1. Introduction and Results

Let $X_1, \dots, X_N, \dots$ be independent identically distributed random variables with the common distribution function F, expectation $\mathbf{E}\, X_1 = \mu$, and finite non-vanishing variance $\sigma^2 = \mathbf{E}\,(X_1 - \mu)^2$. Let

$$\mathbf{t}_N = \frac{\overline{X}_N - \mu}{\hat{\sigma}_N}$$

denote Student's t statistic, where

$$\hat{\sigma}_N^2 = N^{-1} \sum\nolimits_{i=1}^{N} (X_i - \overline{X}_N)^2 \qquad \text{and} \qquad \overline{X}_N = N^{-1}(X_1 + \dots + X_N).$$

It is well known that the statistic $T_N = \sqrt{N}\mathbf{t}_N$ is asymptotically standard normal, i.e., as $N \to \infty$

$$\text{(1.1)} \qquad \sup_x |G_N(x) - \Phi(x)| \to 0, \qquad \text{where} \qquad G_N(x) = \mathbf{P}\,\{T_N \le x\}$$

denotes the distribution function of T_N and where $\Phi(x)$ denotes the standard normal distribution function.

The accuracy of (1.1) was studied by a number of authors, Helmers and van Zwet (1982), Helmers (1985), Slavova (1985), Hall (1988), Praškova (1989), Friedrich (1989), Bentkus and Götze (1996), Bentkus, Bloznelis and Götze (1996), etc.

The Berry–Esseen bound of order $O(N^{-1/2})$ under the minimal moment condition $\mathbf{E}\,|X_1|^3 < \infty$ was obtained by Slavova (1985). Bentkus and Götze (1996) proved the bound $\sup_x |G_N(x) - \Phi(x)| < cN^{-1/2}\beta_3/\sigma^3$, thus extending the classical result of Esseen (1945) to the Studentized sums. Here $\beta_3 := \mathbf{E}\,|X_1 - \mu|^3$.

A higher order approximation to Student's t statistic was considered by Chung (1946), Bhattacharya and Ghosh (1978), Chibisov (1980), Hall (1987), Bentkus, Götze and van Zwet (1997), Putter and van Zwet (1998), etc.

Although increasingly general and precise, none of these results is optimal in the sense of expansions for $G_N(x)$ being established under minimal conditions. Hall (1987) proved the validity of a k-term Edgeworth expansion for Student's t statistic with remainder $o(N^{-k/2})$, for every integer k, provided that $\mathbf{E}\,|X_1|^{k+2} < \infty$ and the distribution F is non-singular. The moment conditions in Hall (1987) are the minimal ones, but the smoothness condition on the distribution F of the observations is too restrictive. What is perhaps more important, Hall's result is only valid for a fixed underlying distribution function. As a result, it cannot be applied to the bootstrap.

The aim of the present paper is to prove the validity of an expansion under minimal conditions, in such a way that an extension to the bootstrap is straightforward. To this end, we approximate $G_N(x)$ by the one-term Edgeworth expansion

$$H_N(x) = \Phi(x) + \frac{\kappa_3}{6\sqrt{N}}\,(2\,x^2 + 1)\,\Phi'(x), \qquad \kappa_3 = \mathbf{E}\,(X_1 - \mu)^3/\sigma^3$$

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and construct explicit bounds for the remainder

$$\Delta_N = \sup_x |G_N(x) - H_N(x)|.$$

The minimal smoothness condition which allows to prove the validity of one-term Edgeworth expansion, i.e., to prove the bound $\Delta_N = o(N^{-1/2})$, is that F is non-lattice.

Theorem 1.1. *Assume that* $\mathbf{E}\,|X_1|^3 < \infty$ *and the distribution of* X_1 *is non-lattice. Then*

$$\Delta_N = o(N^{-1/2}) \qquad \text{as} \quad N \to \infty. \tag{1.2}$$

Remark. If $\mu = 0$, then Theorem 1.1 remains valid if we replace Δ_N by $\tilde{\Delta}_N := \sup_x |\tilde{G}_N(x) - H_N(x)|$, where $\tilde{G}_N(x) := \mathbf{P}\{\tilde{T}_N \le x\}$ denotes the distribution function of the selfnormalized sum

$$\tilde{T}_N = \frac{X_1 + \cdots + X_N}{(X_1^2 + \cdots + X_N^2)^{1/2}}\,.$$

Bootstrap. Given $\mathbb{X}_N := \{X_1, \ldots, X_N\}$, let $X_1^*, \ldots, X_N^*$ denote independent random variables uniformly distributed in $\mathbb{X}$. Write

$$T_N^* = \sqrt{N}\,\frac{\overline{X_N^*} - \overline{X}_N}{\hat{\sigma}_N^*}\,, \qquad\qquad {\sigma_N^*}^2 = N^{-1}\sum_{i=1}^N (X_i^* - \overline{X_N^*})^2$$

and let $G_N^*(x) := \mathbf{P}\{T_N^* \le x \,|\, \mathbb{X}_N\}$ denote the conditional probability of the event $\{T_N^* \le x\}$ given $\mathbb{X}_N$, for $x \in \mathbb{R}$.

Theorem 1.2. *Suppose that* $\mathbf{E}\,|X_1|^3 < \infty$ *and that the distribution of* X_1 *is non-lattice. Then as* $N \to \infty$

$$\sup_{x\in\mathbb{R}} |G_N^*(x) - G_N(x)| = o(N^{-1/2}) \qquad \text{a. s.} \tag{1.3}$$

Theorem 1.2 improves earlier results of Babu and Singh (1983), Helmers (1991) and Putter and van Zwet (1998) where the bound (1.3) was established assuming that F is non-lattice and under increasingly sharp moment conditions, the sharpest to date being $\mathbf{E}\,|X_1|^{3+\varepsilon} < \infty$, for some $\varepsilon > 0$, obtained in the latter paper.

Theorems 1.1 and 1.2 are consequences of an explicit upper bound for Δ_N given in Proposition 2.1, which is formulated in Section 2 below.

A usual proof of the validity of Edgeworth expansions for distribution functions combines Esseen's (1945) smoothing lemma and expansions of characteristic functions. Such scheme applies easily to a linear statistic since its characteristic function has a comparatively simple multiplicative structure. Before to apply it to Student's $\mathbf{t}$ statistic we approximate $\mathbf{t}$ by a statistic which is conditionally linear in the first m observations $X_1, \dots, X_m$, $m < n$, given the remaining part of the sample $X_{m+1}, \dots, X_n$. Here $m \approx \ln N$. Bentkus and Götze (1996) used similar idea to estimate the characteristic function of Student's $\mathbf{t}$ statistic.

The rest of the paper is organized as follows. In Section 2 we prove Theorems 1.1 and 1.2. Several steps of the proofs which are more technical are deferred to Section 3.

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2. Proofs

In the beginning of this section we formulate Proposition 2.1. Then we prove Theorems 1.1 and 1.2, which are corollaries of Proposition 2.1. The proof of our main result, Proposition 2.1, is postponed until after the proofs of Theorems 1.1 and 1.2.

Write

$$\rho_u = 1 - \sup\bigl\{ |\,\mathbf{E}\,\exp\{i\,t\,X_1\}| \,:\; \sigma^2/(9\beta_3) \le |t| \le u/\sigma \bigr\}, \qquad u > 0.$$

For the remainder of this paper we shall assume without loss of generality that $\mathbf{E}\,X_1 = 0$ and $\sigma^2 = \mathbf{E}\,X_1^2 = 1$.

Proposition 2.1. *There exists an absolute constant $c > 0$ such that for each $N = 2, 3, \ldots$ and each $1 < u < N^{1/6}$,*

$$\Delta_N \le \frac{c}{\sqrt{N}}\Bigl(\frac{1}{u} + \frac{A_N}{\rho_u^4} + B_N\Bigr), \tag{2.1}$$
$$A_N = N^{-1/2}\bigl(1 + \mathbf{E}\,X_1^4 \mathbb{I}\{X_1^2 \le N\}\bigr),$$
$$B_N = N^{-1/2}\beta_3^2 + \mathbf{E}\,|X_1|^3 \mathbb{I}\{X_1^2 \ge N\}.$$

Proof of Theorem 1.1. It is easy to show that $\mathbf{E}\,|X_1|^3 < \infty$ implies $A_N = o(1)$ and $B_N = o(1)$ as $N \to \infty$. Furthermore, by the non-lattice property of F, we have $\rho_u > 0$, for every $u > 1$. Hence, one can find an increasing sequence $u_N \to \infty$ as $N \to \infty$ such that $\rho_{u_N}^{-4} A_N = o(1)$ and, therefore, the right-hand side of (2.1) with $u = u_N$ is of order $o(N^{-1/2})$.

Proof of Theorem 1.2. We shall apply (2.1) to $G_N^*(x)$ conditionally, given $\mathbb{X}_N$. Denote $Y_1 = (X_1^* - \overline{X}_N)/\hat{\sigma}_N$ and write $\hat{\kappa}_3 = \mathbf{E}^*Y_1^3$, $\hat{\beta}_3 = \mathbf{E}^*|Y_1|^3$, where $\mathbf{E}^*$ denotes the conditional expectation given $\mathbb{X}_N$.

Define

$$H_N^*(x) = \Phi(x) + \frac{\hat\kappa_3}{6\sqrt{N}}\,(2x^2+1)\Phi'(x).$$

Then by (2.1), we have for every $1 < u < N^{1/6}$,

$$\Delta_N^* := \sup_x |G_N^*(x) - H_N^*(x)| \le \frac{c}{\sqrt{N}}\Big(\frac{1}{u} + \frac{A_N^*}{\hat\rho_u^4} + B_N^*\Big), \tag{2.2}$$
$$A_N^* = N^{-1/2}\big(1 + \mathbf{E}^* Y_1^4 \mathbb{I}\{Y_1^2 \le N\}\big),$$
$$B_N^* = N^{-1/2}\hat\beta_3^2 + \mathbf{E}^*|Y_1|^3\mathbb{I}\{Y_1^2 \ge N\},$$

where we denote

$$\hat\rho_u = 1 - \sup\big\{|\,\mathbf{E}^*\exp\{i\,t\,Y_1\}| : \ b_N \le |t| \le c_N\big\}, \qquad b_N = \hat\sigma_N^2/(9\hat\beta_3), \quad c_N = u/\hat\sigma_N.$$

In order to prove the theorem it suffices to show

$$\Delta_N^* = o(N^{-1/2}) \qquad \text{a.s.} \qquad \text{as} \quad N \to \infty. \tag{2.3}$$

Indeed, by the SLLN, $\hat\kappa_3 \to \kappa_3$ a.s. as $N \to \infty$. Hence, $\sup_x |H_N(x) - H_N^*(x)| = o(N^{-1/2})$ a.s. and this in combination with (1.2) and (2.3) implies (1.3).

Let us show (2.3). Given $a > 1$ write

$$\beta_3(a) = \mathbf{E}\,|X_1|^3\mathbb{I}\{|X_1| \ge a\}, \qquad \hat\beta_3(a) = \mathbf{E}^*|Y_1|^3\mathbb{I}\{|Y_1| \ge a\},$$

and note that

$$A_N^* \le \frac{1}{\sqrt{N}} + \frac{a}{\sqrt{N}}\,\mathbf{E}^*|Y_1|^3\mathbb{I}\{|Y_1| \le a\} + \hat\beta_3(a) \le 2\,\frac{a}{\sqrt{N}}\,\hat\beta_3 + \hat\beta_3(a), \tag{2.4}$$
$$B_N^* \le N^{-1/2}\hat\beta_3^2 + \hat\beta_3(a).$$

Where in the last inequality it is assumed that N is sufficiently large, namely $N \ge a^2$. Since $\mathbf{E}\,|X_1|^3 < \infty$, the SLLN implies that as $N \to \infty$,

$$\hat\beta_3 \to \beta_3 \qquad \text{and} \qquad \limsup_N \hat\beta_3(a) \le \beta_3(a) \qquad \text{a.s.} \tag{2.5}$$

$$b_N \to b := \sigma^2/(9\beta_3) \qquad \text{and} \qquad c_N \to u/\sigma \qquad \text{a.s.} \tag{2.6}$$

Furthermore, Singh (1981) showed that for any, but fixed $d > 0$,

$$\sup_{0\le|t|\le d}\Big|\,\mathbf{E}\,\exp\{itX_1\} - \mathbf{E}^*\exp\{itX_1^*\}\Big| = o(1) \qquad \text{a.s.} \qquad \text{as} \quad N \to \infty,$$

see formula (2.4) ibidem. This fact in combination with the SLLN for $\hat\sigma_N$, b_N and c_N, see (2.6), implies

$$\hat\rho_u \to \rho_u \qquad \text{a.s.} \quad \text{as} \qquad N \to \infty, \tag{2.7}$$

for every fixed $u > 0$. Substitution of (2.4), (2.7) in (2.2) in combination with (2.5) gives

$$\limsup_N \sqrt{N}\,\Delta_N^* \le c\big(u^{-1} + \beta_3(a)(1 + \rho_u^{-4})\big) \qquad \text{a.s.},$$

for arbitrary, but fixed $u > 1$ and $a > 0$. Since $\mathbf{E}\,|X_1|^3 < \infty$, $\beta_3(a) = o(1)$ as $a \to \infty$ and therefore the right-hand side can be made arbitrary small by choosing u and a sufficiently large. We obtain (2.3) thus completing the proof of the theorem.

Proof of Proposition 2.1.

For clarity we start by outlining the main steps of the proof. We first use Lemma 3.1 below to replace the statistic T_N by a statistic S_0, which is conditionally linear in the first m observations $X_1, \dots, X_m$, given the remaining observations of the sample, $X_{m+1}, \dots, X_N$. With $K_0(x) = \mathbf{P}\{S_0 \le x\}$ denoting the distribution function of S_0, an application of Berry–Esseen's smoothing lemma then reduces the problem of bounding $|K_0(x) - H_N(x)|$ to that of bounding the difference $|\hat K_0(t) - \hat H_N(t)|$, where $\hat K_0$ and $\hat H_N$ denote the Fourier transforms of K_0 and H_N respectively. The (conditional) linearity of S_0 produces a multiplicative component in $\hat K_0$ and in combination with the smoothness condition (non-lattice condition on F) guarantees an exponential decay of $|\hat K_0(t)|$, for large t, $|t| \ge c(F)\sqrt{N}$. Finally we bound the difference $|\hat K_0(t) - \hat H_N(t)|$, for $|t| \le c(F)\sqrt{N}$.

Here and in what follows we write $c(a, b, ...)$ to denote a constant that depends only on the quantities $a, b, ...$. By c, c_1, c_2,... we denote generic absolute constants. The expression $\exp\{x\}$ is abbreviated by $\mathrm{e}\{x\}$. We shall write $A \ll B$ to denote the fact that $A \le c\,B$. If Q denotes the sum $q_1 + \cdots + q_k$ and $A \subset \{1, \dots, k\}$ then write $Q_A = \sum_{j \in A} q_j$. Given $A = \{i_1, \dots, i_m\} \subset \{1, \dots, N\}$ we write $\mathbf{E}_{\,i_1,\dots,i_m}$ to denote the conditional expectation given $\{X_j,\ j \notin A\}$. For a function h write $\|h\| = \sup_x |h(x)|$.

Let $g : R \to R$ be a function which is infinitely many times differentiable with bounded derivatives and such that

$$\frac{8}{9} \le g(x) \le \frac{8}{7}, \quad \text{for all} \quad x \in R, \quad \text{and} \quad g(x) = \frac{1}{\sqrt{x}}, \quad \text{for} \quad \frac{7}{8} \le x \le \frac{9}{8}.$$

Write $c_g = \|g\| + \|g'\| + \|g''\| + \|g'''\|$.

First, we replace the random variables $X_1, \dots, X_N$ by truncated random variables

$$Y_i = a^{-1} N^{-1/2} X_i \mathbb{I}\{X_i^2 \le a^2 N\}, \qquad i = 1, \dots, N,$$

where a is the largest non-negative solution of the equation

$$a^2 = \mathbf{E}\, X_1^2\, \mathbb{I}\{X_1^2 \le a^2 N\}.$$

Note that $|Y_i| \le 1$ and $\mathbf{E}\, Y_i^2 = 1/N$. Write

$$\begin{aligned} &Y = Y_1 + \cdots + Y_N, \qquad \eta = \eta_1 + \cdots + \eta_N, \qquad \eta_i = Y_i^2 - \mathbf{E}\, Y_i^2, \\ &b_k = \mathbf{E}\,|Y_1|^k, \quad \mathcal{M} = N b_4, \quad \gamma = N|\,\mathbf{E}\, Y_1|, \quad \gamma_0 = N^{-1/2}\,\mathbf{E}\,|X_1|^3 \mathbb{I}\{X_1^2 \ge N/2\}. \end{aligned}$$

By Hölder's inequality,

$$b_3 \ge b_2^{3/2} = N^{-3/2}, \qquad \mathcal{M} \ge N\, b_2^2 = N^{-1}, \qquad (b_3 N)^2 \le N^2 b_2 b_4 = \mathcal{M}. \tag{2.8}$$

We may and shall assume that for a sufficiently small $c_0 > 0$,

$$\beta_3/\sqrt{N} \le c_0, \qquad \rho_u^{-4} N^{-1} \le c_0, \qquad \gamma_0 \le c_0. \tag{2.9}$$

Indeed, if the first inequality fails, the bound (2.1) follows from the simple inequality $\Delta_N \ll 1 + \beta_3/\sqrt{N} \ll \beta_3^2/N$. Hence, without loss of generality we may assume that $\beta_3/\sqrt{N} \le c_0$. Then $\Delta_N \ll 1 + \beta_3/\sqrt{N} \ll 1$ and each of the remaining inequalities in (2.9) implies (2.1)

Using (2.9) and Lemma 3.2 we obtain $3/4 < a \le 1$ and, hence, $\mathcal{M} \le A_N$. Therefore, in order to prove the proposition it suffices to show that

$$\Delta_N \ll \mathcal{R} + T^{-1}, \qquad \text{where} \qquad \mathcal{R} = \mathcal{M}/\rho_u^4 + \gamma_0, \qquad T = u\sqrt{N}. \tag{2.10}$$

Furthermore, (2.9) implies $\Delta_N \ll 1$ and therefore we may and shall assume without loss of generality that $\mathcal{M} \le c_0$, since otherwise (2.10) follows from $\Delta_N \ll 1 \ll \mathcal{M}$.

Let m be the smallest integer such that $m \ge 4\rho_u^{-1} \ln N$. Write $A = \{1, \dots, m\}$ and $B = \{m+1, \dots, N\}$ and denote $V_B = 1 + \eta_B - Y_B^2/N$. Introduce the statistics

$$S = Y g(1 + \eta - Y^2/N), \qquad S_0 = Y g(V_B) + \eta_A Y_B g'(V_B). \tag{2.11}$$

By Lemma 3.1, see below, the probability $\mathbf{P}\{|S_0 - T_N| \ge N^{-2/3}\}$ is not greater than $c\mathcal{M}/\rho_u^2 + c\gamma_0$. Now Slutzky's argument gives

$$\Delta_N \le \|K_0 - H_N\| + N^{-2/3}\|H_N'\| + c\mathcal{R}.$$

But $\|H_N'\| \le c$, by (2.9). Hence, in order to prove (2.1) it remains to show

$$\|K_0 - H_N\| \ll \mathcal{R} + T^{-1}.$$

We are going to apply Esseen's (1945) smoothing lemma. We have

$$\|K_0 - H_N\| \ll I + T^{-1}, \quad \text{where} \quad I = \int_{0\le|t|\le T} \frac{|\hat K_0(t) - \hat H_N(t)|}{|t|}\, dt.$$

Write $I = I_1 + I_2 + I_3$, where

$$I_1 = \int_{0\le|t|\le L} \frac{|\hat K_0(t) - \mathbf{E}\, \mathrm{e}\{itS\}|}{|t|}\, dt, \quad I_2 = \int_{0\le|t|\le L} \frac{|\mathbf{E}\, \mathrm{e}\{itS\} - \hat H_N(t)|}{|t|}\, dt,$$
$$I_3 = \int_{L\le|t|\le T} \frac{|\hat K_0(t) - \hat H_N(t)|}{|t|}\, dt, \qquad L^{-1} = 4Nb_3.$$

We shall bound I_1, I_2 and I_3. An application of Lemma 3.3 gives $I_2 \ll \mathcal{R}$.
Let us show $I_1 \ll \rho_u^{-2}\mathcal{M} + N^{-2/3}$. By (3.2) and (3.3), see below,

$$|\hat K_0(t) - \mathbf{E}\, \mathrm{e}\{itS\}| \le |t|\, \mathbf{E}\, |S_0 - S| \ll |t|\, \mathbf{E}\, R, \quad R = R_1 + \cdots + R_5,$$

where the random variables R_i are given by (3.2) and (3.3). A simple calculation shows $\mathbf{E}\, R \ll N^{-1/2}(\rho_u^{-2}\mathcal{M} + N^{-2/3})$. We obtain,

$$I_1 \ll L\, \mathbf{E}\, R \ll \sqrt{N}\, \mathbf{E}\, R \ll \rho_u^{-2}\mathcal{M} + N^{-2/3}.$$

It remains to show that $I_3 \ll \mathcal{R}$. We have

$$I_3 \le I_4 + I_5, \qquad I_4 = \int_{L\le|t|\le T} |\hat K_0(t)|/|t| dt, \quad I_5 = \int_{L\le|t|\le T} |\hat H_N(t)|/|t| dt.$$

By (2.9), $|\hat H_N(t)| \le \mathrm{e}\{-t^2/2\}(1 + |t|^3)$. Therefore, $I_5 \le \mathrm{e}\{-(cL)^2\}$. The inequality $\mathrm{e}\{-s^2\} \le s^{-2}$ (which holds for sufficiently large s) applied to $s = cL$ gives $I_5 \ll (Nb_3)^2 \ll \mathcal{M}$, see (2.8). Finally, we shall show

$$I_4 \ll N^{-1}. \tag{2.12}$$

Write $\mathbb{I} = \mathbb{I}\{|Y_B| \le N^{1/20}\}$. We have

$$|\hat K_0(t)| \le \mathbf{E}\, |\psi(t)| + \mathbf{P}\, \{|Y_B| > N^{1/20}\}, \qquad \psi(t) = \mathbf{E}_A\, \mathrm{e}\{itS_0\}\mathbb{I}.$$

By Chebyshev's inequality, the probability is bounded by $\mathbf{E}\, |Y_B|^{40}/N^2 \ll N^{-2}$, (here we use the bound $\mathbf{E}\, |Y_B|^k \le c(k)$, see, e.g., Bentkus and Götze (1996)). Furthermore,

$$|\psi(t)| \le |\varkappa|^m, \qquad \varkappa = \mathbf{E}_1\, \mathrm{e}\{it(Y_1 g(V_B) + \eta_1 Y_B g'(V_B))\}\mathbb{I}.$$

We complete the proof of (2.12) by showing

$$|\varkappa| \le 1 - \rho_u/2. \tag{2.13}$$

Indeed, this inequality implies $|\varkappa|^m \le \mathrm{e}\{-m\rho_u/2\} \le N^{-2}$, by the choice of m. Let us prove (2.13). Expanding in powers of $it\eta_1 Y_B g'(V_B)$ we obtain

$$\begin{aligned} \varkappa &= \mathbf{E}_1\, \mathrm{e}\{itY_1 g(V_B)\}\mathbb{I} + R, \\ |R| &\le 2T\,\mathbf{E}_1 |\eta_1 Y_B| c_g \mathbb{I} \le 2\, c_g T N^{1/20-1} \le 2\, c_g N^{-1/4} < \rho_u/4. \end{aligned} \tag{2.14}$$

In the last step we used (2.9). Furthermore, write $Z = N^{-1/2} a^{-1} X_1 g(V_B)$. We have

$$\begin{aligned} &\big|\,\mathbf{E}_1\, \mathrm{e}\{itY_1 g(V_B)\} - \mathbf{E}_1\, \mathrm{e}\{itZ\}\big| \le 2\,T\ \mathbf{E}\,|Y_1 g(V_B) - Z| \\ &\le 2\,T\,\|g\| N^{-3/2} a^{-3}\,\mathbf{E}\,|X_1|^3 \mathbb{I}\{X_1^2 \ge a^2 N\} \le 6\gamma_0 N^{-1/4} \le N^{-1/4} \le \rho_u/4. \end{aligned} \tag{2.15}$$

Here we used inequalities $T \le N^{3/4}$, $2a^{-3}\|g\| \le 6$ and $\gamma_0 \le c_0$, see (2.9). But $\mathbf{E}_1\, \mathrm{e}\{itZ\} \le \rho_u$. Collecting (2.14) and (2.15) we obtain (2.13) thus completing the proof of the proposition.

3. Appendix

Lemma 3.1. *Assume that (2.9) holds. Let S_0 be given by (2.11). Then*

$$\mathbf{P}\,\{|S_0 - T_N| \ge N^{-2/3}\} \ll \mathcal{R}_1, \qquad \mathcal{R}_1 = \rho_u^{-2}\mathcal{M} + \gamma_0. \tag{3.1}$$

Proof of Lemma 3.1. The inequalities

$$\begin{aligned} \mathbf{P}\,\Big\{\sqrt{N}\,\mathbf{t}_N \ne \frac{Y}{\sqrt{1+\eta - Y^2/N}}\Big\} &\le N\ \mathbf{P}\,\{X_1^2 > a^2 N\} \le (4/3)^3\gamma_0, \\ \mathbf{P}\,\Big\{\frac{Y}{\sqrt{1+\eta - Y^2/N}} \ne S\Big\} &\le \mathbf{P}\,\{|\eta| > 1/4\} + \mathbf{P}\,\{Y^2/N > 1/4\} \end{aligned}$$

and

$$\mathbf{P}\,\{|\eta| > 1/4\} \le 16\,\eta^2 \le 32N\ \mathbf{E}\ Y_1^4, \qquad \mathbf{P}\,\{Y^2/N > 1/4\} \le 4\,N^{-1}\,\mathbf{E}\ Y^2 = 4/N$$

imply $\mathbf{P}\,\{T_N \ne S\} \ll \mathcal{R}_1$.

In order to prove (3.1) it remains to show $\mathbf{P}\,\{|S - S_0| > N^{-2/3}\} \ll \mathcal{R}_1$. Write $S = Yg(V_B + \eta_A + W)$, where $W = -Y_A^2/N - 2Y_AY_B/N$. Expanding g in powers of W we get $|S - Yg(V_B + \eta_A)| \le c_g|YW|$, where $|YW| \le R_1 + R_2 + R_3$ and

$$R_1 = |Y_A|^3/N, \qquad R_2 = 3Y_A^2|Y_B|/N, \qquad R_3 = 2|Y_A|Y_B^2/N. \tag{3.2}$$

Furthermore, expanding g in powers of η_A we obtain

$$|Y g(V_B + \eta_A) - S_0| \le c_g(R_4 + R_5), \qquad R_4 = |Y_A \eta_A|, \quad R_5 = |Y_B|\eta_A^2. \tag{3.3}$$

Then $|S - S_0| \le c_g(R_1 + \cdots + R_5)$ and, by Chebyshev's inequality,

$$\begin{aligned} &\mathbf{P}\{|S - S_0| \ge N^{-2/3}\} \\ &\ll N^{2/3}\,\mathbf{E}\,R_1 + N^{2/3}\,\mathbf{E}\,R_2 + \mathbf{E}\,N^{4/3}R_3^2 + \mathbf{E}\,(N^{2/3}R_4)^{4/3} + N^{2/3}\,\mathbf{E}\,R_5. \end{aligned}$$

A simple calculation shows that the right hand side is bounded by $c\mathcal{R}_1$ thus completing the proof of (3.1)

Lemma 3.2. *Assume that $\gamma_0 \le 1/4$. Then*

$$1 - \sqrt{2}\,\gamma_0 \le a \le 1, \tag{3.4}$$

In particular we have $3/4 \le a \le 1$.

Proof of Lemma 3.2. Clearly, $a^2 \le \sigma^2 = 1$. For $u \ge 0$, write $\phi(u) = \mathbf{E}\,X_1^2\mathbb{I}\{X_1^2 \le u\,N\}$. We have

$$\tau := 1 - \phi(1/2) = \mathbf{E}\,X_1^2\mathbb{I}\{X_1^2 > N/2\} \le \sqrt{2}\,\gamma_0 \le 1/2. \tag{3.5}$$

The function $\phi(u)$ is nondecreasing. Therefore, (3.5) implies $\phi(1-\tau) \ge \phi(1/2) = 1-\tau$. But, $\phi(1) \le \sigma^2 = 1$. Then there exists a solution of the equation $u = \phi(u)$ in the interval $1 - \tau \le u \le 1$. This implies $a^2 \ge 1 - \tau$ and we obtain $a \ge a^2 \ge 1 - \tau$. This inequality in combination with (3.5) yields (3.4) thus completing the proof of the lemma.

Lemma 3.3. *Assume that (2.9) holds and that $\mathcal{M} \le c_0$, where c_0 is a sufficiently small absolute constant. Then*

$$I_2 = \int_{0 \le |t| \le L} \frac{|\,\mathbf{E}\ \mathrm{e}\{itS\} - H_N(t)|}{|t|}\,dt \ll \mathcal{M} + \gamma_0, \qquad L = (4b_3N)^{-1}.$$

Proof of Lemma 3.3.

Write $R = \mathbf{E}\ \mathrm{e}\{itS\} - H_N(t)$. Split $I_2 = I_{2,1} + I_{2,2}$, where

$$I_{2,1} = \int_{0 \le |t| \le c} |t|^{-1}|R|dt, \qquad I_{2,2} = \int_{c \le |t| \le L} |t|^{-1}|R|dt.$$

Let us show that $I_{2,1} \ll \mathcal{M} + \gamma_0$. For an infinitely many times differentiable function with bounded derivatives $h : \mathbb{R} \to \mathbb{C}$ write $c(h) = \|h'\| + \cdots + \|h^{vi}\|$. Let ξ be a standard normal random variable. By Lemma 3.1 of Bloznelis and Putter (1999),

$$|\,\mathbf{E}\,h(S) - \,\mathbf{E}\,h(\xi) + \frac{N}{6}\,\mathbf{E}\,Y_1^3(3\,\mathbf{E}\,h'(\xi) + 2\,\mathbf{E}\,h'''(\xi))| \ll c(h)(\mathcal{M} + \gamma_0).$$

Choosing $h(x) = \mathrm{e}\{itx\}$ we obtain form this bound that $|R| \ll (|t| + \cdots + |t|^6)(\mathcal{M} + \gamma_0)$. Clearly, the last inequality implies $I_{2,1} \ll \mathcal{M} + \gamma_0$.

The proof of the bound $I_{2,2} \ll \mathcal{M} + \gamma_0$ is similar to that of Lemma 3.1 of Bloznelis and Putter (1999), but somewhat more involved. Now the remainder $|R|$ of the expansion needs to be integrable with respect to the measure $dt/|t|$ over the region $\{|t| > c\}$. In order to construct such an estimate for the remainder we use the same argument as Bentkus and Götze (1996). That is, we approximate $\mathbf{E}\,\mathrm{e}\{itS\}$ by the characteristic function of a statistic which is conditionally linear in the first m observations $X_1, \ldots, X_m$ given the remaining part of the sample $X_{m+1}, \ldots, X_n$. Here $m \approx Nt^{-2} \ln t$. The detailed calculations are given in Bloznelis and Putter (1998).

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