1. Introduction and results

Let X, X_1, $X_2, \ldots$ be independent and identically distributed centered random variables with $\mathbf{E}X^2 < \infty$. By the central limit theorem, the distributions F_n of the normalized sums $S_n = n^{-1/2}(X_1 + \cdots + X_n)$ converge to the normal distribution $N(0, \sigma^2)$, where $\sigma^2 = \mathbf{E}X_1^2$. Assume that the distribution of X has an absolutely continuous component (with respect to Lebesgue's measure λ). Then the distribution F_n has an absolute continuous component too. Moreover, this component becomes dominant as $n \to \infty$, and it's density p_n converges to the density g of the limiting normal distribution $N(0, \sigma^2)$ in L_1 metric

$$\int_{-\infty}^{+\infty} |p_n(x) - g(x)| dx \to 0. \tag{1.1}$$

This version of the local limit theorem is due to Prokhorov (1952), see also Ibragimov and Linnik (1971). Prokhorov's local limit theorem extends to random vectors with values in a finite-dimensional Euclidean space, see Mamatov and Halikov (1964).

Note that (1.1) ensures the normal approximation of the probabilities $\mathbf{P}\{S_n \in B\}$ uniformly over the class of all Borel sets:

$$\sup\{|\mathbf{P}\{S_n \in B\} - G(B)| : B \text{ is a Borel set}\} \to 0. \tag{1.2}$$

Here G denotes the limiting normal distribution. Furthermore, it is easy to see that (1.1) and (1.2) are equivalent. Sazonov and Ulyanov (1979), Senatov (1980), Sazonov (1981) provide bounds for the convergence rate in (1.2) in the central limit theorem in R^k.

In the present paper we study the validity of (1.2) in the case of infinite-dimensional central limit theorem (CLT). Let E be a separable Banach space with the norm $\|\cdot\|$. Given a signed measure μ, defined on the class of Borel sets $\mathcal{B}(E)$ of E, let $|\mu|$ denote the total variation

$$|\mu| = \sup\{|\mu(B)| : B \in \mathcal{B}(E)\}.$$

Let $X^1, X^2, \ldots$ be independent random variables with values in E and with the common distribution F. Assume that the distributions F_n of $S_n = n^{-1/2}(X^1 + \cdots + X^n)$ converge weakly to a Gaussian distribution G. We are interested in what extra condition on F would ensure the validity of (1.2). Yu. A. Davydov (1989) poses the question whether the infinite-dimensional analogue of Prokhorov's local limit theorem with λ replaced by G holds: *does the assumption that F has a* *component which is absolutely continuous with respect to* G *implies* $|F_n - G| \to 0$. In the present paper we give the negative answer by constructing an example of l_2-valued random vector which satisfies CLT. Although its distribution F is absolutely continuous with respect to the limiting distribution G, the total variation $|F_{n_k} - G|$ is bounded from below by a positive constant for some sequence $n_k \to \infty$, see Theorem below.

Our result is negative. There are also positive results related to (1.2) for the infinite-dimensional CLT. Rachev and Yukich (1989) showed the bound $|F_n - G| \le 2^{-3/2} n^{-1/2}$ subject to the condition $\nu_3(F,G) \le 1 + 2(3/2)^{3/2}$, where $\nu_t(F,G) = \sup\{|h|^t |(F-G) * G_h| \, : \, h \in R\}$. Here $\mu * \varkappa$ denotes the convolution of signed measures μ and $\varkappa$, and $G_h(B) := G(h^{-1}B)$, for $B \in \mathcal{B}(E)$. Clearly, the above mentioned result provides a sufficient condition for (1.2) to hold. However the relation between infinite-dimensional distributions F and G imposed by this condition is rather complex, see, e.g., Bloznelis (1989). Note that this is not the case for finite-dimensional F and G because in this case the quantity $\nu_t(F,G)$ can be easily handled using the differentiability of the density function (with respect to Lebesgue's measure) of the Gaussian distribution G_h. In particular, ν_t is comparable with Zolotarev's ζ_t metric, for $E = R^k$, see Rachev and Yukich (1989).

Let Y be a centered Gaussian random variable with values in the space l_2 (= the space of real sequences $x = (x_1, x_2, \dots)$ such that $\|x\|^2 = \sum x_i^2 < \infty$). We assume that the distribution G of Y is infinite-dimensional, that is, $\mathbf{P}\{Y \in L\} = 0$ for every finite-dimensional subspace $L \subset l_2$.

Theorem. *For every infinite-dimensional Gaussian random variable* Y *with values in* l_2 *there exists a random variable* X^1 *with values in* l_2 *such that*

$$\mathbf{E}X^1 = 0, \quad cov X^1 = cov Y, \quad \mathbf{E}\|X\|^r < \infty, \quad \textit{for every} \quad r > 0, \tag{1.3}$$

the distribution F *of* X^1 *is absolutely continuous with respect to the distribution* G *and*

$$\limsup_n |F_n - G| > 0. \tag{1.4}$$

Note that (1.3) is sufficient for the CLT (the weak convergence $F_n \Rightarrow G$).

Remark. Theorem extends to random variables with values in an arbitrary (infinite-dimensional) separable Banach space.

2. Proof

Before the proof we introduce necessary notation. Let R^∞ denote the linear space of real sequences $y = (y_1, y_2, \dots)$ and $\pi_k : R^\infty \to R$ denote the k-th coordinate projection, so that $\pi_k(y) = y_k$. The space R^∞ is a measurable space with respect to the σ-algebra $\mathcal{B}$ generated by the cylindric sets $\pi_1^{-1}(B_1)\cap\pi_2^{-1}(B_2)\cap\cdots\cap\pi_k^{-1}(B_k)$, where, $B_1,\dots,B_k$, $k=1,2,\dots$ denote Borel subsets of R. By $e^1,e^2,\dots$ we denote the coordinate vectors of R^∞ so that $\pi_k(e^i)=0$ for $k\neq i$ and $\pi_i(e^i)=1$. For $y\in R^\infty$ we also write $y=\sum_i y_ie^i$.

Let $\eta_1,\eta_2,\dots$ be independent standard normal random variables. The sequence $\eta=(\eta_1,\eta_2,\dots)$ is a random vector with values in R^∞. Let $\mathcal{N}$ denote its distribution (a cylindric probability distribution on R^∞).

Proof. The proof consists of three steps. The first step provides a construction of a probability distribution $\mathcal{F}$ on R^∞ which is absolutely continuous with respect to $\mathcal{N}$ ($\mathcal{N}$-absolutely continuous for short) and satisfies (2.5) and (2.6). In the second step we show (1.4) for sums of R^∞- valued random variables with the distribution $\mathcal{F}$. In the third step using the canonical representation of an l_2-valued Gaussian random variable we construct a linear operator $R^\infty\to l_2$. The image of $\mathcal{F}$ provides the principal component of the distribution of the random variable X^1 from the theorem.

Step 1. In order to keep the presentation simple we prove a variant of the theorem for a distribution $\mathcal{F}$ which has an $\mathcal{N}$-absolutely continuous component only. Obvious changes that lead to $\mathcal{N}$-absolutely continuous distribution are indicated at the end of the proof.

Introduce the probability distribution on R^∞

$$\mathcal{F}=\mathcal{N}_A+\sum_{i\geq 1}p_i(\delta_{x_i}+\delta_{-x_i})/2. \tag{2.1}$$

Here, for $x_i>0$, we denote by δ_{x_i} (respectively δ_{-x_i}) the unit mass placed at the point x_ie^i (respectively $-x_ie^i$). Furthermore, $\mathcal{N}_A$ denotes the restriction of $\mathcal{N}$ on the set A

$$A=\{x\in R^\infty:\,|\pi_i(x)|\leq a_i,\ i=1,2,\dots\},$$

that is, $\mathcal{N}_A(B):=\mathcal{N}(A\cap B)$, for any $B\subset\mathcal{B}$. Here $a_i=4\sqrt{\ln(i+2)}$ for $i\geq 2$ and we choose $0<a_1<1$ such that $\mathcal{N}(A)=1/2$. To see that such a choice is possible write $\mathcal{N}(A)=P_1(a_1)P_2$, where $P_1(a)=\mathbf{P}\{|\eta_1|\leq a\}$ and

$$P_2=\prod_{i\geq 2}\mathbf{P}\{|\eta_i|\leq a_i\}\geq 1-\sum_{i\geq 2}\mathbf{P}\{|\eta_i|>a_i\}$$

and note that $P_1(1)\geq 0.59$ and $P_2\geq 0.99$.

Introduce the sequence

$$\sigma_i^2=\mathbf{E}\eta_i^2\mathbb{I}\{\eta\in A\}=\mathcal{N}(A)\mathbf{E}(\eta_i^2|\,\eta_i^2\leq a_i^2),\qquad i\geq 1.$$

Clearly, $\sigma_i^2\to\mathcal{N}(A)=0.5$ as $i\to\infty$. From the fact that the function $x\to\mathbf{E}(\eta_1^2|\,\eta_1^2\leq x^2)$ is increasing for $x>0$, we conclude that the sequence $\{\sigma_i^2\}$ increases together with the sequence $\{a_i\}$ for $i = 1, 2, \ldots$. Moreover, a simple calculation shows that $\sigma_2^2 \geq 1/3$. Therefore

$$\sigma_i^2 < 0.5, \quad \text{for} \quad i \geq 1, \qquad \text{and} \quad \sigma_i^2 > 1/3, \quad \text{for} \quad i \geq 2. \tag{2.2}$$

We complete the description of the probability distribution (2.1) by specifying the numbers $p_i, x_i > 0$. Denote $N_k = 2^{k+1}$ and introduce the sequence $\{n_k\}$, where $n_1 = 1$ and $n_k = 2^{4(k+5)}$, for $k \geq 2$. Write

$$I_k = \{i : n_k \leq i < n_{k+1}\}, \qquad k = 1, 2, \ldots,$$

and put, for $i \in I_k$,

$$p_i = x_i^{-2}(1 - \sigma_i^2), \tag{2.3}$$

$$x_i = 4N_k a_i(1 + \Delta_i), \tag{2.4}$$

where the numbers $\Delta_i > 0$ are specified in what follows. The function

$$f(\Delta_{n_k}, \ldots, \Delta_{n_{k+1}-1}) = \sum_{i \in I_k} \frac{1 - \sigma_i^2}{16 N_k a_i^2 (1 + \Delta_i)^2}$$

is continuous and decreasing with respect to each of it's arguments and satisfies

$$f(0, \ldots, 0) > 1 > f(n_{k+1}^{1/2}, \ldots, n_{k+1}^{1/2}).$$

The first inequality follows from the inequality $1 - \sigma_i^2 > 1/2$, see (2.2). The second inequality is obvious. Therefore, one can find $0 < \Delta_i < n_{k+1}^{1/2}$, $i \in I_k$, such that

$$f(\Delta_{n_k}, \ldots, \Delta_{n_{k+1}-1}) = 1.$$

We use these values of Δ_i to define x_i by means of (2.4). From (2.3) and (2.4) we obtain

$$\sum_{i \in I_k} p_i = N_k^{-1} f(\Delta_{n_k}, \ldots, \Delta_{n_{k+1}-1}) = N_k^{-1}$$

This implies $\sum_i p_i = 1/2$ and, therefore, the formula (2.1) defines indeed a probability distribution on R^∞.

Let $\xi = (\xi_1, \xi_2, \ldots)$ be a random element with values in R^∞ and with the distribution $\mathcal{F}$. Note that (2.3) implies

$$\mathbf{E}\xi_i^2 = \sigma_i^2 + x_i^2 p_i = 1, \qquad i \geq 1. \tag{2.5}$$

Since the distribution $\mathcal{F}$ is symmetric, we have

$$\mathbf{E}\xi_i = 0, \quad \text{and} \quad \mathbf{E}\xi_i\xi_j = 0, \qquad 1 \le i < j. \tag{2.6}$$

Step 2. Let $\xi^1, \xi^2, \dots$ be independent random variables with values in R^∞ and with the common distribution $\mathcal{F}$. Write $S_r^* = \xi^1 + \dots + \xi^r$. We shall show that

$$\limsup_{k\to\infty} \bigl(\mathcal{N}(T_k) - \mathbf{P}\{S_{N_k}^* \in \sqrt{N}_k\, T_k\}\bigr) \ge (1 - e^{-1})/2, \tag{2.7}$$

for cylindric sets $T_k = \cap_{i\in I_k} \pi_i^{-1}([-a_i, a_i])$. Let $L(y) = \sum_{i\in I_k} \pi_i(y) e^i$ denote the projection of R^∞ on the linear subspace R_k generated by the coordinate vectors e^i, $i \in I_k$. The Borel σ-algebra of R_k is denoted by $\mathcal{B}(R_k)$. By H we denote the probability distribution of the random vector $Z = L(\xi) = \sum_{i\in I_k} \xi_i e^i$ which takes values in R_k. Let $Z_1,\ Z_2, \dots$ be independent random vectors in R_k with the common distribution H. By H^r we denote the r-fold convolution of H. That is, $\mathbf{P}\{Z_1 + \dots + Z_r \in B\} = H^r(B)$, for $B \subset \mathcal{B}(R_k)$. In particular, the probability in (2.7) equals

$$\mathbf{P}\{S_N^* \in \sqrt{N}\, T_k\} = H^N(V), \quad \text{for} \quad V = \sqrt{N} L(T_k). \tag{2.8}$$

Here and below we write $N = N_k$ for brevity. In order to prove (2.7) we shall show that

$$H^N(V) \le (1 + (1 - N^{-1})^N)/2. \tag{2.9}$$

Indeed, this inequality together with $\lim_{k\to\infty} \mathcal{N}(T_k) = 1$ implies (2.7).

Let us prove (2.9). Split $H = H_1 + H_2$, where

$$H_1 = \sum_{i\in I_k} p_i (\delta_{x_i} + \delta_{-x_i})/2,$$

and where $H_2 = H - H_1$ is a positive measure with the support set $L(A_k) = L(T_k)$. Note that, for $i = 1, 2$, the total variation $|H_i| = \sup\{H_i(B) :\ B \in \mathcal{B}(R_k)\}$ satisfy

$$|H_1| = \sum_{i\in I_k} p_i = N^{-1}, \qquad |H_2| = 1 - |H_1| = 1 - N^{-1}.$$

Let us write the probability of (2.8) in the following form

$$H^N(V) = (H_1 + H_2)^N(V) = H_2^N(V) + \sum_{j=1}^{N} \binom{N}{j} H_1^j H_2^{N-j}(V), \tag{2.10}$$

where H_iH_j denotes the convolution $H_i * H_j$. That is, for $B \in \mathcal{B}(R_k)$,

$$H_iH_j\,(B) = H_i * H_j\,(B) = \int_{R_k} \mathbb{I}_{x+y\in B} H_i(dx)H_j(dy).$$

Note that the first summand in the right of (2.10) does not exceed

$$H_2^N(V) \le |H_2^N| = |H_2|^N = (1 - N^{-1})^N.$$

Now consider the measure $H_1^j H_2^{N-j}$ in the case where j is odd. Clearly, $H_1^j(B) = 0$, for every Borel set $B \in \mathcal{B}(R_k)$ such that $B \cap B_0 = \emptyset$, where

$$B_0 = \Big\{ \sum_{i\in I_k} r_i x_i e^i : \ r_i \in \{0, \pm 1, \pm 2, \dots\},\ r_i \ne 0 \text{ for some } i \in I_k \Big\} \subset R_k. \tag{2.11}$$

Furthermore, the support $(N-j)L(T_k)$ of the measure H_2^{N-j} is a subset of $N\,L(T_k)$. Since B_0 does not contain the zero vector $\overline{0} \in R_k$ and, by (2.4), $x_i > 4Na_i$ we conclude that the sets $B_0 + N\,L(T_k)$ and V do not intersect. Therefore,

$$H_1^j H_2^{N-j}(V) = 0. \tag{2.12}$$

Now consider the measure $H_1^j H_2^{N-j}$ for even j. Split $H_1^j = Q + R$, where $Q(B) = H_1^j(\overline{0} \cap B)$, for $B \in \mathcal{B}(R_k)$. Arguing as in the proof of (2.12) we conclude that $H_1^j H_2^{N-j}(V) = QH_2^{N-j}(V)$. Furthermore, the identity $|Q| = H_1^j(\overline{0})$ and the inequality (which is proved at the end of this step)

$$H_1^j(\overline{0}) \le |H_1^j| \binom{j}{j/2} 2^{-j} = \frac{1}{N^j} \binom{j}{j/2} 2^{-j} \tag{2.13}$$

implies

$$H_1^j H_2^{N-j}(V) = QH_2^{N-j}(V) \le |Q|\,|H_2^{N-j}| = \binom{j}{j/2} 2^{-j} \frac{1}{N^j} (1 - \frac{1}{N})^{N-j}. \tag{2.14}$$

This inequality in combination with (2.12) and (2.10) yields

$$H^N(V) \le (1 - \frac{1}{N})^N + \sum_{i=1}^{N/2} \binom{N}{2i} (\frac{1}{N})^{2i} (1 - \frac{1}{N})^{N-2i} \binom{2i}{i} 2^{-2i}.$$

Since $\binom{j}{j/2} 2^{-j} \le 2^{-1}$ the right sum is less than

$$\sum_{j=2}^{N} \binom{N}{j} (\frac{1}{N})^j (1 - \frac{1}{N})^{N-j} \frac{1}{2} \le \frac{1}{2}(1 - (1 - \frac{1}{N})^N).$$

We arrive at (2.9).

Let us prove (2.13). Given an even number j, let $\varkappa^1, \dots, \varkappa^j$ be independent random vectors with values in R_k and with the common distribution $\mathbf{P}\{\varkappa^1 \in B\} = |H_1|^{-1} H_1(B)$, for $B \in \mathcal{B}(R_k)$. Write

$$\mathbb{I}_i = \mathbb{I}_{\varkappa^i \in M}, \qquad M = \{x_i e^i : \, i \in I_k\} \subset R_k.$$

Clearly, $\mathbb{I}_1, \dots, \mathbb{I}_j$ are independent Bernoulli random variables with the success probability $\mathbf{P}\{\mathbb{I}_i = 1\} = 1/2$. The event $\{\varkappa^1 + \dots + \varkappa^j = \overline{0}\}$ is a particular case of the event $\{\mathbb{I}_1^+ + \dots + \mathbb{I}_j^+ = j/2\}$. Therefore,

$$p^* := \mathbf{P}\{\varkappa^1 + \dots + \varkappa^j = \overline{0}\} \le \mathbf{P}\{\mathbb{I}_1 + \dots + \mathbb{I}_j = j/2\} = \binom{j}{j/2} 2^{-j}.$$

This implies (2.13), since $H_1^j(\overline{0}) = |H_1|^j p^*$.

Step 3. Given l_2-valued Gaussian random variable Y one can find a sequence of orthonormal vectors $\{d^i\} \subset l_2$ such that almost surely $Y = \sum_i s_i Y_i d^i$, where $Y_1, Y_2, \dots$ is a sequence of independent standard normal random variables and $\{s_i\}$ is a sequence of square summable non-negative integers. Choose an integer sequence $r_k \uparrow +\infty$ such that $s_{r_k} x_k < 1$ for $k = 1, 2, \dots$. We can assume that $\eta_k = Y_{r_k}$. Define the random variable $X = \sum_i X_i d^i$ with values in l_2, by putting $X_i = s_i Y_i$ for $i \ne \{r_k\}$ and $X_i = s_i \xi_i$ for $i \in \{r_k\}$. Clearly, the random variable X satisfies (1.3) and (1.4).

In order to obtain a distribution which is absolutely continuous with respect to G we replace the point masses δ_{x_i} and δ_{-x_i} by measures H_{x_i} and H_{-x_i} that concentrate around the points $x_i e^i$ and $-x_i e^i$ respectively and are absolutely continuous with respect to $\mathcal{N}$. In this way one obtains a distribution on R^∞ which satisfies (2.5) and (2.6). Further steps of the proof remains unchanged.

Appendix

Here we provide details of the proof in the case of G-absolutely continuous distribution F. In *Step 4.* we construct a probability distribution F_1 on R^∞ such that: F_1 is absolutely continuous with respect to $\mathcal{N}$, the first and the second order moments of F_1 and $\mathcal{N}$ coincide and

$$\limsup_N |F_1^N(\sqrt{N}\, T_N) - \mathcal{N}(T_N)| > 0,$$

for a sequence of cylindric sets $T_N \subset R^\infty$. This inequality is verified in *Step 5.*

Step 4. Let the sequences $\{n_k\}, \{N_k\}$ and sets $I_k \subset \mathbb{N}$ be the same as above. Let $a > 0$ denote the solution of the equation $\mathbf{E}(\eta_1^2 | \, \eta_1^2 \le a^2) = 1/2$. For $k = 1, 2, \dots$ introduce the sets $A_k \subset R^\infty$

$$A_k = A_k' \cap A_k'', \quad A_k' = \bigcap_{i \in I_{k+1}} \pi_i^{-1}([-a, a]), \quad A_k'' = \bigcap_{j \in \mathbb{N} \setminus I_{k+1}} \pi_j^{-1}([-a_j, a_j]).$$

Define positive measures G_k on R^∞. For a Borel set $B \in \mathcal{B}$ put

$$G_k(B) = m_k^{-1}\mathcal{N}(A_k \cap B), \qquad \text{where} \quad m_k = \prod_{i \in I_{k+1}} \frac{\mathbf{P}\{\eta_i^2 \le a_i^2\}}{\mathbf{P}\{\eta_i^2 \le a^2\}}\,.$$

Clearly, $|G_k| = G_k(A_k) = m_k^{-1}\mathcal{N}(A_k) = \mathcal{N}(A) = 1/2$.
Define the probability distribution on R^∞

$$F_1 = \mathcal{N}_A + \sum_{i \ge 1} \frac{p_i}{2}\,(H_{x_i} + H_{-x_i}),$$

where for $i \in I_k$ we put $H_{x_i}(B) = G_k(B - x_i e^i)$ (respectively $H_{-x_i} = G_k(B + x_i e^i)$), for $B \in \mathcal{B}$. The sequences of positive numbers $\{p_i\}$ and $\{x_i\}$ are determined below. Since F_1 is a probability distribution we have $|F_1| = 1$. This together with the identities $|\mathcal{N}_A| = 1/2$ and

$$|H_{x_i}| = |H_{-x_i}| = |G_k| = 1/2, \qquad i \in I_k,\ k = 1, 2, \dots$$

imply $\sum_i p_i = 1$. The last identity is satisfied provided that

$$\sum_{i \in I_k} p_i = 2^{-k}, \qquad k = 1, 2, \dots. \tag{3.2}$$

We shall construct the sequence $\{p_i\}$ so that (3.2) holds.
Firstly, define p_i and x_i for $i \in I_1$. Put

$$p_i^{-1} = 2(n_2 - n_1), \qquad 1 \le i < n_2.$$

This yields (3.2) for $k = 1$. Given $i \in I_1$, we define x_i. The function

$$g_i(x) = \mathcal{N}(A)\mathbf{E}\big((\eta_i + x)^2 \,\big|\, \eta_i^2 \le a_i^2\big) - \sigma_i^2$$

is continuous, $g_i(0) = 0$ and $g_i(x) \to +\infty$ as $x \to \infty$. Therefore, there exists $x_i > 0$ such that

$$g_i(x_i) = p_i^{-1}(1 - 2\sigma_i^2). \tag{3.3}$$

Here we use the fact that $1 - 2\sigma_i^2 > 0$, see (2.2).
Let us show that

$$v_i^2 = 1, \qquad \text{where} \qquad v_i^2 = \int y_i^2 \, dF_1(y). \tag{3.4}$$

Here $y_i = \pi_i(y)$ denotes the i-th coordinate of $y = (y_1, y_2, \dots) \in R^\infty$. Denote

$$s_{ij} = \int y_i^2 dH_{x_j}(y) = \int y_i^2 dH_{-x_j}(y).$$

We have

$$v_i^2 = \sigma_i^2 + \sum_{j=1}^{\infty} p_j s_{ij}. \tag{3.5}$$

For $i \in I_1$ and $j \neq i$ we have $s_{ij} = \sigma_i^2$. Therefore, the right sum in (3.5) equals $\sigma_i^2(1 - p_i)$. From (3.5) we obtain

$$v_i^2 = \sigma_i^2 + \sigma_i^2(1 - p_i) + p_i s_{ii}.$$

Finally, (3.4) follows from the identities (3.3) and $s_{ii} = g_i(x_i) + \sigma_i^2$. For convenience we include the proof of the last identity

$$\begin{aligned} s_{ii} &= \int (x_i + y_i)^2 dG_1(y) \\ &= \frac{G_1(A_1)}{\mathbf{P}\{\eta_i^2 \le a_i^2\}} \mathbf{E}(\eta_i + x_i)^2 \mathbb{I}_{\eta_i^2 \le a_i} \\ &= G_1(A_1)\mathbf{E}((\eta_i + x_i)^2 \,\big|\, \eta_i^2 \le a_i^2) \\ &= g_i(x_i) + \sigma_i^2. \end{aligned}$$

In the last step we use $G_1(A_1) = \mathcal{N}(A)$.
Let $k = 2, 3, \dots$. For $i \in I_k$, write

$$p_i = 2 r_i x_i^{-2}, \tag{3.6}$$

$$x_i = 4 N_k a_i (1 + \Delta_i), \tag{3.7}$$

$$r_i = 1 - 2\sigma_i^2 + 2^{1-k}(\sigma_i^2 - 4^{-1}),$$

where the positive numbers Δ_i are specified in what follows. The function

$$h(\Delta_{n_k}, \dots, \Delta_{n_{k+1}-1}) = \sum_{i \in I_k} \frac{r_i}{16\, a_i^2 (1 + \Delta_i)^2 N_k}.$$

is continuous. Using the inequalities (which follow from (2.2)) $6^{-1}2^{-k} < r_i < 1$, it is easy to show that

$$h(n_{k+1}^{1/2}, \dots, n_{k+1}^{1/2}) < 1 < h(0, \dots, 0).$$

Therefore, there exist numbers $0 < \Delta_i < n_{k+1}^{1/2}$, for $i \in I_k$, such that

$$h(\Delta_{n_k}, \dots, \Delta_{n_{k+1}-1}) = 1. \tag{3.8}$$

We use these numbers Δ_i to define x_i in (3.7). It follows from (3.6), (3.7) and (3.8) that (3.2) holds

$$\sum_{i \in I_k} p_i = \frac{2}{N_k}\, h(\Delta_{n_k}, \dots, \Delta_{n_{k+1}-1}) = \frac{2}{N_k} = 2^{-k}.$$

Let us show (3.4), for $i \in I_k$. To this aim, given $i \in I_k$, we evaluate s_{ij}, $j \in \mathbb{N}$. We have

$$\begin{aligned}
s_{ij} &= \sigma_i^2, \qquad j \in \mathbb{N} \setminus I_{k-1}, \qquad j \neq i,\\
s_{ij} &= G_{k-1}(A_{k-1})\mathbf{E}(\eta_i^2 \big|\, \eta_i^2 \le a^2) = 4^{-1}, \qquad j \in I_{k-1},\\
s_{ii} &= G_k(A_k)\mathbf{E}\big((\eta_i + x_i)^2 \big|\, \eta_i^2 \le a_i^2\big) = \sigma_i^2 + x_i^2/2.
\end{aligned}$$

Here we use the identities $G_k(A_k) = \mathcal{N}(A) = 1/2$, for every $k = 1, 2, 3, \dots$. Collecting these expressions in (3.5) we obtain (3.4).

Step 5. Let $Z = (Z_1, Z_2, \dots)$ be a random variable with values in R^∞ and with the distribution F_1. By the construction of the distribution F_1 we have $\mathbf{E}Z_i^2 = 1$ and $\mathbf{E}Z_i = 0$ and $\mathbf{E}Z_iZ_j = 0$, for $j > i \ge 1$. The latter two identities follows from the fact that F_1 is symmetric.

Let $Z^1, Z^2, \dots$ be independent random variables with values in R^∞ and with the common distribution F_1. Write $S_r^* = Z^1 + \dots + Z^r$. We shall show that

$$\limsup_{k \to \infty} (\mathcal{N}(T_k) - \mathbf{P}\{S_{N_k}^* \in \sqrt{N_k}\, T_k\}) \ge (1 - e^{-1})/2. \tag{3.9}$$

By $\tilde{H}$ we denote the image $L(F_1)$ of the distribution F_1. We have $\tilde{H} = \tilde{H}_1 + \tilde{H}_2$, where

$$\tilde{H}_1 = \sum_{i \in I_k} \frac{p_i}{2} \big(L(H_{x_i}) + L(H_{-x_i})\big)$$

and $\tilde{H}_2 = L(F_1) - \tilde{H}_1$. Here $L(H_{x_i})$ denotes the image of the measure H_{x_i}. Note that for large i we have $a_i > a$. We assume that k is large enough so that the latter inequality holds. Therefore, the support of the measure $\tilde{H}_2$ is the set $L(T_k)(= T_k^*$ for brevity). The support of the measure $L(H_{x_i})$ is the set $x_j e^j + T_k^*$ Note that

$$\begin{aligned}
|\tilde{H}_1| &= \sum_{i \in I_k} \frac{p_i}{2} \left(|H_{x_i}| + |H_{-x_i}|\right)\\
&= \sum_{i \in I_k} \frac{p_i}{2} = \frac{1}{2^{k+1}} = \frac{1}{N_k}\,.
\end{aligned}$$

Therefore,

$$|\tilde{H}_2| = 1 - |\tilde{H}_1| = 1 - N_k^{-1}. \tag{3.10}$$

In order to prove (3.9) we proceed as in the proof of (2.7). That is, we shall show (2.9). To this aim we prove that (2.12) and (2.14) holds in the present situation as well.

In what follows B denotes a Borel set in R_k. To prove (2.12) note that, for j odd, $\tilde{H}_1^j(B) = 0$, provided that $B \cap B_j^* = \emptyset$, where $B_j^* = B_0 + jT_k^*$, and where $B_0 \subset R_k$ is defined by (2.11). Similarly, $\tilde{H}_2^{N-j}(B) = 0$, for every B such that $B \cap (N-j)T_k^* = \emptyset$. Therefore, $\tilde{H}_1^j \tilde{H}_2^{N-j}(B) = 0$, provided that $B \cap (B_j^* + (N-j)T_k^*) = \emptyset$. Since $|x_i| > 4Na_i$, the sets $B_j^* + (N-j)T_k^*$ and $V = \sqrt{N}\, T_k^*$ do not intersect. Thus, (2.12) follows.

Let us prove (2.14). For even j we have $\tilde{H}_1^j(B) = 0$, provided that $B \cap (jT_k^* \cup B_j^*) = \emptyset$. Split $\tilde{H}_1^j = \tilde{Q} + \tilde{R}$, where $\tilde{Q}$ (respectively $\tilde{R}$) denote the restriction of $\tilde{H}_1^j$ on the set jT_k^* (respectively B_j^*). In particular, $\tilde{Q}(B) = \tilde{H}_1^j(B \cap jT_k^*)$. The same argument as above shows $\tilde{R}\, \tilde{H}_2^{N-j}(V) = 0$. Therefore,

$$\tilde{H}_1^j \tilde{H}_2^{N-j}(V) = (\tilde{Q} + \tilde{R})\tilde{H}_2^{N-j}(V) = \tilde{Q}\tilde{H}_2^{N-j}(V). \tag{3.11}$$

Proceeding as in the proof of (2.13) we obtain $|\tilde{Q}| \le N^{-j}\binom{j}{j/2}2^{-j}$. It follows from this inequality, (3.11) and (3.10) that

$$\tilde{H}_1^j \tilde{H}_2^{N-j}(V) \le |\tilde{Q}|\, |\tilde{H}_2|^{N-j} \le \binom{j}{j/2} 2^{-j} \, \frac{1}{N^j} \, (1 - \, \frac{1}{N} \,)^{N-j}.$$

We arrive at (2.14), thus, completing the proof of (3.9).

References

Bloznelis, M. 1989., *Lower bound for the rate of convergence in the CLT in Hilbert space*, Lithuan. Math. J. **29**, 333–338.

Davydov Yu. A. 1988., *Local limit theorems for functionals in random processes*, Theory Probab. Appl. **33**, 732–738.

Davydov, Yu. A. 1989., *A variant of an infinitele–dimensional local limit theorem*, Zap. Nauchn. Sem. Leningrad. Otdel. Math. Inst. Steklov. (LOMI). Problemy Teorii Veroyatnost. Raspred. XI **177**, 46–50 (in Russian).

Ibragimov, I.A. and Linnik, Yu. V. 1971, *Independent and stationary sequences of random variables*, Wolters–Noordhoff, The Netherlands.

Mamatov, M. and Halikov, M.K. 1964., *Global limit theorems for distribution functions in the higher dimensional case*, Izv. Akad. Nauk UzSSR, Ser. Fiz. Mat. Nauk **1**, 13–21 (in Russian).

Prokhorov, Yu.V. 1952., *A local theorem for densities*, Doklady Akad. Nauk SSSR (N.S) **83**, 797–800 (in Russian).

Rachev, S.T. and Yukich, J.E. 1989., *Rates for the CLT via new ideal metrics*, Ann. Probab. **17**, 775–788.

Sazonov, V.V. 1981., *Normal approximation - some recent advances*, Lecture Notes Math Vol 879, Springer, Heidelberg.

Sazonov, V.V. and Ulyanov V.V. 1979., *On the speed of convergence in the central limit theorem*, Adv. Appl.Probab. **11**, 269–270.

Senatov, V.V. 1980., *Uniform estimates of the rate of convergence in the multi-dimensional central limit theorem*, Theory Probab. Appl. **25**, 745–759.