EMPIRICAL EDGEWORTH EXPANSION FOR FINITE POPULATION STATISTICS.II

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Abstract. For symmetric asymptotically linear statistics based on simple random samples, we construct a one-term empirical Edgeworth expansion, where the moments defining the true Edgeworth expansion are replaced by their jackknife estimators. In order to establish the validity of the empirical Edgeworth expansion (in probability) we prove the consistency of the jackknife estimators.

4. Lemmas

In the first part of the section, we collect the statements of lemmas. Proofs are given aftewards.

Lemma 4.1. *Under conditions of Lemma 2.3, the bounds (3.29), (3.30), (3.32), and (3.39) hold. If in addition, $n \le N/2$, then the bound (3.38) holds.*

Let r_i, $\tilde r_i$, r_{ij}, and w_{ij} be random variables defined in (3.19), (3.26), (3.28), and (3.21), respectively.

Lemma 4.2. *We have*

$$\mathbf{E}\, r_i^2 \le 2^{-1} n_*^{-1} \delta_2, \qquad 1 \le i \le n+1, \tag{4.1}$$

$$\mathbf{E}\, \tilde r_i \le 2^{-1} n_*^{-1} \delta_2, \qquad 1 \le i \le n+2, \tag{4.2}$$

$$\mathbf{E}\, r_{ij}^2 \le c\,\frac{q\sigma_1^2}{n^3} + c\,\frac{q\sigma_2^2}{n} + c\,\frac{\delta_3}{n_*^3}\,, \qquad 1 \le i < j \le n+2. \tag{4.3}$$

Furthermore, under conditions of Lemma 2.3, the following bounds hold:

$$\mathbf{E}\,|w_{ij}T_iT_j| = O(\tau^{-5}), \qquad \mathbf{E}\,|w_{ij}T_iT_j|^{6/5} = O(\tau^{-6}). \tag{4.4}$$

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Lemma 4.3. *Assume that (2.1) is satisfied. Then we have*

$$\mathbf{E}\, w_{ij}^2 \le 3C_2\tau^{-6}\gamma_2 + 6C_2\tau^{-2}n^{-2}, \tag{4.5}$$

$$\mathbf{E}\, \tilde r_i^2 \le 2^5 C_2\tau^{-4}\gamma_2 + 2^{-2}\tau^{-4}\delta_3, \tag{4.6}$$

$$\mathbf{E}\, |w_{ij}\tilde r_j| \le 2^4\tau^{-5}(1+C_2)(1+\gamma_2+\delta_3), \tag{4.7}$$

$$\mathbf{E}\, |r_{ij}\tilde r_i| \le c\tau^{-5}\varkappa(C_2\gamma_2+\delta_3)^{1/2}, \tag{4.8}$$

$$\mathbf{E}\, |r_{ij}T_iT_j| \le c\tau^{-5}C_2\varkappa, \qquad \mathbf{E}\, |r_{ij}T_i| \le c\tau^{-4}C_2^{1/2}\varkappa, \tag{4.9}$$
$$\mathbf{E}\, |r_{ij}T_*| \le c\tau^{-5}C_2^{1/2}\varkappa,$$

where $\varkappa^2 = \delta_3 + \tau^{-2}C_2 + \tau^{-2}C_2\gamma_2$ *and where* T_* *is defined in (3.21).*

Proof of Lemma 3.1. Let us prove (3.14) and (3.15). From (3.12) and (3.6) it follows that that $\mathbf{E}\, U_1^2 \le \sigma_T^2 \le \mathbf{E}\, U_1^2 + \delta_2$. In particular, we have $\mathbf{E}\, U_1^2 \le C_2$. Furthermore, by (3.10),

$$\mathbf{E}\, U_1^2 = \zeta^2(1+(N-1)^{-1}) \le \zeta^2 + (N-1)^{-1}\mathbf{E}\, U_1^2.$$

Combining these inequalities, we obtain (3.14) and the estimate

$$0 \le \sigma_T^2 - \zeta^2 = (\sigma_T^2 - \mathbf{E}\, U_1^2) + (\mathbf{E}\, U_1^2 - \zeta^2) \le \delta_2 + (N-1)^{-1}C_2.$$

Let us prove (3.16). By (3.8),

$$\mathbf{E}\,(\mathbb{D}_2 T)^2 = \mathbf{E}\, U_2^2(\mathbb{D}_2T) + \mathbf{E}\, U_3^2(\mathbb{D}_2T) + \cdots + \mathbf{E}\, U_{n_*}^2(\mathbb{D}_2T). \tag{4.10}$$

Combining (3.11) and (3.10), we get

$$\mathbf{E}\, U_2^2(\mathbb{D}_2T) = 4\sigma_2^2(N-1)(N-3)^{-1}. \tag{4.11}$$

Furthermore, by (3.11), for $j = 3, \dots, n_*$, we have

$$\mathbf{E}\, U_j^2(\mathbb{D}_2T) = 2^{-1}h_{3,j}h_{2,j}^{-1}\mathbf{E}\, U_j^2(\mathbb{D}_3T) \le 2^{-1}n_*\mathbf{E}\, U_j^2(\mathbb{D}_3T). \tag{4.12}$$

In the last step, we used the inequality $h_{3,j} \le n_* h_{2,j}$. From (4.12) it follows that

$$\sum_{j=3}^{n_*}\mathbf{E}\, U_j^2(\mathbb{D}_2T) \le 2^{-1}n_*\sum_{j=3}^{n_*}\mathbf{E}\, U_j^2(\mathbb{D}_3T) = 2^{-1}n_*\mathbf{E}\,(\mathbb{D}_3T)^2 = 2^{-1}n_*^{-3}\delta_3. \tag{4.13}$$

Combining (4.10), (4.11), and (4.13), we obtain

$$\delta_2 = n_*^2\mathbf{E}\,(\mathbb{D}_2T)^2 \le 4\,\frac{N-1}{N-3}\,n_*^2\sigma_2^2 + 2^{-1}n_*^{-1}\delta_3 \le 2^6C_2n_*^{-1}\gamma_2 + 2^{-1}n_*^{-1}\delta_3.$$

In the last step, we used the identity $\sigma_2^2 = \tau^{-6}\zeta^2\gamma_2$ and the inequality $\zeta^2 \le C_2$.

Proof of Lemma 4.1. We shall use the same notation as in proof of Lemma 2.3.

From (3.13) and (3.14) it follows that $\mathbf{E}\,T_*^2 \le \zeta^2 n^{-2} \le C_2 n^{-2}$. Furthermore, by Chebyshev's inequality, $\mathbf{P}\{|T_*| > n^{-3/4}\} \le C_2 n^{-1/2}$. Therefore, in what follows, we can assume that

$$|T_*| \le n^{-3/4}. \tag{4.14}$$

By (2.3) and (3.14), we have, for $0 < r \le s$,

$$\mathbf{E}\,|T_i|^r = \tau^{-r}\zeta^r\mathbf{E}\,|T_i\sigma_1^{-1}|^r = O(\tau^{-r}), \qquad 1 \le i \le N. \tag{4.15}$$

Proof of (3.29). Let us prove (3.29) for $i = 5$. The inequality (4.14) and $|\tilde{v}_i| \le |T_i| + |T_*|$ imply

$$|\tilde{R}_5| \le R_1^* + n^{-3/4}R_2^* + n^{-3/2}R_3^*, \qquad R_1^* = \sum{}' |r_{ij}T_iT_j|, \tag{4.16}$$

$$R_2^* = \sum{}' |r_{ij}|(|T_i| + |T_j|), \qquad R_3^* = \sum{}' |r_{ij}|.$$

Furthermore, by symmetry and (4.9),

$$\begin{aligned}
\mathbf{E}\,R_1^* &= 2^{-1}[n+2]_2\mathbf{E}\,|r_{1n}T_1T_n| = o(n^2\tau^{-5}),\\
\mathbf{E}\,R_2^* &= [n+2]_2\mathbf{E}\,|r_{1n}T_1| = o(n^2\tau^{-4}),\\
\mathbf{E}\,R_3^* &\le 2^{-1}[n+2]_2\mathbf{E}\,|r_{1n}| = o(n^2\tau^{-3}).
\end{aligned}$$

Therefore, we have

$$q^2\tau R_1^* = o_P(1), \qquad q^2\tau R_2^* = o_P(\tau), \qquad q^2\tau R_3^* = o_P(\tau^2).$$

Invoking these bounds in (4.16), we obtain $\tau q^2 \tilde{R}_5 = o_P(1)$.

Let us prove (3.29) for $1 \le i \le 4$ in the case where $n \le N/2$. Note that $n \le N/2$ implies $\tau^2 \ge n/2$ and $q \ge 1/2$.

Consider $\tilde{R}_1$. We have $\tau|\tilde{R}_1| \le (A_1 + |T_*|)B_1$, where

$$A_1 = \max_{1\le j\le n+2} |T_j|, \qquad B_1 = \tau\sum{}' |w_{ij}|(|\tilde{r}_i| + |\tilde{r}_j|).$$

By symmetry,

$$\begin{aligned}
\mathbf{E}\,A_1^s &\le \mathbf{E}\,|T_1|^s + \cdots + \mathbf{E}\,|T_{n+2}|^s = (n+2)\mathbf{E}\,|T_1|^s,\\
\mathbf{E}\,B_1 &= \tau[n+2]_2\mathbf{E}\,|w_{n1}||\tilde{r}_n|.
\end{aligned}$$

Now (4.15) implies $\mathbf{E}\, A_1^s = o(1)$, and (4.7) implies $\mathbf{E}\, B_1 = O(1)$. Therefore, $A_1 = o_P(1)$ and $B_1 = O_P(1)$. Finally, invoking (4.14), we obtain $(A_1 + |T_*|)B_1 = o_P(1)$.

Consider $\tilde{R}_3$. We have $\tau|\tilde{R}_3| \le A_2 B_2$, where

$$A_2 = \max_{1\le j\le n+2} |\tilde{r}_j|, \qquad B_2 = \tau \sum{}' |w_{ij}||\tilde{r}_i|.$$

By symmetry,

$$\mathbf{E}\, A_2^2 \le \mathbf{E}\, \tilde{r}_1^2 + \cdots + \mathbf{E}\, \tilde{r}_{n+2}^2 = (n+2)\mathbf{E}\, \tilde{r}_1^2,$$
$$\mathbf{E}\, B_2 = \tau 2^{-1}[n+2]_2 \mathbf{E}\, |w_{n1}||\tilde{r}_n|.$$

Now (3.25) combined with (3.16) implies $\mathbf{E}\, A_2^2 = o(1)$, and (4.7) implies $\mathbf{E}\, B_2 = O(1)$. Therefore, $A_2 = o_P(1)$ and $B_2 = O_P(1)$ and, hence, $\tau q^2 \tilde{R}_3 = o_P(1)$.

Proofs of (3.29) for $i = 2$ and $i = 4$ are similar to those in the cases where $i = 1$ and $i = 3$. The only difference is that now, instead of $\mathbf{E}\, B_1$ and $\mathbf{E}\, B_2$, one estimates the expectations of

$$\tau \sum{}' |r_{ij}|(|\tilde{r}_i| + |\tilde{r}_j|) \quad \text{and} \qquad \tau \sum{}' |r_{ij}\tilde{r}_j|,$$

using (4.8).

Let us prove (3.29) for $1 \le i \le 4$ in the case where $n > N/2$. We first prove (3.29) for $i = 1$. By symmetry, it suffices to show that

$$q^2 \tau R^* = o_P(1), \qquad R^* = \sum{}' w_{ij}\tilde{v}_i\tilde{r}_j. \tag{4.17}$$

The inequalities $|\tilde{v}_i| \le |T_i| + |T_*|$ and (4.14) imply

$$|R^*| \le R_1^* + n^{-3/4} R_2^*, \qquad R_1^* = \sum{}' |w_{ij}T_i\tilde{r}_j|, \qquad R_2^* = \sum{}' |w_{ij}\tilde{r}_j|.$$

In order to prove (4.17) we shall show that

$$q^2 \tau \mathbf{E}\, R_1^* = o(1) \qquad \text{and} \qquad q^2 \tau n^{-3/4} \mathbf{E}\, R_2^* = o(1). \tag{4.18}$$

Let us prove the first bound. By Cauchy-Schwarz,

$$\mathbf{E}\, R_1^* \le A_1^{1/2} B_1^{1/2}, \qquad A_1 = \mathbf{E} \sum{}' w_{ij}^2, \quad B_1 = \mathbf{E} \sum{}' T_i^2 \tilde{r}_j^2. \tag{4.19}$$

The inequalities

$$T_1^2 + \cdots + T_{n+2}^2 \le N\mathbf{E}\,T_1^2, \qquad \tilde r_1^2 + \cdots + \tilde r_{n+2}^2 \le N\mathbf{E}\,\tilde r_1^2, \tag{4.20}$$

combined with (4.15), (3.25), and (3.16) imply

$$B_1 \le N^2\mathbf{E}\,T_1^2\mathbf{E}\,\tilde r_1^2 = N^2 o(\tau^{-4}). \tag{4.21}$$

Furthermore, by (4.5),

$$A_1 = [n+2]_2 2^{-1}\mathbf{E}\,w_{n1}^2 = n^2 O(\tau^{-6}). \tag{4.22}$$

Invoking (4.21) and (4.22) in (4.19) and using the inequality $N \le 2n$, we obtain the first bound of (4.18). The second bound follows from (4.7).

Let us prove that $\tau q^2\mathbf{E}\,|\tilde R_3| = o(1)$. By Cauchy–Schwarz, $\mathbf{E}\,|\tilde R_3| \le A_1^{1/2}B_2^{1/2}$, where $B_2 = \mathbf{E}\sum' \tilde r_i^2\tilde r_j^2$ and where A_1 is defined in (4.19). Furthermore, by (4.20), (3.25), and (3.16) we have $B_2 \le N^2(\mathbf{E}\,\tilde r_1^2)^2 = N^2 o(\tau^{-4})$. This bound in combination with (4.22) implies $\tau q^2\mathbf{E}\,|\tilde R_3| = o(1)$.

The bounds $\tau q^2\mathbf{E}\,|\tilde R_k| = o(1)$ for $k = 2, 4$ are proved in a similar way. The only difference is that, instead of A_1, one estimates the expectation $\mathbf{E}\sum' r_{ij}^2 = n^2 o(\tau^{-6})$ by (3.25).

Proof of (3.30) and (3.32). Let us prove (3.30). It suffices to show that $q^2\tau\Delta' = o_P(1)$, where $\Delta' = (n+1)^2 n^{-2}s - s^*$. We have

$$\Delta' = s_1^* - s_2^*, \qquad s_1^* = T_*^2\sum{}' w_{ij}, \qquad s_2^* = T_*\sum{}' w_{ij}(T_i + T_j). \tag{4.23}$$

By (4.14),

$$|s_1^*| \le n^{-3/2}\sum{}' |w_{ij}|, \qquad |s_2^*| \le n^{-3/4}\sum{}' |w_{ij}|(|T_i| + |T_j|).$$

Invoking the bounds (which follow from (4.5), by Cauchy–Schwarz) $\mathbf{E}\,|w_{1n}| = O(\tau^{-3})$ and $\mathbf{E}\,|w_{1n}T_1| = O(\tau^{-4})$, we obtain $\mathbf{E}\,|s_1^*| = n^{1/2}O(\tau^{-3})$ and $\mathbf{E}\,|s_2^*| = n^{5/4}O(\tau^{-4})$. Now the bound $\mathbf{E}\,q^2\tau|\Delta'| = o(1)$ follows from (4.23).

Let us prove (3.32). By symmetry,

$$2q^2\tau n^2[n+2]_2^{-1}\mathbf{E}\,s^* = q^2 n^2\tau\mathbf{E}\,w_{1n}T_1T_n = \tau^5\mathbf{E}\,w_{1n}T_1T_n.$$

Therefore, $\Delta = \tau^5\mathbf{E}\,(w_{1n} - T_{1n})T_1T_n$. Finally, by symmetry,

$$\tau^5\mathbf{E}\,(w_{1n} - T_{1n})T_1T_n = -2\tau^5(n+1)^{-1}\mathbf{E}\,T_1^2T_n = O(N^{-1}).$$

In the last step, we use (4.15) and the identity

$$\mathbf{E}\,T_1^2 T_n = \mathbf{E}\,T_1^2 \mathbf{E}\,(T_n|X_1) = \mathbf{E}\,T_1^2\Big(\frac{N\mathbf{E}\,T_n - T_1}{N-1}\Big) = \frac{-1}{N-1}\,\mathbf{E}\,T_1^3,$$

which follows from $\mathbf{E}\,T_n = 0$.

Before the proof of (3.38) and (3.39) we introduce some more notation. Write

$$s_{ij} = w_{ij} T_i T_j, \qquad s_0 = \mathbf{E}\,s_{ij}, \qquad t_i^2 = \mathbf{E}\,(T_j^2|X_i)$$

and note that, by (3.1) and (4.15), $t_i^2 \le 2\mathbf{E}\,T_j^2 = O(\tau^{-2})$.

Proof of (3.38). Using the property (3.2) and the identity $\mathbb{I}_i\mathbb{I}_j = \bar{\mathbb{I}}_i\bar{\mathbb{I}}_j + \mathbb{I}_i + \mathbb{I}_j - 1$ we obtain

$$\mathbf{E}\,g_2^*(X_i, X_j)\mathbb{I}_i\mathbb{I}_j = \mathbf{E}\,g_2^*(X_i, X_j)\bar{\mathbb{I}}_i\bar{\mathbb{I}}_j.$$

Here we denote

$$\bar{\mathbb{I}}_i = 1 - \mathbb{I}_i.$$

Furthermore, invoking the identity

$$g_2^*(X_i, X_j) = s_{ij} - s_0 + (N-1)(N-2)^{-1}\big(2s_0 - \mathbf{E}\,(s_{ij}\,|\,X_i) - \mathbf{E}\,(s_{ij}\,|\,X_j)\big), \tag{4.24}$$

we get

$$\begin{aligned}
&\mathbf{E}\,g_2^*(X_i, X_j)\bar{\mathbb{I}}_i\bar{\mathbb{I}}_j = R_1 - R_2 + (N-1)(N-2)^{-1}(2R_2 - R_3 - R_4),\\
&R_1 = \mathbf{E}\,s_{ij}\bar{\mathbb{I}}_i\bar{\mathbb{I}}_j, \qquad R_2 = s_0\mathbf{E}\,\bar{\mathbb{I}}_i\bar{\mathbb{I}}_j,\\
&R_3 = \mathbf{E}\,\bar{\mathbb{I}}_i\bar{\mathbb{I}}_j\mathbf{E}\,(s_{ij}|X_i), \qquad R_4 = \mathbf{E}\,\bar{\mathbb{I}}_i\bar{\mathbb{I}}_j\mathbf{E}\,(s_{ij}|X_j).
\end{aligned}$$

In order to prove (3.38) we show that $R_k = O(\tau^{1/4-6})$ for $k = 1, 2, 3$.

By Chebyshev's inequality and (4.15),

$$\mathbf{E}\,\bar{\mathbb{I}}_i\bar{\mathbb{I}}_j \le \mathbf{E}\,\bar{\mathbb{I}}_i, \qquad \mathbf{E}\,\bar{\mathbb{I}}_i \le \eta^{-2}\mathbf{E}\,T_i^2 = \tau^{1/2}\sigma_1^2 = O(\tau^{-3/2}), \tag{4.25}$$

$$\mathbf{E}\,T_i^2\bar{\mathbb{I}}_i \le \eta^{-1}\mathbf{E}\,|T_i|^3 = O(\tau^{1/4-3}). \tag{4.26}$$

Furthermore, using (3.1), we obtain from (4.26) that

$$\mathbf{E}\,T_i^2 T_j^2\bar{\mathbb{I}}_i\bar{\mathbb{I}}_j \le N(N-1)^{-1}(\mathbf{E}\,T_i^2\bar{\mathbb{I}}_i)^2 = O(\tau^{1/2-6}). \tag{4.27}$$

Combining (4.4) and (4.25), we obtain $R_2 = O(\tau^{-1/2-6})$. Furthermore, by Cauchy–Schwarz, we derive from (4.5) and (4.27)

$$|R_1| \le (\mathbf{E}\,w_{ij}^2)^{1/2}(\mathbf{E}\,T_i^2 T_j^2\bar{\mathbb{I}}_i\bar{\mathbb{I}}_j)^{1/2} = O(\tau^{1/4-6}).$$

Consider $R_3 = \mathbf{E}\, T_i \bar{\mathbb{I}}_i \bar{\mathbb{I}}_j \mathbf{E}\,(w_{ij} T_j | X_i)$. We apply Cauchy–Schwarz conditionally given X_i:

$$|R_3| \le \mathbf{E}\, A_0 A_1^{1/2} t_i, \qquad A_0 = |T_i \bar{\mathbb{I}}_i \bar{\mathbb{I}}_j|, \qquad A_1 = \mathbf{E}\,(w_{ij}^2 | X_i).$$

Therefore, $|R_3| = O(\tau^{-1})\mathbf{E}\, A_0 A_1^{1/2}$. Furthermore, by Cauchy–Schwarz and (4.5),

$$\mathbf{E}\, A_0 A_1^{1/2} \le (\mathbf{E}\, A_0^2)^{1/2} (\mathbf{E}\, A_1)^{1/2} = O(\tau^{-3})(\mathbf{E}\, A_0^2)^{1/2}.$$

Finally, by (3.1), (4.25), and (4.26),

$$\mathbf{E}\, A_0^2 = \mathbf{E}\, \bar{\mathbb{I}}_j \mathbf{E}\,(T_i^2 \bar{\mathbb{I}}_i | X_j) \le N(N-1)^{-1} \mathbf{E}\, \bar{\mathbb{I}}_j \mathbf{E}\, T_i^2 \bar{\mathbb{I}}_i = O(\tau^{-1/4-4}).$$

Hence, we obtain

$$\mathbf{E}\, |R_3| = O(\tau^{-1}) O(\tau^{-3}) O(\tau^{-1/8-2}) = O(\tau^{-1/8-6}).$$

Proof of (3.39). A simple calculation gives

$$\begin{aligned}
&\tilde{g}_2(X_i, X_j) = r_1 - r_2 + (N-1)(N-2)^{-1}(2r_2 - r_3 - r_4),\\
&r_1 = g_2^*(X_i, X_j)\mathbb{I}_i \mathbb{I}_j, \qquad r_2 = \mathbf{E}\, \mathbb{I}_i \mathbb{I}_j g_2^*(X_i, X_j),\\
&r_3 = \mathbb{I}_i \mathbf{E}\,(\mathbb{I}_j g_2^*(X_i, X_j) \,|\, X_i), \qquad r_4 = \mathbb{I}_j \mathbf{E}\,(\mathbb{I}_i g_2^*(X_i, X_j) \,|\, X_j).
\end{aligned}$$

In order to prove (3.39) we shall show that $\mathbf{E}\, r_i^2 = O(\tau^{-7})$ for $i = 1, 2, 3$. Note that, by (3.38), we have $r_2^2 = o(\tau^{1/2-12})$.

Let us prove the bound $\mathbf{E}\, r_1^2 = O(\tau^{-7})$. In view of (4.24) and the bound $s_0^2 = O(\tau^{-10})$ (see (4.4)), it sufices to show that

$$\mathbf{E}\, \mathbb{I}_i \mathbb{I}_j s_{ij}^2 = O(\tau^{-7}), \qquad \mathbf{E}\, \mathbb{I}_i \mathbb{I}_j \big(\mathbf{E}\,(s_{ij} \,|\, X_i)\big)^2 = O(\tau^{-7}). \tag{4.28}$$

Invoking the inequality $s_{ij}^2 \mathbb{I}_i \mathbb{I}_j \le \eta^4 w_{ij}^2$, from (4.5) we obtain the first bound of (4.28). In order to prove the second bound write $\mathbf{E}\,(s_{ij} \,|\, X_i) = T_i \mathbf{E}\,(w_{ij} T_j \,|\, X_i)$ and apply Cauchy–Schwarz conditionally given X_i:

$$|\mathbf{E}\,(w_{ij} T_j \,|\, X_i)|^2 \le \big(\mathbf{E}\,(w_{ij}^2 \,|\, X_i)\big) t_i^2.$$

Furthermore, invoking the inequality $T_i^2 \mathbb{I}_i \le \eta^2$ and the bound $t_i^2 = O(\tau^{-2})$, we obtain

$$\mathbf{E}\, \mathbb{I}_i \mathbb{I}_j \big(\mathbf{E}\,(s_{ij} \,|\, X_i)\big)^2 = O(\tau^{-2}) \eta^2 \mathbf{E}\, w_{ij}^2 = O(\tau^{-1/2-8}).$$

Let us prove $\mathbf{E}\, r_3^2 = O(\tau^{-7})$. Denote

$$r_{1,*} = \mathbb{I}_i \mathbf{E}\,(\mathbb{I}_j s_{ij}|X_i), \qquad r_{2,*} = \mathbb{I}_i \mathbf{E}\,(s_{ij}|X_i)\mathbf{E}\,(\mathbb{I}_j|X_i),$$
$$r_{3,*} = \mathbb{I}_i \mathbf{E}\,\big(\mathbb{I}_j \mathbf{E}\,(s_{ij}|X_j)\big|X_i\big).$$

In view of (4.24) and the bound $s_0^2 = O(\tau^{-10})$, it suffices to show that $\mathbf{E}\, r_{i,*}^2 = O(\tau^{-7})$ for $i = 1, 2, 3$.

Let us prove these bounds. We have

$$r_{1,*} = \mathbb{I}_i T_i \mathbf{E}\,(\mathbb{I}_j T_j w_{ij}|X_i), \qquad r_{2,*} = \mathbb{I}_i T_i \mathbf{E}\,(T_j w_{ij}|X_i)\mathbf{E}\,(\mathbb{I}_j|X_i).$$

By the inequality $T_i^2 \mathbb{I}_i \le \eta^2$ and Cauchy–Schwarz (applied conditionally given X_i),

$$r_{1,*}^2 \le \eta^4 \mathbf{E}\,(w_{ij}^2|X_i), \qquad r_{2,*}^2 \le \eta^2 \mathbf{E}\,(w_{ij}^2|X_i) t_i^2.$$

Finally, invoking (4.5), we obtain $\mathbf{E}\, r_{1,*}^2 = O(\tau^{-7})$ and $\mathbf{E}\, r_{2,*}^2 = O(\tau^{-1/2-8})$.

Consider $r_{3,*}$. Write

$$w := \mathbf{E}\,\big(\mathbb{I}_j \mathbf{E}\,(s_{ij}|X_j)\big|X_i\big) = \mathbf{E}\,(\mathbb{I}_j T_j H\big|X_i), \quad H = \mathbf{E}\,(w_{ij} T_i|X_j).$$

We apply Cauchy–Schwarz conditionally given X_i:

$$w^2 \le t_i^2 \mathbf{E}\,(H^2\big|X_i) \quad \text{and} \quad H^2 \le t_j^2 \mathbf{E}\,(w_{ij}^2|X_j).$$

Furthermore, invoking the bound $t_i^2 = O(\tau^{-2})$, we obtain

$$w^2 = \mathbf{E}\,\big(\mathbf{E}\,(w_{ij}^2|X_j)\big|X_i\big) O(\tau^{-4}).$$

Finally, using (4.5), we get

$$\mathbf{E}\, r_{3,*}^2 \le \mathbf{E}\, w^2 = \mathbf{E}\, w_{ij}^2 O(\tau^{-4}) = O(\tau^{-10}).$$

Proof of Lemma 4.3. By (3.14) we have

$$\mathbf{E}\, T_{ij}^2 = \sigma_2^2 = \sigma_1^2 \tau^{-4} \gamma_2 = \tau^{-6} \zeta^2 \gamma_2 \le \tau^{-6} C_2 \gamma_2, \tag{4.29}$$

$$\mathbf{E}\, |T_i|^r = \sigma_1^r \beta_r = \tau^{-r} \zeta^r \beta_r \le \tau^{-r} C_2^{r/2} \beta_r. \tag{4.30}$$

Let us prove (4.5). By the inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2)$,

$$w_{ij}^2 \le 3T_{ij}^2 + 3(n+1)^{-2}(T_i^2 + T_j^2).$$

Now apply (4.29) and (4.30).

The bound (4.6) follows from (4.2) and (3.16).
The bound (4.7) follows from (4.5) and (4.6) by Cauchy-Schwarz.
Let us prove (4.8). By (4.3) and (4.29),

$$\mathbf{E}\, r_{ij}^2 \le c\tau^{-6}\varkappa^2. \tag{4.31}$$

Now (4.8) follows from (4.31) and (4.6) by Cauchy–Schwarz.
Let us prove (4.9). By Cauchy–Schwarz,

$$\mathbf{E}\,|r_{ij}T_iT_j| \le (\mathbf{E}\, r_{ij}^2)^{1/2}(\mathbf{E}\, T_i^2T_j^2)^{1/2}.$$

Invoking (4.31) and the bound $\mathbf{E}\, T_j^2T_i^2 \le 2\tau^{-4}C_2^2$, which follows from (3.1) and (4.30), we obtain the first inequality of (4.9). The remaining two inequalities are proved in a similar way.

Before the proof of Lemma 4.2 we introduce some more notation. For $2 \le i \le n_*$ and $3 \le j \le n_*$, write

$$K_i = \sum_{A_{i-1}\subset\Omega_n} T_{A_{i-1}\cup\{n+1\}}, \qquad L_j = \sum_{A_{j-2}\subset\Omega_n} T_{A_{j-2}\cup\{n+1,n+2\}}. \tag{4.32}$$

$$M_i = \sum_{A_{i-1}\subset\Omega_{n+1}} T_{A_{i-1}\cup\{n+2\}}, \qquad Q_i = \sum_{A_i\subset\Omega_{n+1}} T_{A_i}.$$

Proof of Lemma 4.2. Let us prove (4.1). By symmetry, it suffices to prove (4.1) for $i = n+1$. Write

$$a_j = \frac{n-j+1}{n+1}, \qquad b_j = \frac{j}{n+1}, \qquad d_j = \frac{N-2j+1}{N-n}, \qquad d_j' = \frac{N-j+1}{N-n}.$$

We have $r_{n+1} = \varkappa_2 + \cdots + \varkappa_{n_*}$, where

$$\varkappa_j = \sum_{A_j\subset\Omega_{n+1}} \big(\mathbb{I}_{n+1\in A_j} - \frac{j}{n+1}\big)T_{A_j} = a_jK_j - b_jU_j.$$

Note that, by (3.3), $\varkappa_i$ and $\varkappa_j$ are uncorrelated for $i \ne j$. Therefore, the inequality, which we prove below,

$$\mathbf{E}\,\varkappa_j^2 \le 2^{-1}n_*\mathbf{E}\,U_j^2(\mathbb{D}_2T), \qquad 2 \le j \le n_*, \tag{4.33}$$

implies the bound (4.1). Indeed, we have

$$\mathbf{E}\, r_{n+1}^2 = \sum_{j=2}^{n_*}\mathbf{E}\,\varkappa_j^2 \le \frac{n_*}{2}\sum_{j=2}^{n_*}\mathbf{E}\,U_j^2(\mathbb{D}_2T) = \frac{n_*}{2}\,\mathbf{E}\,(\mathbb{D}_2T)^2 = \frac{\delta_2}{2n_*}. \tag{4.34}$$

It remains to prove (4.33). By (5.2), $a_j^2 \mathbf{E}\, K_j^2 = a_j b_j d_j \mathbf{E}\, U_j^2$. This identity in combination with the inequalities $d_j \le d_j'$ and $1 \le d_j'$ implies

$$\begin{aligned}\mathbf{E}\, \varkappa_j^2 &\le 2a_j^2 \mathbf{E}\, K_j^2 + 2b_j^2 \mathbf{E}\, U_j^2 = 2\, b_j(a_j d_j + b_j) \mathbf{E}\, U_j^2 \\ &\le 2\, b_j(a_j + b_j) d_j' \mathbf{E}\, U_j^2 = 2^{-1} b_j d_j' h_{2,j} \mathbf{E}\, U_j^2(\mathbb{D}_2 T),\end{aligned}$$

where, in the last step, we used the identity $a_j + b_j = 1$ and replaced $\mathbf{E}\, U_j^2$ by $4^{-1} h_{2,j} \mathbf{E}\, U_j^2(\mathbb{D}_2 T)$ (see (3.11)). Finally, invoking the inequality

$$b_j d_j' h_{2,j} = \frac{n-1}{n+1} \, \frac{n(N-n-1)}{(j-1)(N-j)} \le n_*$$

we complete the proof of (4.33).

Let us prove (4.2). By symmetry, it suffices to prove (4.2) for $i = n+2$. Since, by (3.3), the summands of (3.26) are uncorrelated, the bound (4.2) follows from the inequalities

$$\mathbf{E}\, \tilde V_{n+2,k}^2 \le 2^{-1} n_* \mathbf{E}\, U_k^2(\mathbb{D}_2 T), \qquad 2 \le k \le n_* \tag{4.35}$$

(cf. (4.34)). Let us prove (4.35). Denote

$$l_k = \frac{n+1-k}{n+1}, \quad \tilde a_k = \frac{n+2-k}{n+2}, \quad \tilde b_k = \frac{k}{n+2}, \quad e_k = \frac{(N-k+1)(N-n-k)}{[N-n]_2}.$$

From the identity $\tilde V_{n+2,k} = l_k \tilde a_k M_k - l_k \tilde b_k Q_k$ it follows that

$$\mathbf{E}\, \tilde V_{n+2,k}^2 \le 2 l_k^2 \tilde a_k^2 \mathbf{E}\, M_k^2 + 2 l_k^2 \tilde b_k^2 \mathbf{E}\, Q_k^2. \tag{4.36}$$

Furthermore, by (5.4) and (5.5),

$$\begin{aligned} l_k^2 \tilde a_k^2 \mathbf{E}\, M_k^2 &= l_k \tilde a_k \tilde b_k \, \frac{(N-2k+1)(N-n-k)}{[N-n]_2} \, \mathbf{E}\, U_k^2 \le l_k \tilde a_k \tilde b_k e_k \mathbf{E}\, U_k^2, \\ l_k^2 \tilde b_k^2 \mathbf{E}\, Q_k^2 &= l_k \tilde b_k^2 \, \frac{N-n-k}{N-n} \, \mathbf{E}\, U_k^2 \le l_k \tilde b_k^2 e_k \mathbf{E}\, U_k^2. \end{aligned}$$

Invoking these inequalities in (4.36) and using the identity $\tilde a_k + \tilde b_k = 1$, we get $\mathbf{E}\, \tilde V_{n+2,k}^2 \le 2\, l_k \tilde b_k e_k \mathbf{E}\, U_k^2$. Finally, the identity $\mathbf{E}\, U_k^2 = 4^{-1} h_{2,k} \mathbf{E}\, U_k^2(\mathbb{D}_2 T)$ (see (3.11)), in combination with the inequality

$$l_k \tilde b_k e_k h_{2,k} \le \frac{(n+1-k)(N-n-k)}{N-k} \le n_*,$$

completes the proof of (4.35).

Let us prove (4.3). By (4.7), $\mathbf{E}\, r_{ij}^2 \le 5(\mathbf{E}\, R_0^2 + \cdots + \mathbf{E}\, R_4^2)$. We shall bound the expectations $\mathbf{E}\, R_i^2$ for $0 \le i \le 4$. By (3.13), $\mathbf{E}\, R_1^2 \le 4qn^{-3}\sigma_1^2$. By (3.10), $\mathbf{E}\, R_2^2 \le 18q^2 n^{-2}\sigma_2^2$. Furthermore, invoking (5.6), we obtain

$$\mathbf{E}\, R_3^2 = \mathbf{E}\, R_4^2 \le 4qn^{-1}\sigma_2^2.$$

We complete the proof of (4.3) by showing that

$$\mathbf{E}\, R_0^2 \le 2^4 n_*^{-3}\delta_3.$$

Recall that $R_0 = Z_3 + \cdots + Z_{n_*}$ (see (3.27) and (3.28)). Note that, by (3.3), the random variables Z_k and Z_r are uncorrelated for $k \ne r$. Therefore, the inequality

$$\mathbf{E}\, Z_k^2 \le 2^4 \eta_k, \qquad \eta_k = n_* \mathbf{E}\, U_k^2(\mathbb{D}_3 T), \qquad 3 \le k \le n_*. \tag{4.37}$$

implies

$$\mathbf{E}\, R_0^2 = \sum_{k=3}^{n_*} \mathbf{E}\, Z_k^2 \le 2^4 n_* \sum_{k=3}^{n_*} \mathbf{E}\, U_k^2(\mathbb{D}_3 T) = 2^4 n_* \mathbf{E}\, (\mathbb{D}_3 T)^2 = 2^4 n_*^{-3}\delta_3.$$

It remains to prove (4.37). By symmetry, we can assume without loss of generality that $i = n+1$ and $j = n+2$. A simple calculation gives

$$Z_k = u_1 U_k + u_2(K_k + W) + u_3 L_k, \tag{4.38}$$

where the random variables K_k and L_k are introduced in (4.32) and where

$$W = \sum_{A_{k-1} \subset \Omega_n} T_{A_{k-1} \cup \{n+2\}}, \qquad u_1 = \frac{[k+1]_2}{[n+2]_2}\,,$$
$$u_2 = u_1 - \frac{k}{n+1}\,, \qquad u_3 = u_1 - \frac{2k}{n+1} + 1.$$

From (4.38) it follows that

$$\mathbf{E}\, Z_k^2 \le 4u_1^2 \mathbf{E}\, U_1^2 + 4u_2^2(\mathbf{E}\, K_k^2 + \mathbf{E}\, W^2) + 4u_3^2 \mathbf{E}\, L_k^2.$$

Note that $\mathbf{E}\, W^2 = \mathbf{E}\, K_k^2$. Therefore, in order to prove (4.37) it suffices to show that

$$u_1^2 \mathbf{E}\, U_k^2 \le \eta_k, \qquad u_2^2 \mathbf{E}\, K_k^2 \le \eta_k, \qquad u_3^3 \mathbf{E}\, L_k^2 \le \eta_k.$$

The first inequality follows from (3.11). Remaining two inequalities are consequences of (5.2), (3.11) and (5.3), (3.11), respectively.

Let us prove (4.4). By Hölder's inequality,

$$\mathbf{E}\, |w_{ij} T_i T_j|^r \le \big(\mathbf{E}\, w_{ij}^2\big)^{r/2} \big(\mathbf{E}\, |T_i T_j|^{2r/(2-r)}\big)^{(2-r)/2},$$

where we choose $r = 1$ and $r = 6/5$. Combining (4.5) and the bound $\mathbf{E}\, |T_i T_j|^{2r/(2-r)} = O(\tau^{-4r/(2-r)})$, which follows from (4.15) via (3.1), we obtain (4.4).

5. Estimates for moments

In what follows, we shall use the formula (see, e.g., Zhao and Chen [3])

$$\sum_{v=0}^{\min\{s,t\}} \binom{s}{v}\binom{t}{v}\binom{u}{v}^{-1}(-1)^v = \binom{u-t}{s}\binom{u}{s}^{-1}, \tag{5.1}$$

where s, t, u are nonnegative integers such that $u \ge \max\{s,t\}$.

Let K_j, L_j, M_j, and Q_j be the random variables introduced in (4.32).

Lemma 5.1. *We have*

(5.2)
$$\mathbf{E}\,K_j^2 = \frac{j}{n-j+1}\,\frac{N-2j+1}{N-n}\,\mathbf{E}\,U_j^2, \qquad 2 \le j \le n_*,$$

(5.3)
$$\mathbf{E}\,L_j^2 = \frac{[j]_2\,[N-2j+2]_2}{[n-j+2]_2\,[N-n]_2}\,\mathbf{E}\,U_j^2, \quad 3 \le j \le n_*,$$

(5.4)
$$\mathbf{E}\,M_j^2 = \frac{j(n+1)}{[n-j+2]_2}\,\frac{(N-2j+1)(N-n-j)}{[N-n]_2}\,\mathbf{E}\,U_j^2, \quad 2 \le j \le n_*,$$

(5.5)
$$\mathbf{E}\,Q_j^2 = \frac{n+1}{n-j+1}\,\frac{N-n-j}{N-n}\,\mathbf{E}\,U_j^2, \quad 2 \le j \le n_*,$$

(5.6)
$$\mathbf{E}\,\Big(\sum_{k\in\Omega_{n+2}\setminus\{i\}} T_{\{i,k\}}\Big)^2 = \frac{(n+1)(N-n-2)}{N-2}\,\sigma_2^2.$$

Proof of Lemma 5.1. Let us prove (5.2). By symmetry,

$$\mathbf{E}\,K_j^2 = \binom{n}{j-1}\varkappa_K, \qquad \varkappa_K = \mathbf{E}\,T_{\Omega_{j-1}\cup\{n+1\}}K_j. \tag{5.7}$$

A straightforward calculation gives

$$\varkappa_K = \sum_{v=0}^{v_0}\binom{j-1}{v}\binom{n-j+1}{v}s_{j,j-v}, \qquad v_0 = \min\{j-1,\, n-j+1\}.$$

Invoking (3.9) and the identity (5.1), we obtain $\varkappa_K = \sigma_j^2[N-n-1]_{j-1}/[N-j]_{j-1}$. Substituting this expression into (5.7), we obtain an explicit formula for $\mathbf{E}\,K_j^2$. A comparison of this formula and (3.10) yields (5.2).

The proof of (5.3) is similar. By symmetry,

$$\mathbf{E}\, L_j^2 = \binom{n}{j-2} \varkappa_L, \qquad \varkappa_L = \mathbf{E}\, T_{\Omega_{j-2}\cup\{n+1,n+2\}} L_j. \tag{5.8}$$

A straightforward calculation gives

$$\varkappa_L = \sum_{v=0}^{v_0} \binom{j-2}{v} \binom{n-j+2}{v} s_{j,j-v}, \qquad v_0 = \min\{j-2,\, n-j+2\}.$$

Invoking (3.9) and the identity (5.1), we obtain $\varkappa_L = \sigma_j^2 [N-n-2]_{j-2}/[N-j]_{j-2}$. Substituting this expression into (5.8), we obtain an explicit formula for $\mathbf{E}\, L_j^2$. A comparison of this formula and (3.10) yields (5.3).

Let us prove (5.4). By symmetry,

$$\mathbf{E}\, M_j^2 = \binom{n+1}{j-1} \varkappa_M, \qquad \varkappa_M = \mathbf{E}\, T_{\Omega_{j-1}\cup\{n+2\}} M_j. \tag{5.9}$$

A calculation shows

$$\varkappa_M = \sum_{v=0}^{v_0} \binom{n-j+2}{v} \binom{j-1}{v} s_{j,j-v}, \qquad v_0 = \min\{j-1,\, n-j+2\}.$$

Invoking (3.9) and then using (5.1), we find $\varkappa_M = \sigma_j^2 [N-n-2]_{j-1}/[N-j]_{j-1}$. This expression combined with (5.9) leads to an explicit formula for $\mathbf{E}\, M_j^2$. Comparison of this formula and (3.10) yields (5.4).

Let us prove (5.5). By symmetry,

$$\mathbf{E}\, Q_j^2 = \binom{n+1}{j} \varkappa_Q, \qquad \varkappa_Q = \mathbf{E}\, T_{\Omega_k} Q_j. \tag{5.10}$$

A calculation shows

$$\varkappa_Q = \sum_{v=0}^{v_0} \binom{n-j+1}{v} \binom{j}{v} s_{j,j-v}, \qquad v_0 = \min\{j,\, n-j+1\}.$$

Invoking (3.9) and then using (5.1), we find $\varkappa_Q = \sigma_j^2 [N-n-1]_j/[N-j]_j$. This expression combined with (5.10) leads to an explicit formula for $\mathbf{E}\, Q_j^2$. Comparison of this formula and (3.10) yields (5.5).

The identity (5.6) follows from (3.9). We have

$$\mathbf{E}\,\Big(\sum_{k\in\Omega_{n+2}\setminus\{i\}} T_{\{i,k\}}\Big)^2 = (n+1)\sigma_2^2 + (n+1)n s_{2,1} = \frac{(n+1)(N-n-2)}{N-2}\,\sigma_2^2.$$

6. Appendix

Here we give an estimate for the convergence rate in probability of finite population sample means. Strong law of large numbers was established in Rosén [2].

Given $f_\nu : \mathcal{X}_\nu \to \mathbb{R}$, introduce the random variables $Z_{i,\nu} = f_\nu(X_{i,\nu})$, $i = 1, \dots, n_\nu$. Denote $V_\nu(\varepsilon) = \mathbf{E}\,|Z_{1,\nu}|\mathbb{I}_{|Z_{1,\nu}|>\varepsilon\tau_\nu^2}$.

Lemma 6.1. *Assume that there exists an absolute constant $C_3 > 0$ and a sequence $\varepsilon_\nu \downarrow 0$ such that $\mathbf{E}\,|Z_{1,\nu}| \le C_3$ and $V_\nu(\varepsilon_\nu) < \varepsilon_\nu$. Then, for $\nu = 1, 2, \dots,$*

$$\mathbf{P}\{|\sum_{i=1}^{n_\nu}(Z_{i,\nu} - \mathbf{E}\,Z_{1,\nu})| > \tau_\nu^2\psi_\nu\} \le \psi_\nu, \qquad \psi_\nu = 2(C_3+1)\varepsilon_\nu^{1/3} + 2V_\nu(1). \tag{6.1}$$

Proof of Lemma 6.1. Write $\tilde{Z}_{i,\nu} = Z_{i,\nu}\mathbb{I}_{|Z_{i,\nu}|\le\tau_\nu^2}$,

$$Z_\nu = \sum_{i=1}^{n_\nu} Z_{i,\nu}, \qquad Z'_\nu = \sum_{i=n_\nu+1}^{N_\nu} Z_{i,\nu}, \qquad \tilde{Z}_\nu = \sum_{i=1}^{n_\nu}\tilde{Z}_{i,\nu}.$$

We can assume that the number of summands $n_\nu \le N_\nu/2$. Otherwise, using the identity $Z_\nu - \mathbf{E}\,Z_\nu = \mathbf{E}\,Z'_\nu - Z'_\nu$, we turn to the sum Z'_ν having less than $N_\nu/2$ summands. Note that the inequality $n_\nu \le N_\nu/2$ implies $n_\nu \le 2\tau_\nu^2$. In order to prove (6.1) we replace $Z_{i,\nu}$ by $\tilde{Z}_{i,\nu}$ for $i = 1, \dots, n_\nu$ and bound the error of this replacement by

$$\sum_{i=1}^{n_\nu}\mathbf{P}\{|Z_{i,\nu}| > \tau_\nu^2\} = n_\nu\mathbf{P}\{|Z_{1,\nu}| > \tau_\nu^2\} \le \frac{n_\nu}{\tau_\nu^2}V_\nu(1) \le 2V_\nu(1). \tag{6.2}$$

In the next step, we replace $\mathbf{E}\,Z_\nu$ by $\mathbf{E}\,\tilde{Z}_\nu$. We have

$$|\mathbf{E}\,Z_\nu - \mathbf{E}\,\tilde{Z}_\nu| \le n_\nu\mathbf{E}\,|Z_{1,\nu} - \tilde{Z}_{1,\nu}| = n_\nu\mathbf{E}\,|Z_{1,\nu}|\,\mathbb{I}_{|Z_{1,\nu}|>\tau_\nu^2} \le 2\tau_\nu^2V_\nu(1). \tag{6.3}$$

From (6.2) and (6.3) it follows that, given $\delta > 0$,

$$\mathbf{P}\{|Z_\nu - \mathbf{E}\,Z_\nu| > \tau_\nu^2\delta\} \le \mathbf{P}\{|\tilde{Z}_\nu - \mathbf{E}\,\tilde{Z}_\nu| > \tau_\nu^2(\delta - 2V_\nu(1))\} + 2V_\nu(1). \tag{6.4}$$

Choose $\delta = \varkappa + 2V_\nu(1)$ and $\varkappa = \varepsilon_\nu^{1/3}$ and apply the bound

$$\mathbf{P}\{|\tilde{Z}_\nu - \mathbf{E}\,\tilde{Z}_\nu| > \tau_\nu^2\varkappa\} \le \varkappa^{-2}2(C_3+1)\varepsilon_\nu. \tag{6.5}$$

From (6.4) and (6.5) we obtain

$$\mathbf{P}\{|Z_\nu - \mathbf{E}\,Z_\nu| > \tau_\nu^2\delta\} \le \psi_\nu.$$

Now (6.1) follows from the inequality $\psi_\nu \geq \delta$.

It remains to prove (6.5). This inequality follows by Chebyshev's inequality from the bound

$$\mathbf{Var}\,\tilde{Z}_\nu \leq 2\tau_\nu^4(C_3\eta + V_\nu(\eta)), \tag{6.6}$$

where we choose $\eta = \varepsilon_\nu$. Let us prove (6.6). A direct calculation gives

$$\mathbf{Var}\,\tilde{Z}_\nu = \tau_\nu^2 N_\nu (N_\nu - 1)^{-1}\mathbf{Var}\,\tilde{Z}_{1,\nu} \leq 2\tau_\nu^2 \mathbf{Var}\,\tilde{Z}_{1,\nu}. \tag{6.7}$$

Write $\mathbf{Var}\,\tilde{Z}_{1,\nu} \leq \mathbf{E}\,\tilde{Z}_{1,\nu}^2 = W_1 + W_2$, where

$$W_1 = \mathbf{E}\,Z_{1,\nu}^2 \mathbb{I}_{|Z_{1,\nu}| \leq \eta\tau_\nu^2} \leq \eta\tau_\nu^2 \mathbf{E}\,|Z_{1,\nu}| \leq \eta\tau_\nu^2 C_3.$$
$$W_2 = \mathbf{E}\,Z_{1,\nu}^2 \mathbb{I}_{\eta\tau_\nu^2 < |Z_{1,\nu}| \leq \tau_\nu^2} \leq \tau_\nu^2 V_\nu(\eta).$$

Hence, we obtain $\mathbf{Var}\,\tilde{Z}_{1,\nu} \leq \tau_\nu^2(\eta C_3 + V_\nu(\eta))$. Invoking this inequality in (6.7), we obtain (6.6), thus, completing the proof.

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