^2(Q_x)\frac{1}{(1+s_2^2)^{10}}. \qquad (90)$$

Furthermore, we have $R_2 = t^2(R_{21}+R_{22}+R_{23})$, where

$$\begin{aligned} R_{21} :&= \mathbf{E}J^y(\Omega_{m_2})\mathbb{Q}_x^2(D_3^x) \ll \sigma^2(Q_x)\mathbf{E}J^y(\Omega_{m_2}) \ll \frac{\sigma^2(Q_x)}{(1+s_2^2)^{10}}, \qquad (91) \\ R_{22} :&= \mathbf{E}J^y(\Omega_{m_2})\mathbb{Q}_y^2(D_3^y) \ll \tau_2^4\mathbf{E}\psi_y^2 J^y(\Omega_{m_2}) \ll \frac{\sigma^2(Q_y)}{(1+s_2^2)^{10}}, \\ R_{23} :&= \mathbf{E}J^y(\Omega_{m_2})\mathbb{Q}_{xy}^2(B_4) \ll \tau_1^2\tau_2^2\mathbf{E}\psi_{xy}^2 J^y(\Omega_{m_2}) \ll \frac{\sigma^2(Q_{xy})}{(1+s_2^2)^{10}}. \end{aligned}$$

Here we used bounds (133). Collecting inequalities (90) and (91) in (89) we obtain $t^2 R \prec \mathcal{R}$ thus proving the second formula of (81). We arrive at (65). The proof of (66) is almost the same. The proof of (50) is complete.

Let us prove (49). Using the notation (67) write

$$\begin{aligned} F_3 &= H_1^* + t\mathbb{Q}_x(D_2^x), \qquad H_1^* := H_2^* + t\mathbb{Q}_y(D_2^y), \\ H_2^* :&= H_3^* + t\mathbb{Q}_{xy}(B_2), \qquad H_3^* := t\mathbb{Q}_{xy}(B_3) \end{aligned}$$

and denote, for $k = 1, 2, 3$,

$$f_k^* = \mathbf{E} J_k, \qquad J_k = J_0 \exp\{iH_k^*\}, \qquad \text{and} \qquad f_4^* = \mathbf{E} J_0. \tag{92}$$

In order to prove (49) we shall show that

$$\begin{aligned} f_2 &\sim f_1^* + f_5^*, \qquad f_5^* := it\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_x(D_2^x), && (93)\\ f_1^* &\sim f_2^* + f_6^*, \qquad f_6^* := it\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_y(D_2^y), && (94)\\ f_2^* &\sim f_3^* + f_7^*, \qquad f_7^* := it\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_{xy}(B_2), && (95)\\ f_3^* &\sim f_4^* + f_8^*, \qquad f_8^* := it\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_{xy}(B_3), && (96)\\ f_4^* &\sim \mathbf{E}\exp\{i\tilde{\mathbb{L}}\} + i\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}F_2. && (97) \end{aligned}$$

Let us prove (93). Expanding the exponent in powers of $it\mathbb{Q}_x(D_2^x)$ we obtain

$$f_2 = f_1^* + it\mathbf{E}J_1\mathbb{Q}_x(D_2^x) + t^2 R, \qquad |R| \le \mathbf{E}J_{xy}^y(\Omega_{m_2})\mathbb{Q}_x^2(D_2^x).$$

We shall show that

$$t^2 R \prec \mathcal{R} \qquad \text{and} \qquad it\mathbf{E}J_1\mathbb{Q}_x(D_2^x) \sim f_5^*. \tag{98}$$

In order to prove the first bound write $R = R_1 + R_2$, where we denote

$$\begin{aligned} R_1 &= \mathbf{E}J_{xy}^y(\Omega_{m_2})\mathbb{I}_y\mathbb{Q}_x^2(D_2^x), \qquad R_2 = \mathbf{E}J_{xy}^y(\Omega_{m_2})\bar{\mathbb{I}}_y\mathbb{Q}_x^2(D_2^x),\\ \mathbb{I}_y &= \mathbb{I}_{\{\alpha\varkappa_{xy}^y(\Omega_{m_2})<\sigma_y^2\}}, \qquad \bar{\mathbb{I}}_y = 1 - \mathbb{I}_y. \end{aligned}$$

It follows from (133) and (119) by symmetry, that

$$R_1 \ll (3 + s_2^2)^{-10}\mathbf{E}\mathbb{Q}_x^2(D_2^x) \ll (3 + s_2^2)^{-10}(m_1/N_1)\sigma^2(Q_x). \tag{99}$$

Therefore, we have $t^2 R_1 \prec \mathcal{R}$. In order to show that $t^2 R_2 \prec \mathcal{R}$ we apply Chebyshev's inequality and symmetry and use the inequality $J_{xy}^y(\Omega_{m_2}) \le 1$. We have

$$\begin{aligned} R_2 &\ll \sigma_y^{-2}\mathbf{E}\mathbb{Q}_x^2(D_2^x)\varkappa_{xy}^y(\Omega_{m_2}) = R_{21} + R_{22},\\ R_{21} &:= \sigma_y^{-2}\mathbf{E}\mathbb{Q}_x^2(D_2^x)(v_1^y)^2 = \sigma_y^{-2}\tau_2^{-2}\frac{m_1}{N_1}\sigma^2(Q_x)\sigma^2(Q_y),\\ R_{22} &:= \sigma_y^{-2}\mathbf{E}\mathbb{Q}_x^2(D_2^x)(u_1^y)^2 = \sigma_y^{-2}m_1 p_x q_x \mathbf{E}(v_1^x)^2(u_1^y)^2. \end{aligned}$$

Invoking the bound (121) we obtain

$$R_{22} \ll (m_1/N_1)\tau_2^{-2}\sigma_y^{-2}(\tau_1^{-2}\gamma_x + \sigma^2(Q_x)\sigma^2(Q_{xy})).$$

Since $t^2 R_{21} \prec \mathcal{R}$ and $t^2 R_{22} \prec \mathcal{R}$ we obtain $t^2 R_2 \prec \mathcal{R}$. This completes the proof of the first bound of (98).

Let us prove the second part of (98). By symmetry, it suffices to show that

$$it\mathbf{E}J_1\mathbb{Q}_x(D_*^x) \sim \tilde{f}_5^*, \qquad \tilde{f}_5^* := it\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_x(D_*^x), \tag{100}$$

where $D_*^x = (\Omega_{m_1} \setminus \Omega_{m_3}) \times (\Omega_{N_1} \setminus \Omega_{m_1})$. We shall prove (100). Expanding in powers of $it\mathbb{Q}_y(D_2^y)$ we obtain

$$\begin{aligned} it\mathbf{E}J_1\mathbb{Q}_x(D_*^x) &= \tilde{f}_6^* + (it)^2 R_1 + t^3 R_2, \qquad \tilde{f}_6^* := it\mathbf{E}J_2\mathbb{Q}_x(D_*^x), \qquad (101)\\ R_1 : &= \mathbf{E}J_2\mathbb{Q}_x(D_*^x)\mathbb{Q}_y(D_2^y), \qquad |R_2| \le \mathbf{E}J_y^x(\Omega_{m_3})|\mathbb{Q}_x(D_*^x)|\mathbb{Q}_y^2(D_2^y). \end{aligned}$$

Let us show that $t^3 R_2 \prec \mathcal{R}$. Using inequalities $a \le 1 + a^2$ and $J_y^x(\Omega_{m_3}) \le 1$ we write

$$|R_2| \le R_{21} + R_{22}, \qquad R_{21} := \mathbf{E}J_y^x(\Omega_{m_3})\mathbb{Q}_y^2(D_2^y), \qquad R_{22} := \mathbf{E}\mathbb{Q}_x^2(D_*^x)\mathbb{Q}_y(D_2^y).$$

We have $R_{22} \ll (m_1/N_1)(m_2/N_2)\sigma^2(Q_x)\sigma^2(Q_y)$. Therefore, $t^3 R_{22} \prec \mathcal{R}$. Furthermore, write

$$R_{21} = R_{21}' + R_{21}^*, \qquad R_{21}' := \mathbf{E}J_y^x(\Omega_{m_3})\mathbb{I}_y\mathbb{Q}_y^2(D_2^y), \qquad R_{21}^* := \mathbf{E}J_y^x(\Omega_{m_3})\bar{\mathbb{I}}_y\mathbb{Q}_y^2(D_2^y),$$

where we denote $\mathbb{I}_y = \mathbb{I}_{\{\alpha\varkappa_y^x(\Omega_{m_3}) < \sigma_x^2\}}$ and $\bar{\mathbb{I}}_y = 1 - \mathbb{I}_y$. Invoking (132) we obtain

$$R_{21}' \ll (3 + s_1^2)^{-10}\mathbf{E}\mathbb{Q}_y^2(D_2^y) \ll (3 + s_1^2)^{-10}(m_2/N_2)\sigma^2(Q_y).$$

Therefore, we have $t^3 R_{21}' \prec \mathcal{R}$. Furthermore, by Chebyshev's inequality and symmetry we write

$$\begin{aligned} R_{21}^* &\ll \sigma_x^{-2}\mathbf{E}\mathbb{Q}_y^2(D_2^y)\varkappa_y^x(\Omega_{m_3}) = \sigma_x^{-2}\mathbf{E}\mathbb{Q}_y^2(D_2^y)(u_1^x)^2\\ &= \sigma_x^{-2} m_2 p_y q_y \mathbf{E}(v_1^y)^2(u_1^x)^2\\ &\ll \sigma_x^{-2}(m_2/N_2)\tau_1^{-2}(\tau_2^{-2}\gamma_y + \sigma^2(Q_y)\sigma^2(Q_{xy})). \end{aligned}$$

In the last step we used (121). It follows that $t^3 R_{21}^* \prec \mathcal{R}$. Finally, we have

$$|t|^3|R_2| \le |t|^3(R_{21}' + R_{21}^* + R_{22}) \prec \mathcal{R}. \tag{102}$$

Let us show that $t^2 R_1 \prec \mathcal{R}$ for R_1 in (101). By symmetry, it suffices to show that

$$t^2 R_1' \prec \mathcal{R} \qquad \text{for} \qquad R_1' := \mathbf{E}J_2\mathbb{Q}_x(D_*^x)\mathbb{Q}_y(D_*^y), \tag{103}$$

where $D^y_* = (\Omega_{m_2} \setminus \Omega_{m_4}) \times (\Omega_{N_2} \setminus \Omega_{m_2})$. We have

$$\begin{aligned} |R'_1| &\le \mathbf{E} J^x_y(\Omega_{m_3}) J^y_x(\Omega_{m_4}) |\mathbb{Q}_x(D^x_*) \mathbb{Q}_y(D^y_*)| \le R'_{11} + R'_{12}, \\ R'_{11} : &= \mathbf{E} J^x_y(\Omega_{m_3}) \mathbb{Q}^2_y(D^y_*), \qquad R'_{12} := \mathbf{E} J^y_x(\Omega_{m_4}) \mathbb{Q}^2_x(D^x_*). \end{aligned}$$

Proceeding as in the proof of the bound $t^3 R_{21} \prec \mathcal{R}$ above we obtain $t^2 R'_{11} \prec \mathcal{R}$ and $t^2 R'_{12} \prec \mathcal{R}$, thus proving (101).

It follows from (101), (102), (103) that

$$it \mathbf{E} J_1 \mathbb{Q}_x(D^x_*) \sim \tilde{f}^*_6.$$

Let us show that

$$\tilde{f}^*_6 \sim \tilde{f}^*_7, \qquad \tilde{f}^*_7 := it \mathbf{E} J_3 \mathbb{Q}_x(D^x_*). \tag{104}$$

Expanding the exponent in powers of $it \mathbb{Q}_{xy}(B_2)$ we obtain

$$\tilde{f}^*_6 = \tilde{f}^*_7 + t^2 R, \qquad |R| \le \mathbf{E} J^y_x(\Omega_{m_2}) |\mathbb{Q}_x(D^x_*) \mathbb{Q}_{xy}(B_2)|.$$

We have

$$|R| \le R_1 + R_2, \qquad R_1 := \mathbf{E} J^y_x(\Omega_{m_2}) \mathbb{Q}^2_x(D^x_*), \qquad R_2 := \mathbf{E} J^y_x(\Omega_{m_2}) \mathbb{Q}^2_{xy}(B_2).$$

Proceeding as in the proof of the bound $t^3 R_{21} \prec \mathcal{R}$ above we obtain the bound $t^2 R_1 \prec \mathcal{R}$. In order to prove the bound $t^2 R_2 \prec \mathcal{R}$ write, by symmetry,

$$R_2 = m_1 (N_2 - m_2) p_x q_x p_y q_y \mathbf{E} V \le (m_1/N_1) \tau_1^2 \tau_2^2 \mathbf{E} V, \qquad V := \mathbf{E} J^y_x(\Omega_{m_2}) \psi^2_{xy}(X_1, Y_{N_2}).$$

Split

$$V = V_1 + V_2, \qquad V_1 := \mathbf{E} J^y_x(\Omega_{m_2}) \mathbb{I}_y \psi^2_{xy}(X_1, Y_{N_2}), \qquad V_2 := \mathbf{E} J^y_x(\Omega_{m_2}) \bar{\mathbb{I}}_y \psi^2_{xy}(X_1, Y_{N_2}),$$

where we denote $\mathbb{I}_y = \mathbb{I}_{\{\alpha \varkappa^y_x(\Omega_{m_2}) < \sigma^2_y\}}$ and $\bar{\mathbb{I}}_y = 1 - \mathbb{I}_y$. It follows from (133) that

$$V_1 \ll (3 + s_2^2)^{-10} \mathbf{E} \psi^2_{xy}(X_1, Y_{N_2}) \ll (3 + s_2^2)^{-10} \tau_1^{-2} \tau_2^{-2} \sigma^2(Q_{xy}).$$

By Chebyshev's inequality and symmetry,

$$\begin{aligned} V_2 &\ll \sigma_y^{-2} \mathbf{E} \psi^2_{xy}(X_1, Y_{N_2}) \varkappa^y_x(\Omega_{m_2}) = \sigma_y^{-2} \mathbf{E} \psi^2_{xy}(X_1, Y_{N_2}) (u^y_1)^2 \\ &\ll \sigma_y^{-2} \tau_1^2 \sigma^4_{xy} \ll \sigma_y^{-2} \tau_1^{-2} \tau_2^{-4} \sigma^4(Q_{xy}). \end{aligned}$$

Since $t^2 (m_1/N_1) \tau_1^2 \tau_2^2 V_k \prec \mathcal{R}$ for $k = 1, 2$, we obtain $t^2 R_2 \prec \mathcal{R}$, thus completing the proof of (104).

Let us show that

$$\tilde{f}_7^* \sim \tilde{f}_8^*, \qquad \tilde{f}_8^* := it\mathbf{E}J_0\mathbb{Q}_x(D_*^x). \tag{105}$$

Expanding the exponent in powers of $it\mathbb{Q}_{xy}(B_3)$ we obtain

$$\begin{aligned}\tilde{f}_7^* &= \tilde{f}_8^* + (it)^2 R_1 + t^3 R_2,\\ R_1 : &= \mathbf{E}J_0\mathbb{Q}_x(D_*^x)\mathbb{Q}_{xy}(B_3), \qquad |R_2| \le \mathbf{E}J^x(\Omega_{m_3})|\mathbb{Q}_x(D_*^x)|\mathbb{Q}_{xy}^2(B_3).\end{aligned}$$

In order to prove (105) we shall show that

$$t^2R_1 \prec \mathcal{R} \qquad \text{and} \qquad t^3R_2 \prec \mathcal{R}. \tag{106}$$

Let us prove the second bound. Using the inequality $a \le 1 + a^2$ write

$$|R_2| \le R_{21}+R_{22}, \qquad R_{21} := \mathbf{E}J^x(\Omega_{m_3})\mathbb{Q}_{xy}^2(B_3), \qquad R_{22} := \mathbf{E}\mathbb{Q}_x^2(D_*^x)\mathbb{Q}_{xy}^2(B_3).$$

By symmetry and (132), we have

$$R_{21} \le (m_2/N_2)\tau_1^2\tau_2^2\mathbf{E}J^x(\Omega_{m_3})\psi_{xy}^2(X_{N_1}, Y_1) \ll (m_2/N_2)(3 + s_1^2)^{-10}\sigma^2(Q_{xy}).$$

Therefore, $t^3R_{21} \prec \mathcal{R}$. Furthermore, by symmetry and (121), we have

$$R_{22} = (m_1-m_3)m_2p_xq_xp_yq_y\mathbf{E}(v_1^x)^2(u_1^y)^2 \ll (m_1m_2/N_1N_2)(\tau_1^{-2}\gamma_x+\sigma^2(Q_x)\sigma^2(Q_{xy})).$$

Therefore, $t^3R_{22} \prec \mathcal{R}$. We obtain the second bound of (106). In order to prove the first bound it suffices to show that

$$t^2R_1' \prec \mathcal{R} \qquad \text{for} \qquad R_1' := \mathbf{E}J_0\mathbb{Q}_x(D_*^x)\mathbb{Q}_{xy}(B_*),$$

where $B_* = (\Omega_{N_1} \setminus \Omega_{m_1}) \times (\Omega_{m_2} \setminus \Omega_{m_4})$. We have

$$\begin{aligned}|R_1'| &\le \mathbf{E}J^x(\Omega_{m_3})J^y(\Omega_{m_4})|\mathbb{Q}_x(D_*^x)\mathbb{Q}_{xy}(B_*)| \le R_{11}' + R_{12}',\\ R_{11}' : &= \mathbf{E}J^x(\Omega_{m_3})\mathbb{Q}_{xy}^2(B_*), \qquad R_{12}' := \mathbf{E}J^y(\Omega_{m_4})\mathbb{Q}_x^2(D_*^x).\end{aligned}$$

Invoking (132) and (133) and using symmetry we obtain $t^2R_{11}' \prec \mathcal{R}$ and $t^2R_{12}' \prec \mathcal{R}$, thus completing the proof of (106).

Finally expanding the exponent in $\tilde{f}_8^*$ in powers of iF_2 and using symmetry we obtain

$$\tilde{f}_8^* = \tilde{f}_5^* + tR, \qquad |R| \le \mathbf{E}J^x(\Omega_{m_1})J^y(\Omega_{m_2})|\mathbb{Q}_x(D_*^x)F_2|. \tag{107}$$

We have $|R| \ll |t|(R_1 + R_2 + R_3 + R_4)$, where

$$\begin{aligned}R_1 : &= \mathbf{E}J^y(\Omega_{m_2})\mathbb{Q}_x^2(D_*^x), \qquad R_2 := \mathbf{E}J^x(\Omega_{m_3})J^y(\Omega_{m_2})\mathbb{Q}_x^2(D_3^x),\\ R_3 : &= \mathbf{E}J^x(\Omega_{m_3})J^y(\Omega_{m_2})\mathbb{Q}_y^2(D_3^y), \qquad R_4 := \mathbf{E}J^x(\Omega_{m_3})J^y(\Omega_{m_2})\mathbb{Q}_{xy}^2(B_4).\end{aligned}$$

Invoking (132) and (133) and using symmetry we obtain $t^2 R_k \prec \mathcal{R}$ for $k = 1, 2, 3, 4$. This implies $\tilde{f}_8^* \sim \tilde{f}_5^*$ and completes the proof of (100). We arrive to (93). The proof of (94) is almost the same.

Let us prove (95). Expanding the exponent in powers of $it\mathbb{Q}_{xy}(B_2)$ we obtain

$$f_2^* = f_3^* + it\mathbf{E}J_3\mathbb{Q}_{xy}(B_2) + t^2 R, \qquad |R| \le \mathbf{E}J_x^y(\Omega_{m_2})\mathbb{Q}_{xy}^2(B_2).$$

In view of the bound (which is shown in the proof of (104)) $t^2|R| \prec \mathcal{R}$, we need to prove that $it\mathbf{E}J_3\mathbb{Q}_{xy}(B_2) \sim f_7^*$. By symmetry, it suffices to show that

$$it\mathbf{E}J_3\mathbb{Q}_{xy}(B_*) \sim \tilde{f}_9^*, \qquad \tilde{f}_9^* := it\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_{xy}(B_*), \tag{108}$$

where $B_* = (\Omega_{m_1} \setminus \Omega_{m_3}) \times (\Omega_{N_2} \setminus \Omega_{m_2})$. We shall show that

$$it\mathbf{E}J_3\mathbb{Q}_{xy}(B_*) \sim it\mathbf{E}J_0\mathbb{Q}_{xy}(B_*) \qquad \text{and} \qquad it\mathbf{E}J_0\mathbb{Q}_{xy}(B_*) \sim \tilde{f}_9^*. \tag{109}$$

In order to prove the first relation of (109) we expand the exponent in powers of $it\mathbb{Q}_{xy}(B_3)$,

$$\begin{aligned} it\mathbf{E}J_3\mathbb{Q}_{xy}(B_*) &= it\mathbf{E}J_0\mathbb{Q}_{xy}(B_*) + (it)^2 R_1 + t^3 R_2, \\ R_1 :&= \mathbf{E}J_0\mathbb{Q}_{xy}(B_*)\mathbb{Q}_{xy}(B_3), \\ |R_2| &\le \mathbf{E}J^x(\Omega_{m_3})|\mathbb{Q}_{xy}(B_*)|\mathbb{Q}_{xy}^2(B_3) \end{aligned}$$

and show that $t^2 R_1 \prec \mathcal{R}$ and $t^3|R_2| \prec \mathcal{R}$. The proof of these bounds is similar to that of (106). The proof of the second relation of (109) is similar to that of the relation $\tilde{f}_8^* \sim \tilde{f}_5^*$, see (107) above. We obtain (108) thus completing the proof of (95). The proof of (96) and (97) is similar. The proof of (45) is complete.

8.2.2. Proof of (46). We shall show that

$$\begin{aligned} it\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_{xy} &\sim \tau_1^2\tau_2^2\kappa_{xy}(it)^3\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}, && (110) \\ it\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_{x} &\sim 2^{-1}\tau_1^4\kappa_{x}(it)^3\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}, && (111) \\ it\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_{y} &\sim 2^{-1}\tau_2^4\kappa_{y}(it)^3\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}. && (112) \end{aligned}$$

Write

$$\begin{aligned} &\tilde{\mathbb{L}} = \mathbb{T} + \mathbb{Z}, \qquad \mathbb{T} = \sum_{k=1}^{N_1} t_k w_k, \qquad \mathbb{Z} = \sum_{j=1}^{N_2} z_j \xi_j, && (113) \\ &t_k = tg_x(X_k) + s_1/\tau_1, \qquad z_j = tg_y(Y_j) + s_2/\tau_2. \end{aligned}$$

Let us prove (110). By symmetry,

$$it\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_{xy} = N_1 N_2 h_1, \qquad h_1 = it\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}\psi_{xy}(X_1, Y_1)\omega_1\xi_1.$$

Split $\tilde{\mathbb{L}} = \tilde{\mathbb{L}}' + t_1\omega_1 + z_1\xi_1$. Expanding the exponent $\exp\{i\tilde{\mathbb{L}}' + it_1\omega_1 + iz_1\xi_1\}$ in powers of $it_1\omega_1$ and $iz_1\xi_1$ we obtain

$$\begin{aligned} h_1 &= h_2 + R_1 + R_2, \\ h_2 : &= it\mathbf{E}\exp\{i\tilde{\mathbb{L}}'\}w_1, \qquad w_1 := -\psi_{xy}(X_1, Y_1)t_1 z_1 \omega_1^2 \xi_1^2, \\ |R_1| &\le |t|\mathbf{E}J^x(\Omega'_{N_1})J^y(\Omega'_{N_2})|w_1 t_1 \omega_1|, \\ |R_2| &\le |t|\mathbf{E}J^x(\Omega'_{N_1})J^y(\Omega'_{N_2})|w_1 z_1 \xi_1|. \end{aligned} \tag{114}$$

Here $\Omega'_{N_k} = \Omega_{N_k} \setminus \{1\}$. We have

$$h_2 = it\mathbf{E}\exp\{i\tilde{\mathbb{L}}'\}w_2, \qquad \text{where} \qquad w_2 := -p_x^2 q_x^2 p_y^2 q_y^2 \psi_{xy}(X_1, Y_1)t_1 z_1.$$

Proceeding as in (114) we obtain

$$\begin{aligned} h_2 &= h_3 + R_3 + R_4, \qquad h_3 = it\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}w_2, \\ |R_3| &\le |t|\mathbf{E}J^x(\Omega'_{N_1})J^y(\Omega'_{N_2})|w_2 t_1 \omega_1|, \\ |R_4| &\le |t|\mathbf{E}J^x(\Omega'_{N_1})J^y(\Omega'_{N_2})|w_2 z_1 \xi_1|. \end{aligned} \tag{115}$$

Let $\mathbf{E}_{\nu,\xi}$ denote the conditional expectation given all the random variables, but $\{\nu_k,\ k = 1, \dots, N_1\}$ and $\{\xi_r,\ r = 1, \dots, N_2\}$. Note that $(t, s_1, s_2) \to \mathbf{E}_{\nu,\xi}\exp\{i\tilde{\mathbb{L}}\}$ is a non-random function. Therefore, we have

$$\begin{aligned} h_3(t) &= \mathbf{E}\mathbf{E}_{\nu,\xi}\exp\{i\tilde{\mathbb{L}}\}w_2 = \mathbf{E}_{\nu,\xi}\exp\{i\tilde{\mathbb{L}}\}\mathbf{E}w_2 \\ &= \mathbf{E}\exp\{i\tilde{\mathbb{L}}\}w_3, \end{aligned} \tag{116}$$

where $w_3 := \mathbf{E}w_2 = (it)^2 p_x q_x p_y q_y \kappa_{xy}$. The last identity uses (25). From (114), (115) and (116) we obtain

$$it\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_{xy} = \tau_1^2\tau_2^2\kappa_{xy}(it)^3\mathbf{E}\exp\{i\tilde{\mathbb{L}}\} + N_1 N_2 R, \qquad R := R_1 + R_2 + R_3 + R_4.$$

We complete the proof of (110) by showing that $R \prec \mathcal{R}$.

Invoking (132) and (133) we obtain

$$|R| \ll |t|(3 + s_1^2)^{-10}(3 + s_2^2)^{-10}\mathbf{E}(|w_1| + |w_2|)(|t_1| + |z_1|). \tag{117}$$

Using Cauchy-Schwartz we obtain for $k = 1, 2$,

$$\begin{aligned} \mathbf{E}|w_k t_1| &\ll (|t|^3 + |s_1|^3 + |s_2|^3)N_1^{-1}N_2^{-1}\tau_1^{-1}\sigma(Q_{xy})B_{x4}^{1/2}, \\ \mathbf{E}|w_k z_1| &\ll (|t|^3 + |s_1|^3 + |s_2|^3)N_1^{-1}N_2^{-1}\tau_2^{-1}\sigma(Q_{xy})B_{y4}^{1/2}. \end{aligned} \tag{118}$$

These bounds in combination with (117) imply $R \prec \mathcal{R}$. We arrived to (110).

The proof of (111) and (112) is almost the same. We show that

$$|it\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_x - 2^{-1}\tau_1^4\kappa_x(it)^3\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}| \prec |t|(1+|t|^3)\tau_1^{-1}\sigma(Q_x)B_{x4}^{1/2},$$
$$|it\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}\mathbb{Q}_y - 2^{-1}\tau_2^4\kappa_y(it)^3\mathbf{E}\exp\{i\tilde{\mathbb{L}}\}| \prec |t|(1+|t|^3)\tau_2^{-1}\sigma(Q_y)B_{y4}^{1/2}.$$

The proof of (46) is complete. Lemma 8.1 is proved.

9 Moment inequalities

It follows from (27) that

$$\sigma_x^2 \ll \frac{\sigma^2(L_x)}{\tau_1^2}, \qquad \sigma_y^2 \ll \frac{\sigma^2(L_y)}{\tau_2^2}, \qquad \sigma_{xx}^2 \ll \frac{\sigma^2(Q_x)}{\tau_1^4}, \tag{119}$$
$$\sigma_{yy}^2 \ll \frac{\sigma^2(Q_y)}{\tau_2^4}, \qquad \sigma_{xy}^2 \ll \frac{\sigma^2(Q_{xy})}{\tau_1^2\tau_2^2}.$$

Lemma 9.1. *We have*

$$\mathbf{E}\mathbb{Q}_{xy}^2(B_1) \ll \frac{m_1}{N_1}\frac{m_2}{N_2}\sigma^2(Q_{xy}), \tag{120}$$
$$\mathbf{E}\mathbb{Q}_x^2(D_1^x) \ll \frac{m_1^2}{N_1^2}\sigma^2(Q_x), \qquad \mathbf{E}\mathbb{Q}_y^2(D_1^y) \ll \frac{m_2^2}{N_2^2}\sigma^2(Q_y),$$

$$\mathbf{E}(v_1^x)^2(u_1^y)^2 \ll \tau_1^{-2}\tau_2^{-2}(\tau_1^{-2}\gamma_x + \sigma^2(Q_x)\sigma^2(Q_{xy})), \tag{121}$$
$$\mathbf{E}(v_1^y)^2(u_1^x)^2 \ll \tau_1^{-2}\tau_2^{-2}(\tau_2^{-2}\gamma_y + \sigma^2(Q_y)\sigma^2(Q_{xy})).$$

Proof of Lemma 9.1. Inequalities (120) follow from (119) and the identities

$$\mathbf{E}\mathbb{Q}_{xy}^2(B_1) = m_1m_2\mathbf{E}\omega_1^2\xi_1^2\sigma_{xy}^2, \qquad \mathbf{E}\mathbb{Q}_x^2(D_1^x) = 2^{-1}m_1(m_1-1)\mathbf{E}\omega_1^2\omega_2^2\sigma_{xx}^2,$$
$$\mathbf{E}\mathbb{Q}_y^2(D_1^y) = 2^{-1}m_2(m_2-1)\mathbf{E}\xi_1^2\xi_2^2\sigma_{yy}^2.$$

Let us prove the first bound of (121). The proof of the second bound is almost the same. Denote for brevity $a_i = \psi_x(X_1, X_i)\omega_i$ and $b_j = \psi_{xy}(X_j, Y_1)\omega_j$. By independence, we have

$$\mathbf{E}(\sum_i a_i)^2(\sum_j b_j)^2 = \mathbf{E}S_1 + \mathbf{E}S_2 + 4\mathbf{E}S_3, \tag{122}$$
$$S_1 = \sum_i a_i^2 b_i^2, \qquad S_2 = \sum_{i<j}(a_i^2b_j^2 + a_j^2b_i^2), \qquad S_3 = \sum_{i<j} a_ia_jb_ib_j.$$

Here $\sum_i$ denotes the sum over $m_1 < i \le N_1$ and $\sum_{i<j}$ denotes the sum over $m_1 < i < j \le N_1$. By symmetry we have

$$\mathbf{E}S_1 = (N_1 - m_1)\mathbf{E}a_{N_1}^2 b_{N_1}^2 \ll \tau_1^2 \mathbf{E}\psi_x^2(X_1, X_{N_1})\psi_{xy}^2(X_{N_1}, Y_1) \le \tau_1^{-4}\tau_2^{-2}\gamma_x.$$

Furthermore,

$$\begin{aligned}\mathbf{E}S_2 &= \binom{N_1 - m_1}{2}\mathbf{E}a_{N_1}^2 b_{N_1-1}^2 \le \tau_1^4 \mathbf{E}\psi_x^2(X_1, X_{N_1})\psi_{xy}^2(X_{N_1+1}, Y_1) \\ &\ll \tau_1^4 \sigma_{xx}^2 \sigma_{xy}^2 \ll \tau_1^{-2}\tau_2^{-2}\sigma^2(Q_x)\sigma^2(Q_{xy}).\end{aligned}$$

Finally, by Cauchy-Schwartz,

$$\mathbf{E}S_3 \le \sum_{i<j}(\mathbf{E}a_i^2 b_j^2 + \mathbf{E}a_j^2 b_i^2) = \mathbf{E}S_2.$$

Collecting the bounds for $\mathbf{E}S_1$ and $\mathbf{E}S_2 = \mathbf{E}S_3$ in (122) we obtain the first bound of (121).

10 Auxiliary inequalities

Denote

$$\begin{aligned}\beta_x(t) &= \mathbf{E}e^{i\omega_1 t}, \qquad \beta_y(t) = \mathbf{E}e^{i\xi_1 t}, \\ \Theta(t) &= (\frac{2}{\pi}\frac{\pi - t}{\pi + t})^2, \qquad \alpha = \frac{32\pi}{\Theta(1)}(1 + \frac{4}{\Theta(1)}), \qquad (123) \\ u_{\{d,x\}}(t) &= 1 - p_x q_x \frac{\Theta(d)}{2} t^2 \mathbb{I}_{\{|t|<d+\pi\}}, \qquad v_{\{d,x\}}(t) = 1 + p_x q_x \frac{2\pi}{d}(\frac{4}{\Theta(d)} + 1)t^2, \\ u_{\{d,y\}}(t) &= 1 - p_y q_y \frac{\Theta(d)}{2} t^2 \mathbb{I}_{\{|t|<d+\pi\}}, \qquad v_{\{d,y\}}(t) = 1 + p_y q_y \frac{2\pi}{d}(\frac{4}{\Theta(d)} + 1)t^2.\end{aligned}$$

In proofs we use the following result, see Lemma 5.2 in Bloznelis and Götze (2000), *Bernoulli*, **6**, 729-760.

Lemma 10.1. *For each $d \in (0, \pi)$ and $t, s \in \mathbb{R}$ we have*

$$|\beta_x(s+t)|^2 \le u_{\{d,x\}}(t)v_{\{d,x\}}(s), \qquad |\beta_y(s+t)|^2 \le u_{\{d,y\}}(t)v_{\{d,y\}}(s). \quad (124)$$

Denote

$$\varepsilon_x = 8(N_1^{2/3}B_{x3}^{2/3} + 1)/(N_1 - 4), \quad \varepsilon_y = 8(N_2^{2/3}B_{y3}^{2/3} + 1)/(N_2 - 4). \quad (125)$$

Lemma 10.2. *Let $c_T > 2$ and $d \in (0, \pi)$. Assume that $|t| \le T$, where $T^{-1} = c_T B_{x3} \sigma_x$. For $c' = \min\{c_T d, c_T^2 d^2\}$ we have*

$$\mathbf{E}(tg_x(X_1) + \frac{s_1}{\tau_1})^2 \mathbb{I}_{\{|tg_x(X_1)|<d\}} \ge (\sigma_x^2 t^2 + \frac{s_1^2}{\tau_1^2})(1 - 2/c'), \tag{126}$$

$$\mathbf{E}^{[4]}(tg_x(X_1) + \frac{s_1}{\tau_1})^2 \mathbb{I}_{\{|tg_x(X_1)|<d\}} \ge (\sigma_x^2 t^2 + \frac{s_1^2}{\tau_1^2})(1 - \varepsilon - 2/c'). \tag{127}$$

Here $\mathbf{E}^{[4]}$ denotes the conditional expectation given X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}, where $1 \notin \{i_1, i_2, i_3, i_4\}$.

Proof of lemma 10.2. The proof is similar to the proof of Lemma 5.3 ibidem. Let us prove (126). Denote

$$\mathbb{I} = \mathbb{I}_{\{T|g_x(X_1)|<d\}}, \qquad \bar{\mathbb{I}} = 1 - \mathbb{I}, \qquad Z_1 = tg_x(X_1) + s_1/\tau_1, \qquad A = \mathbf{E} Z_1^2 \bar{\mathbb{I}}.$$

For $|t| \le T$ we have

$$\begin{aligned} \mathbf{E} Z_1^2 \mathbb{I}_{\{|tg_x(X_1)|<d\}} &\ge \mathbf{E} Z_1^2 \mathbb{I} = \mathbf{E} Z_1^2 - A \\ &= \sigma_x^2 t^2 + \tau_1^{-2} s_1^2 - A. \end{aligned}$$

In the last identity we use $\mathbf{E} g_x(X_1) = 0$. Therefore, in order to prove (126) it suffices to show that

$$A \le (2/c')(\sigma_x^2 t^2 + \tau_1^{-2} s_1^2). \tag{128}$$

Introduce the set $K \subset \mathcal{X}$, $K = \{x_j : T|g_x(x_j)| \ge d\}$. Write

$$A = N_1^{-1} \sum_{x_j \in K} (tg_x(x_j) + \tau_1^{-1} s_1)^2 \le \frac{2}{N_1} t^2 W + \frac{2}{N_1} \frac{s_1^2}{\tau_1^2} |K|, \tag{129}$$

where $W = \sum_{x_j \in K} g_x^2(x_j)$. By Chebyshev's inequality, for $r > 0$,

$$|K| \le \sum_{x_j \in K} (T/d)^r |g_x(x_j)|^r \le (T/d)^r N_1 \mathbf{E}|g_x(X_1)|^r = (T/d)^r N_1 B_{xr} \sigma_x^r. \tag{130}$$

In particular, we have

$$|K| \le (T/d)^3 N_1 B_{x3} \sigma_x^3, \qquad |K| \le (T/d)^2 N_1 \sigma_x^2. \tag{131}$$

Furthermore, by Minkovski's inequality,

$$W \le (\sum_{x_j \in K} |g_x(x_j)|^3)^{2/3} |K|^{1/3} \le (N_1 \mathbf{E}|g_x(X_1)|^3)^{2/3} |K|^{1/3}.$$

Invoking the first (respectively second) bound of (131) and the identity $Tc_T B_{x3}\sigma_x = 1$ we obtain

$$W \le N_1(T/d)B_{x3}\sigma_x^3 = N_1\sigma_x^2/c_T d, \qquad |K| \le N_1/c_T^2 d^2 B_{x3}^2 \le N_1/c_T^2 d^2.$$

In the last step we invoke the simple bound $B_{x3} \ge 1$. Substitution of these bounds in (129) gives (128).

Let us prove (127). Assume without loss of generality that $\{i_1, i_2, i_3, i_4\} = \{2, 3, 4, 5\}$. We have

$$\mathbf{E}^{[4]}(tg_x(X_1) + \frac{s_1}{\tau_1})^2 \mathbb{I}_{\{|tg_x(X_1)|<d\}} = \frac{N_1}{N_1 - 4}\mathbf{E}Z_1^2 \mathbb{I}_{\{|tg_x(X_1)|<d\}} - \frac{1}{N_1 - 4}B,$$

where $B = \sum_{j=2}^{5}(tg_x(X_j)+s_1/\tau_1)^2$. Therefore, (127) follows from (126) and the inequalities

$$B \le 2\sum_{j=2}^{5}(t^2 g_x^2(X_j) + s_1^2/\tau_1^2), \qquad \sum_{j=2}^{5} g_x^2(X_j) \le 4^{1/3} N_1^{2/3} B_{3x}^{2/3}\sigma_x^2.$$

The last inequality follows by Minkovski inequality,

$$\sum_{j=2}^{5} g_x^2(X_j) \le 4^{1/3}(\sum_{j=2}^{5}|g_x^2(X_j)|^{3/2})^{2/3} \le 4^{1/3}N_1^{2/3}(\frac{1}{N_1}\sum_{j=1}^{N_1}|g_x(X_j)|^3)^{2/3}.$$

Proof of Lemma 10.2.is complete.

Let $\mathbf{E}_X$ (respectively $\mathbf{E}_Y$) denote the conditional expectation given all the random variables but X^* (respectively Y^*). By $\mathbf{E}_X^{i_1,\dots,i_k}$ (respectively $\mathbf{E}_Y^{i_1,\dots,i_k}$), we denote the conditional expectation given all the random variables but $\{X_1, \dots, X_{N_1}\}\backslash\{X_{i_1}, \dots, X_{i_k}\}$ (respectively $\{Y_1, \dots, Y_{N_2}\}\backslash\{Y_{i_1}, \dots, Y_{i_k}\}$).

Lemma 10.3 *Assume that $|t| \le T$, where the number T is defined in (43) and the constant c_T in (43) is larger than 8. Assume that numbers $\varepsilon_x, \varepsilon_y$ (defined in (125)) are smaller than $1/4$. Assume that the constant C' in (47) is larger than $10^3/\Theta(1)$. Then for every $B \subset \Omega_{m_1}$, $|B| \ge m_1/4$ and $D \subset \Omega_{m_2}$, $|D| \ge m_2/4$, and for every $\{ij\} \subset \Omega_{N_1} \setminus B$, and $\{k,l\} \subset \Omega_{N_2} \setminus D$ we have almost surely*

$$\mathbf{E}_X^{ij} J_{xy}^x(B)\mathbb{I}_{\{\alpha\varkappa_{xy}^x(B)<\sigma_x^2\}} \ll (3+s_1^2)^{-10}, \quad \mathbf{E}_X^{ij} J^x(B) \ll (3+s_1^2)^{-10}, \tag{132}$$

$$\mathbf{E}_X^{ij} J_x^x(B)\mathbb{I}_{\{\alpha\varkappa_x^x(B)<\sigma_x^2\}} \ll (3+s_1^2)^{-10}, \quad \mathbf{E}_X^{ij} J_y^x(B)\mathbb{I}_{\{\alpha\varkappa_y^x(B)<\sigma_x^2\}} \ll (3+s_1^2)^{-10},$$

$$\mathbf{E}_Y^{kl} J_{xy}^y(D)\mathbb{I}_{\{\alpha\varkappa_{xy}^y(D)<\sigma_y^2\}} \ll (3+s_2^2)^{-10}, \quad \mathbf{E}_Y^{kl} J^y(D) \ll (3+s_2^2)^{-10}, \tag{133}$$

$$\mathbf{E}_Y^{kl} J_x^y(D)\mathbb{I}_{\{\alpha\varkappa_x^y(D)<\sigma_y^2\}} \ll (3+s_2^2)^{-10}, \quad \mathbf{E}_Y^{kl} J_y^y(D)\mathbb{I}_{\{\alpha\varkappa_y^y(D)<\sigma_y^2\}} \ll (3+s_2^2)^{-10}.$$

The integer valued functions $m_1 = m_1(s_1)$ and $m_2 = m_2(s_2)$ are defined in (47).

Remark. Inequalities (132) and (133) remain valid if we replace the expectation E_X^{ij} (respectively $\mathbf{E}_Y^{kl}$) by $\mathbf{E}_X$ or $\mathbf{E}_X^i$ (respectively $\mathbf{E}_Y$ or $\mathbf{E}_Y^k$).

Proof of Lemma 10.3. We shall prove the first inequality of (132) only. Denote $Z_k = tg_x(X_k) + \tau_1^{-1}s_1$. It follows from (124) that

$$J_{xy}^x(B) \le W_1 W_2, \qquad W_1^2 := \prod_{k\in B} u_{\{1,x\}}(Z_k), \qquad W_2^2 := \prod_{k\in B} v_{\{1,x\}}(tv_k^x + tu_k^x). \tag{134}$$

The inequality $1 + z \le e^z$ implies almost surely

$$\begin{aligned} W_2 \mathbb{I}_{\{\alpha\varkappa_{xy}^x(B)<\sigma_x^2\}} &\le \mathbb{I}_{\{\alpha\varkappa_{xy}^x(B)<\sigma_x^2\}} \exp\{2\pi(1+4/\Theta(1))|B|p_x q_x \varkappa_{xy}^x(B)t^2\} \\ &\le \mathbb{I}_{\{\alpha\varkappa_{xy}^x(B)<\sigma_x^2\}} \exp\{16^{-1}\Theta(1)\alpha\varkappa_{xy}^x(B)(|B|/N_1)\tau_1^2 t^2\} \\ &\le \exp\{16^{-1}\Theta(1)(|B|/N_1)\sigma_x^2\tau_1^2 t^2\} =: r. \end{aligned} \tag{135}$$

Therefore, we have almost surely

$$J_{xy}^x(B)\mathbb{I}_{\{\alpha\varkappa_{xy}^x(B)<\sigma_x^2\}} \le W_1 r. \tag{136}$$

Hoeffding's (1963) Theorem 4 implies $\mathbf{E}_X^{ij} W_1^2 \le (\mathbf{E}_X^{ij} u_{\{1,x\}}(Z_1))^{|B|}$. Therefore,

$$\mathbf{E}_X^{ij} W_1 \le (\mathbf{E}_X^{ij} W_1^2)^{1/2} \le (\mathbf{E}_X^{ij} u_{\{1,x\}}(Z_1))^{|B|/2}. \tag{137}$$

For $|s_1/\tau_1| \le \pi$ and $|t| \le T$ we obtain from (127) that

$$\begin{aligned} \mathbf{E}_X^{ij} Z_1^2 \mathbb{I}_{\{|tg_x(X_1)+s_1/\tau_1|<\pi+1\}} &\ge \mathbf{E}_X^{ij} Z_1^2 \mathbb{I}_{\{|tg_x(X_1)|<1\}} \\ &\ge (\sigma_x^2 t^2 + s_1^2/\tau_1^2)(1 - \varepsilon_x - 2/c_T), \end{aligned}$$

where ε_x is defined in (125). Therefore, we have

$$\mathbf{E}_X^{ij} u_{\{1,x\}}(Z_1) \le 1 - 2^{-1} p_x q_x \Theta(1)(\sigma_x^2 t^2 + s_1^2/\tau_1^2)(1 - \varepsilon_x - 2/c_T). \tag{138}$$

The inequality $1 + z \le e^z$ combined with (137) and (138) implies

$$\begin{aligned} \mathbf{E}_X^{ij} W_1 &\le \exp\{-4^{-1}(1-\varepsilon_x - 2/c_T)\Theta(1)p_x q_x |B|(\sigma_x^2 t^2 + s_1^2/\tau_1^2)\} \\ &\le \exp\{-4^{-1}(1-\varepsilon_x - 2/c_T)\Theta(1)(|B|/N_1)(\sigma_x^2\tau_1^2 t^2 + s_1^2)\} \\ &\le \exp\{-8^{-1}\Theta(1)(|B|/N_1)(\sigma_x^2\tau_1^2 t^2 + s_1^2)\}. \end{aligned} \tag{139}$$

Combining this bound and (136) we obtain

$$\begin{aligned} \mathbf{E}_X^{ij} J_{xy}^x(B)\mathbb{I}_{\{\alpha\varkappa_{xy}^x(B)<\sigma_x^2\}} &\le \exp\{-16^{-1}\Theta(1)(|B|/N_1)(\sigma_x^2\tau_1^2 t^2 + s_1^2)\} \\ &\le \exp\{-64^{-1}\Theta(1)(m_1/N_1)(\sigma_x^2\tau_1^2 t^2 + s_1^2)\} \\ &\le \exp\{-\frac{\Theta(1)}{64}\frac{m_1}{N_1}s_1^2\} \ll \frac{1}{(3+s_1^2)^{10}}. \end{aligned}$$

The proof of the first inequality of (132) is complete. The proof of the remaining inequalities of the lemma is much the same.

11 References

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