ONE- AND TWO-TERM EDGEWORTH EXPANSIONS FOR FINITE POPULATION SAMPLE MEAN. EXACT RESULTS. II

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Abstract. We prove the validity of one- and two-term Edgeworth expansions under optimal conditions (a Cramer-type smoothness condition and the minimal moment conditions) and provide precise bounds for the remainders of expansions. The bounds depend explicitly on the ratio $p = N/n$, where N respectively n denotes the sample size respectively the population size.

3. Proofs

Here we prove Theorem 2 formulated in Sec. 1 of the first part of the paper.

Proof of Theorem 2. Given $j \in \{1, 2\}$, we shall show that

$$\Delta_{j,n} = o(\tau_n^{-j}) \qquad \text{as} \qquad \tau_n \to \infty. \tag{3.49}$$

The bounds $\Delta^{\star}_{j,n} = o(\tau_n^{-j})$ are derived from (3.49) using the same argument as that of the proof of Corollary 1.

Let us prove (3.49). We can assume without loss of generality that

$$\operatorname{Var} X_{1,n} = \mathbf{E}X_{1,n}^2 = \tau_n^{-2}.$$

Then $\Delta_{j,n} = \|\mathbf{P}\{S_n \le x\} - G_{j,n}(x)\|$, where $S_n = X_{1,n} + \cdots + X_{N_n,n}$. Furthermore, we can assume that the sum S_n consists of no more than $n/2$ summands, i.e., that $p_n \le q_n$. Indeed, for $p_n > q_n$, we have $S_n = \tilde{S}_n$, since $\mathbf{E}X_{1,n} = 0$. Here

$$\tilde{S}_n = \tilde{X}_{1,n} + \cdots + \tilde{X}_{\tilde{N}_n,n} \quad \text{with} \quad \tilde{X}_{k,n} = -X_{n-k+1,n} \quad \text{and} \quad \tilde{N}_n = n - N_n.$$

In this way, we represent S_n by the sum $\tilde{S}_n$ consisting of less than $n/2$ summands. It is easy to verify that one- and two-term Edgeworth expansions of $P\{\tilde{S}_n \le x\}$ (written in terms of the moments of $\tilde{X}_{1,n}$ and with the parameter $\tilde{p}_n(= q_n)$) coincide with $G_{1,n}$ and $G_{2,n}$, respectively.

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In order to prove (3.49), we shall show that, for every $\varepsilon > 0$,

$$\limsup_n \tau_n^j \Delta_{j,n} < \varepsilon, \qquad j = 1, 2. \tag{3.50}$$

We shall prove (3.50) for $j = 2$ only. The proof for $j = 1$ is almost the same.

Assume that (1.7), (1.8), and (1.9) hold for $j = 2$. Since $p_n \le q_n$, we have $N_n \le \tau_n^2 q_n^{-1} \le 2\tau_n^2$.

In the first step, we replace S_n by S' defined by

$$S' = X_1' + \cdots + X_{N_n}', \qquad X_k' = X_{k,n}\mathbb{I}\{X_{k,n}^2 \le 1\}.$$

Note that, by Chebyshev's inequality and (1.8),

$$\mathbf{P}\{X_1 \ne X_1'\} \le \mathbf{P}\{X_{1,n}^2 > 1\} \le \tau_n^{-4}\beta_{4,n}^\star(1) = o(\tau_n^{-4}) \qquad \text{as} \qquad \tau_n \to \infty.$$

Therefore,

$$\mathbf{P}\{S \ne S'\} \le \sum_{k=1}^{N_n} \mathbf{P}\{X_k \ne X_k'\} \le N_n \mathbf{P}\{X_1 \ne X_1'\} = 2\tau_n^2 \mathbf{P}\{X_1 \ne X_1'\} = o(\tau_n^{-2}),$$

and we obtain

$$\|\mathbf{P}\{S \le x\} - \mathbf{P}\{S' \le x\}\| \le \mathbf{P}\{S \ne S'\} = o(\tau_n^{-2}). \tag{3.51}$$

In the next step, we replace S' by $S'' = S' - \mathbf{E}S'$. Since $\mathbf{E}S' = N_n \mathbf{E}X_1'$ and, by $\mathbf{E}X_{1,n} = 0$,

$$|\mathbf{E}X_1'| = |\mathbf{E}X_{1,n}\mathbb{I}\{X_{1,n}^2 > 1\}| \le \tau_n^{-4}\beta_{4,n}^\star(1) = o(\tau_n^{-4}), \tag{3.52}$$

we have $|\mathbf{E}S'| = o(\tau_n^{-2})$. Therefore,

$$\begin{aligned} \|P\{S' \le x\} - G_{2,n}(x)\| \le& \|P\{S'' \le x\} - G_{2,n}(x)\| + |\mathbf{E}S'| \max_x |G_{2,n}^{(1)}(x)| \\ =& \|P\{S'' \le x\} - G_{2,n}(x)\| + o(\tau_n^{-2}). \qquad (3.53) \end{aligned}$$

In the last step, we used the relation

$$\limsup_n \|G_{2,n}^{(1)}(x)\| \ll \limsup_n \beta_{4,n} < \infty,$$

which follows from (3.10) and (1.7).

Now we replace S'' by S_Y defined by

$$S_Y = Y_1 + \cdots + Y_{N_n}, \qquad Y_k = \sigma_*^{-1}\tau_n^{-1}(X'_k - \mathbf{E}X'_k), \qquad \sigma_*^2 = \operatorname{Var} X'_1.$$

Clearly, $S'' = S_Y\sigma_*\tau_n$ and, thus,

$$\mathbf{P}\{S'' \le x\} = \mathbf{P}\{S_Y \le x + \varkappa x\}, \qquad \varkappa = \sigma_*^{-1}\tau_n^{-1} - 1.$$

Therefore, we can write

$$\begin{aligned} &\|\mathbf{P}\{S'' \le x\} - G_{2,n}(x)\| \le I_1 + I_2 + I_3, \quad I_1 = \|G_Y(x) - G_{2,n}(x)\|, \\ &I_2 = \|\mathbf{P}\{S_Y \le x\} - G_Y(x)\|, \qquad I_3 = \|G_{2,n}(x + \varkappa x) - G_{2,n}(x)\|. \end{aligned} \tag{3.54}$$

Here G_Y is defined in the same way as G_2 but with α_i replaced by $\alpha'_i = \tau_n^i\mathbf{E}Y_1^i$.

In the remaining part of the proof, we show that

$$I_1 = o(\tau_n^{-2}), \qquad\qquad I_3 = o(\tau_n^{-2}), \tag{3.55}$$

and that there exists $c > 0$ such that, for every $0 < \varepsilon < 1$, we have

$$\limsup_n I_2\tau_n^2 \le c\varepsilon. \tag{3.56}$$

Note that (3.55), (3.56), and (3.54), in combination with (3.51) and (3.53), yield (3.50) for $j = 2$. Therefore, it remains to prove (3.55) and (3.56).

Write $\beta'_j = \tau_n^j\mathbf{E}|Y_1|^j$, $j = 1, 2, \ldots$. In the first step of the proof of (3.55) and (3.56), we show that

$$\alpha'_j = \alpha_{j,n} + o(\tau_n^{j-4}), \qquad j = 3, 4, \tag{3.57}$$

$$\beta'_j \le 2^{j-1}\beta_{j,n} + o(\tau_n^{-2}), \qquad j = 3, 4, \tag{3.58}$$

$$\beta'_5 \le 2^4\varepsilon\tau_n\beta_{4,n} + 2^4\tau_n\beta^\star_{4,n}(\varepsilon) + o(\tau_n^{-1}), \qquad \forall \quad 0 < \varepsilon < 1. \tag{3.59}$$

Note that (1.8) implies

$$\sigma_*^2 = \tau_n^{-2} + o(\tau_n^{-4}), \qquad \sigma_*^{-j} - \tau_n^j = o(\tau_n^{j-2}), \qquad j = 1, 2, \ldots. \tag{3.60}$$

To prove the first relation of (3.60), note that, by (3.52),

$$\tau_n^{-2} = \mathbf{E}X_{1,n}^2 \ge \sigma_*^2 = \mathbf{E}(X'_1)^2 - (\mathbf{E}X'_1)^2 = \mathbf{E}(X'_1)^2 + o(\tau_n^{-8}),$$

and, by Chebyshev's inequality and (1.8),

$$\mathbf{E}(X'_1)^2 = \mathbf{E}X_{1,n}^2 - \mathbf{E}X_{1,n}^2\mathbb{I}\{X_{1,n}^2 > 1\} \ge \mathbf{E}X_{1,n}^2 - \tau_n^{-4}\beta^\star_{4,n}(1) = \tau_n^{-2} + o(\tau_n^{-4}).$$

The second relation of (3.60) follows from the first one.
In order to prove (3.57), we write $\alpha'_j = \alpha_{j,n} + R_1 + R_2 + R_3$, where

$$R_1 = \tau_n^j \mathbf{E} Y_1^j - \sigma_*^{-j} \mathbf{E}(X'_1)^j, \quad R_2 = (\sigma_*^{-j} - \tau_n^j)\mathbf{E}(X'_1)^j, \quad R_3 = \tau_n^j \mathbf{E}\big((X'_1)^j - X_{1,n}^j\big),$$

and show that

$$R_1 = o(\tau_n^{-3}), \qquad R_2 = o(\tau_n^{-2}), \qquad R_3 = o(\tau_n^{j-4}). \tag{3.61}$$

Note that the relation (which follows from (1.7))

$$|\mathbf{E}(X'_1)^j| \le \mathbf{E}|X'_1|^j \le \mathbf{E}|X_{1,n}|^j = \tau_n^{-j}\beta_{j,n} = O(\tau_n^{-j}), \qquad j = 1,2,3,4, \tag{3.62}$$

in combination with (3.60), yields $R_2 = o(\tau_n^{-2})$.
The bound $R_3 = o(\tau_n^{j-4})$ follows from (1.8) by Chebyshev's inequality:

$$|R_3| = \tau_n^j |\mathbf{E} X_{1,n}^j \mathbb{I}\{X_{1,n}^2 > 1\}| \le \tau_n^{j-4}\beta^\star_{4,n}(1) = o(\tau_n^{j-4}).$$

To prove $R_1 = o(\tau_n^{-3})$, write $R_1 = \sigma_*^{-j}\mathbf{E}V$, where

$$V = (X'_1 - \mathbf{E}X'_1)^j - (X'_1)^j = (-\mathbf{E}X'_1)\, j\, (X'_1 - \theta^* \mathbf{E}X'_1)^{j-1}$$

for some $0 \le \theta^* \le 1$, by the mean value theorem. Furthermore, invoking the inequality

$$(a+b)^k \le 2^{k-1}(a^k + b^k), \qquad a, b > 0, \quad k = 1, 2, \dots, \tag{3.63}$$

and using (3.52) and (3.62), we obtain

$$\mathbf{E}|V| \ll |\mathbf{E}X'_1|\big(\mathbf{E}|X'_1|^{j-1} + |\mathbf{E}X'_1|^{j-1}\big) = o(\tau_n^{-4})\big(O(\tau_n^{1-j}) + o(\tau_n^{4(j-1)})\big) = o(\tau_n^{-3-j}).$$

This relation together with (3.60) implies

$$R_1 = \sigma_*^{-j}\mathbf{E}V = \big(\tau_n^j + o(\tau_n^{j-2})\big)\mathbf{E}V = o(\tau_n^{-3}),$$

thus completing the proof of (3.61). We arrive at (3.57).
Let us prove (3.58). Note that (3.60) implies

$$\sigma_*^{-1}\tau_n^{-1} = 1 + o(\tau_n^{-2}). \tag{3.64}$$

Combining (3.63) with (3.64), (3.62), and (3.52), we obtain, for $j = 3, 4, 5$,

$$\mathbf{E}|Y_1|^j \le 2^{j-1}\sigma_*^{-j}\tau_n^{-j}\big(\mathbf{E}|X'_1|^j + |\mathbf{E}X'_1|^j\big) \le 2^{j-1}\tau_n^{-j}\beta_{j,n} + o(\tau_n^{-j-2}). \tag{3.65}$$

Since $\beta'_j = \tau_n^j \mathbf{E}|Y_1|^j$, from (3.65) we obtain (3.58).

In order to prove (3.59), we combine the first inequality of (3.65) with (3.64) and (3.52) and obtain

$$\beta'_5 = \tau_n^5 \mathbf{E}|Y_1|^5 \le 2^4(1 + o(\tau_n^{-2}))\tau_n^5 \mathbf{E}|X'_1|^5 + o(\tau_n^{-15}).$$

Finally, invoking the inequality (which holds for an arbitrary $0 < \varepsilon < 1$)

$$\begin{aligned}\mathbf{E}|X'_1|^5 = \mathbf{E}|X_{1,n}|^5 \mathbb{I}\{X_{1,n}^2 \le 1\} &\le \varepsilon \mathbf{E} X_{1,n}^4 + \mathbf{E} X_{1,n}^4 \mathbb{I}\{X_{1,n}^2 > \varepsilon\} \\ &\le \varepsilon \tau_n^{-4} \beta_{4,n} + \tau_n^{-4} \beta^\star_{4,n}(\varepsilon),\end{aligned}$$

we obtain

$$\beta'_5 \le 2^4 \tau_n (1 + o(\tau_n^{-2}))\big(\varepsilon \beta_{4,n} + \beta^\star_{4,n}(\varepsilon)\big) + o(\tau_n^{-15}).$$

This inequality together with (1.7) implies (3.59).

Now we are going to prove (3.55). The bound $I_1 = o(\tau_n^{-2})$ is an immediate consequence of (3.57). To prove the second bound of (3.55), note that, by the mean value theorem and the exponential decay of $G_{2,n}^{(1)}(x)$ as $|x| \to +\infty$, we have, for $|\varkappa| \le 1$,

$$I_3 \ll |\varkappa| \max_x |G_{2,n}^{(1)}(x)|(1 + |x|). \tag{3.67}$$

It is easy to show (see (3.10)) that $\||G_{2,n}^{(1)}(x)|(1 + |x|)\| \ll \beta_{4,n}$. This inequality, in combination with (3.67) and (3.64), yields $I_3 \ll o(\tau_n^{-2})\beta_{4,n} = o(\tau_n^{-2})$ by (1.7).

In order to prove (3.56), we fix $\varepsilon \in (0, 1)$ and apply (3.5) with $T = T_n = \tau_n^2 \varepsilon^{-1}$ and $H = H_n = b_1 \tau_n / \beta'_3$. Therefore, it suffices to show that

$$\limsup_n \tau_n^2 T_n^{-1} \|G_Y^{(1)}(x)\| \le c\,\varepsilon \tag{3.68}$$

$$\limsup_n \tau_n^2 I_{[0,H_n]}(\hat{F}_Y - \hat{G}_Y) \le c\,\varepsilon, \tag{3.69}$$

$$I_{[H_n,T_n]}(\hat{G}_Y) = o(\tau_n^{-2}), \tag{3.70}$$

$$I_{[H_n,T_n]}(\hat{F}_Y) = o(\tau_n^{-2}). \tag{3.71}$$

Here $\hat{G}_Y$ and $\hat{F}_Y$ denote the Fourier–Stieltjes transforms of $G_Y(x)$ and $F_Y(x) = \mathbf{P}\{S_Y \le x\}$, respectively.

Let us prove (3.68) – (3.71). Note that (3.69) is a consequence of (3.8) combined with (3.59) and (1.7), (1.8).

The relation (3.68) follows from the inequalities

$$\limsup_n \|G_Y^{(1)}(x)\| \ll \limsup_n \beta'_4 \ll \limsup_n \beta_{4,n} < \infty,$$

using (3.10), (3.58), and (1.7).

Let us prove (3.70). It follows from (3.11) and (3.12) that

$$I_{[H_n,T_n]}(\hat{G}_Y) \ll H_n^{-3}\big(1 + (\beta_3')^2 + \beta_4'\big) = O(\tau_n^{-3}).$$

In the last step, we used the inequality $\limsup_n \beta_j' < \infty$ for $j = 3, 4$, which follows from (3.58) and (1.7).

To prove (3.71) we proceed as in proof of (3.9) and obtain

$$\begin{aligned} &I_{[H_n,T_n]}(\hat{F}_Y) \le \tau_n \ln T_n \, \exp\{-\tau_n^2 \rho_n'\}, \\ &\rho_n' = 1 - \sup\big\{|\mathbf{E} \exp\{it\tau_n Y_1\}| : \, b_1/\beta_3' \le |t| \le \tau_n \varepsilon^{-1}\big\} \end{aligned} \tag{3.72}$$

Note that (3.72) and

$$\liminf_n \rho_n' > 0 \tag{3.73}$$

imply (3.71). It remains to show (3.73). For this purpose, we replace Y_1 by $X_{1,n}$ and then apply (1.9). Note that, for every t,

$$\begin{aligned} &|\mathbf{E} \exp\{it\tau_n Y_1\}| = |\mathbf{E} \exp\{it\sigma_*^{-1} X_1'\}| \le |\mathbf{E} \exp\{it\tau_n X_{1,n}\}| + R_n(t), \\ &R_n(t) = \Big|\mathbf{E}\big(\exp\{it\sigma_*^{-1} X_1'\} - \exp\{it\tau_n X_{1,n}\}\big)\Big| \le |t| \mathbf{E}|\sigma_*^{-1} X_1' - \tau_n X_{1,n}|. \end{aligned}$$

Therefore, with ρ_n'' defined in the same way as ρ_n', but with Y_1 replaced by $X_{1,n}$ in the exponent, we have

$$\liminf_n \rho_n' \ge \liminf_n \rho_n'' - \limsup_n R_n, \quad R_n = \sup_{|t| \le \tau_n/\varepsilon} |R_n(t)|. \tag{3.74}$$

We claim that (1.9) implies

$$\liminf_n \rho_n'' > 0. \tag{3.75}$$

Indeed, (3.75) follows from (1.9) and the fact that $b_1 \ge \liminf_n b_1/\beta_3' > 0$, where the last inequality is a consequence of $\limsup_n \beta_3' \ll \limsup_n \beta_{3,n} < \infty$ by (3.58) and (1.7).

Finally, we shall show that

$$\limsup_n R_n = 0. \tag{3.76}$$

Clearly, (3.76) together with (3.75) yield (3.73) via (3.74). In order to prove (3.76), we write

$$\mathbf{E}|\sigma_*^{-1} X_1' - \tau_n X_{1,n}| \le \sigma_*^{-1} \mathbf{E}|X_1' - X_{1,n}| + |\sigma_*^{-1} - \tau_n| \mathbf{E}|X_{1,n}|. \tag{3.77}$$

To estimate the first summand, we apply Chebyshev's inequality and use (3.60) and (1.8) to get

$$\sigma_*^{-1}\mathbf{E}|X_1' - X_{1,n}| = \sigma_*^{-1}\mathbf{E}|X_{1,n}|\mathbb{I}\{X_{1,n}^2 > 1\} \le \sigma_*^{-1}\tau_n^{-4}\beta_{4,n}^\star(1) = o(\tau_n^{-3}). \quad (3.78)$$

Furthermore, by (3.60) and (1.7), we have

$$|\sigma_*^{-1} - \tau_n|\mathbf{E}|X_{1,n}| = |\sigma_*^{-1} - \tau_n|\tau_n^{-1}\beta_{1,n} = o(\tau_n^{-2}). \quad (3.79)$$

Collecting (3.78) and (3.79) in (3.77), we obtain $|R_n(t)| \le |t|o(\tau_n^{-2})$, thus, proving (3.76). The proof of Theorem 2 is complete.

4. Appendix I

Introduce the function $\Theta(x) = \Big(\,\frac{2}{\pi}\;\frac{\pi - x}{\pi + x}\,\Big)^2$ and write

$$c_2 = 2\,\frac{b_1}{b}\, + c_1, \qquad c_1 = \frac{3}{2}\,(1 - \Theta(b)) + \frac{2b_1}{b}\,\Theta(b), \qquad b = 0.075. \quad (4.1)$$

Note that the constants $b_1 < b$ (see also (1.2)) are chosen so that $c_1 < 1$ and $c_2 < 1$.

Given a number $L > 0$ and a function $f(s,t)$, we write $f \prec L$ if

$$\tau\int_{|t|\le H}\frac{dt}{|t|}\int_{|s|\le\pi}|f(s,t)|e^{-\zeta_0/2}ds \ll L.$$

Recall that $\zeta_0 = \tau^2 s^2 + t^2$.

Proof of (3.19). In order to prove (3.19), we shall show that

$$R_i \prec \mathcal{R}, \qquad i = 1,2,3, \qquad \mathcal{R} = \beta_5\tau^{-3}. \quad (4.2)$$

Let us prove (4.2) for $i = 1$. By the inequality (5.2) below, we have

$$|\hat{v}_{3,j}^*| \ll pq(1 + |t|^5)\exp\{\,\frac{b_1}{b}\,t^2\}|tx_j|\xi_j^4.$$

This bound, in combination with the inequality (5.4) below, gives

$$|R_1| \ll pq(1 + |t|^5)\exp\{\,\frac{c_2}{2}\,\zeta_0\}\sum_{j=1}^n |tx_j|\xi_j^4. \quad (4.3)$$

The inequalities $\xi_j^4 \ll s^4 + t^4x_j^4$ and $s^4|x_j| \ll |s|^5 + |x_j|^5$ imply

$$\sum_{j=1}^n |tx_j|\xi_j^4 \ll |t|\sum_{j=1}^n(|s|^5 + (1 + t^4)|x_j|^5) = |t|n\big(|s|^5 + (1 + t^4)\kappa_5\big).$$

Combining this bound with (4.3), we obtain

$$|R_1| \ll \tau^2 |t|(1+|t|^5)\big(|s|^5 + (1+t^4)\kappa_5\big) \exp\{\tfrac{c_2}{2}\,\zeta_0\}.$$

This inequality, in combination with $c_2 < 1$, implies $R_1 \prec \tau^{-3} + \tau^2\kappa_5 \ll \mathcal{R}$.

Let us prove (4.2) for $i = 3$. Since $|\gamma^{(3)}(0)| \le pq$ by Lemma 5.1, we have $|\hat{v}_{1,j}| \le pq|tx_j|\xi_j^2$. Furthermore, by (5.2) we have

$$|\hat{v}^*_{1,j}| \le pq|tx_j|\xi_j^2(1+|t|^3)\exp\{\tfrac{b_1}{b}\,t^2\}.$$

From these inequalities, (5.4), and the simple inequality $|x_j|\xi_j^2 \ll w_j := |x_j|(s^2 + t^2x_j^2)$, it follows that

$$|R_3| \le p^3q^3|t|^3(1+|t|^3)\exp\{\tfrac{c_2}{2}\,\zeta_0\}\sum_{j=3}^{n}\sum_{k=2}^{j-1}\sum_{r=1}^{k-1} w_jw_kw_r. \tag{4.4}$$

Note that the (triple) sum equals the expectation

$$\binom{n}{3}\mathbf{E}W_1W_2W_3, \qquad \text{where} \qquad W_i = |X_i|(s^2+t^2X_i^2), \quad i = 1,2,3.$$

To bound this expectation, note that

$$\mathbf{E}(W_3|X_1X_2) \le \frac{n}{n-2}\,\mathbf{E}W_3, \qquad \mathbf{E}(W_2|X_1) \le \frac{n}{n-1}\,\mathbf{E}W_2. \tag{4.5}$$

Furthermore, by the inequality $|ad^2| \ll |a|^3 + |d|^3$, we have

$$\mathbf{E}W_1 \ll \mathbf{E}\big(|s|^3 + |X_1|^3 + t^2|X_1|^3\big) = |s|^3 + (1+t^2)\kappa_3. \tag{4.6}$$

From (4.5) and (4.6) it follows that

$$\binom{n}{3}\mathbf{E}W_1W_2W_3 \ll n^3(\mathbf{E}W_1)^3 \ll n^3\big(|s|^3 + (1+t^2)\kappa_3\big)^3 \ll n^3\big(|s|^9 + (1+t^2)^3\kappa_3^3\big).$$

Combining these inequalities with (4.4), we obtain

$$|R_3| \le \tau^6|t|^3(1+|t|^3)\exp\{\tfrac{c_2}{2}\,\zeta_0\}\big(|s|^9 + (1+t^2)^3\kappa_3^3\big).$$

Now a simple calculation shows that $R_3 \prec \tau^{-3} + \tau^6\kappa_3^3 \ll \mathcal{R}$. In the last step, we estimated $\kappa_3^3 \le \beta_5\tau^{-9}$ using (3.2).

The proof of (4.2) for $i = 2$ is much the same.

Proof of (3.34). In order to prove (3.34), we shall show that

$$\gamma^{n-1}(s)R_2^\star \prec \mathcal{R}, \qquad \gamma^{n-2}(s)R_3^\star \prec \mathcal{R}, \qquad \mathcal{R} = \beta_5\tau^{-3}. \tag{4.7}$$

Let us prove the first inequality of (4.7). By the inequality (5.13) below, we have

$$\gamma^{n-1}(s)R_2^\star = R_2^\star + r \qquad \text{with} \qquad |r| \ll |R_2^\star| r_1^*.$$

Invoking the inequality $|\gamma^{(4)}(0)| \ll pq$ of Lemma 5.1, we obtain

$$|r| \ll \tau^4 s^4 |t|^3 \kappa_3 \exp\{\tfrac{c_1}{2}\,\zeta_0\}.$$

Now a straightforward calculation gives $|r| \prec \kappa_3 \ll \mathcal{R}$. The proof of $R_2^\star \prec \mathcal{R}$ is much the same as that of $S_{1,1}^\star \sim 0$ in (3.41) above. We obtain (4.7) for $k = 2$.

In order to prove the second inequality of (4.7), we show that

$$n^2\delta R_{3,j}^\star \gamma^{n-2}(s) \prec \mathcal{R}, \qquad j = 1, 2, 3, \qquad \text{where} \quad \delta := \Big(\frac{\gamma^{(3)}(0)}{2}\Big)^2 \frac{\binom{n}{2}}{n^2}, \tag{4.8}$$

satisfies $\delta \ll p^2q^2$. To prove (4.8) for $j = 1$, we apply (5.13) and obtain

$$n^2\delta R_{3,1}^\star \gamma^{n-2}(s) = n^2\delta R_{3,1}^\star + r \qquad \text{with} \quad |r| \ll n^2\delta |R_{3,1}^\star| r_1^*.$$

A simple calculation gives

$$|r| \ll \tau^6 s^4 |t|^5 |\mathbf{E}X_1^3X_2^2| \exp\{\tfrac{c_1}{2}\,\zeta_0\} \prec \tau^2 |\mathbf{E}X_1^3X_2^2| \ll \mathcal{R}.$$

In the last step, we estimated $|\mathbf{E}X_1^3X_2^2| \ll \kappa_5$ using Hölder's inequality.

To show $n^2\delta R_{3,1}^\star \prec \mathcal{R}$, we proceed as in the proof of $S_{1,1}^\star \sim 0$ in (3.41) above. The proof of (4.8) for $j = 1$ is complete.

Let us prove (4.8) for $j = 2, 3$. By Lemma 5.6, we have

$$R_{3,2}^\star = -s^4t^2\,\frac{\varkappa_2}{n-1} - 2s^3t^3\,\frac{\varkappa_3}{n-1} - \frac{2}{3}\,s^2t^4\,\frac{\varkappa_4}{n-1}\,. \tag{4.9}$$

By (3.1), $\varkappa^6 \le (pq)^{-1/2}\kappa_5$ and, therefore,

$$|R_{3,3}^\star| \ll s^2t^4\,\frac{\kappa_4}{n-1} + \frac{t^6}{9}\,\frac{\kappa_5}{n^{1/2}\tau}\,. \tag{4.10}$$

Finally, we estimate $n^2\delta\gamma^{n-2}(s) \ll \tau^4\exp\{c_1\zeta_0/2\}$ using (5.5), and apply (4.9) and (4.10) to obtain (4.8) for $j = 2, 3$.

Proof of (3.38). By (5.13) and (5.5) we have, for $j = 1, 2$,

$$|\gamma^{n-j}(s) - \gamma^n(s)| = |\gamma^{n-j}(s)||1 - \gamma^j(s)| \ll pq|s|^3 \exp\{\tfrac{c_1}{2}\,\zeta_0\}.$$

This implies

$$|\gamma^{n-1}(s) - \gamma^n(s)||S_j^\star| \ll pq|s|^3 \exp\{\tfrac{c_1}{2}\,\zeta_0\}|S_j^\star|, \qquad j = 1, 2,$$
$$|\gamma^{n-2}(s) - \gamma^n(s)||S_3^\star| \ll pq|s|^3 \exp\{\tfrac{c_1}{2}\,\zeta_0\}|S_3^\star|.$$

A simple calculation shows that the right hand sides are $\prec \mathcal{R}$. We obtain (3.38).

Proof of (3.40) *and* (3.42). By (5.14) we have

$$\gamma^n(s)S_{1,1}^\star = S_{1,1}^\star + S_{1,1}^* + r, \qquad |r| \ll |S_{1,1}^\star|r_2^*. \tag{4.11}$$

Furthermore, by (5.13) we have

$$|S_{j,1}^\star - \gamma^n(s)S_{j,1}^\star| \ll r_1^*|S_{j,1}^\star|, \qquad j = 2, 3. \tag{4.12}$$
$$|g_{j,1} - \gamma^n(s)g_{j,1}| \ll r_1^*|g_{j,1}|, \qquad j = 1, 2, 3. \tag{4.13}$$

A simple calculation shows that

$$|S_{1,1}^\star|r_2^* \prec \mathcal{R}, \qquad |S_{j,1}^\star|r_1^* \prec \mathcal{R}, \qquad |g_{k,1}|r_1^* \prec \mathcal{R}, \qquad j = 1, 2, \quad k = 1, 2, 3.$$

These bounds, in combination with (4.11), (4.12), and (4.13), yield (3.40) and (3.42).

5. Appendix II

Recall that

$$b_1 = 0.001, \qquad b = 0.075, \qquad \Theta(x) = \Big(\frac{2}{\pi}\ \frac{\pi - x}{\pi + x}\Big)^2, \qquad \zeta_0 = \tau^2 s^2 + t^2,$$
$$\beta(x) = p\exp\{iqx\} + q\exp\{-ipx\}, \qquad \gamma(x) = \beta(x)\exp\{\tfrac{pq}{2}\,x^2\}.$$

Denote

$$\mathcal{K} = \{x_k : |Hx_k| > b\}, \qquad H = b_1\tau\beta_3^{-1}.$$

Below we use the following inequality proved in Höglund [10]. For all $z_0 \in [0, \pi)$ and z satisfying $|z| \le \pi + z_0$, we have

$$|\beta(z)|^2 \le 1 - pq(z)^2\Theta(z_0). \tag{5.1}$$

Lemma 5.1. *The function γ satisfies*

$$\gamma(0) = 1, \quad \gamma'(0) = \gamma''(0) = 0, \quad \gamma^{(3)}(0) = i^3 pq(q-p), \quad \gamma^{(4)}(0) = pq(1-6pq),$$
$$|\gamma^{(k)}(u)| \ll \alpha(u^2)(pq + |u|^k p^k q^k), \qquad k = 3,4,5.$$

Proof of Lemma 5.1. The proof is elementary.

Lemma 5.2. *Assume that $|t| \le H$. For all $0 \le \theta_1^*, \theta_2^* \le 1$ and $j \in \Omega_n$, we have*

$$|\gamma^{(k)}(\theta_1^*(s + \theta_2^* t x_j))| \ll \exp\{\, \frac{b_1}{b}\, t^2\}(pq + |t|^k (pq)^{k/2}), \qquad k = 3,4,5. \tag{5.2}$$

Proof of Lemma 5.2. We prove the lemma only in the case where $k = 5$. Write, for short,

$$\xi := \theta_1^*(s + \theta_2^* t x_k), \qquad \alpha(x) := \exp\{pqx/2\}$$

and note that $\gamma(x) = \beta(x)\alpha(x^2)$.

By the inequality $|a+b|^i \le 2^{i-1}(|a|^i + |b|^i)$, we have

$$|\xi|^i \ll 2^{i-1}(|s|^i + |t|^i |x_k|^5),$$

Since $|s| \le \pi$ and, by (3.1), $|x_k| \le (pq)^{-1/2}$, we obtain

$$|\xi|^5 \ll 1 + |t|^5 (pq)^{-5/2}, \qquad |\xi|^2 \le 2\pi^2 + 2t^2 x_k^2.$$

The second inequality implies $\alpha(\xi^2) \ll \alpha(2t^2 x_k^2)$. Therefore, by Lemma 5.1, we have

$$|\gamma^{(5)}(\xi)| \ll \alpha(2t^2 x_k^2)\big(pq + |t|^5 (pq)^{5/2}\big).$$

It remains to show that

$$\alpha(2t^2 x_k^2) \ll \exp\{t^2 b_1/b\}. \tag{5.3}$$

For $x_k \notin \mathcal{K}$, we have $|t x_k| \le b$. Therefore, $\alpha(2t^2 x_k^2) \ll 1$, and (5.3) is satisfied. For $x_k \notin \mathcal{K}$, we have $x_k^2 \le b_1 (pqb)^{-1}$ by Lemma 5.4. Therefore, $\alpha(2t^2 x_k^2) \le \exp\{t^2 b_1/b\}$. We obtain (5.3), thus, proving the lemma.

Lemma 5.3. *Assume that $|t| \le H$. For every $i = 1,2,3$ and $j \in \Omega_n$, we have*

$$|\Psi_{i,j}| \le \exp\{\, \frac{c_1}{2}\, \zeta_0\}, \quad \text{where} \quad \Psi_{i,j} = \gamma^{n-j-i+1}(s) \prod_{k=1}^{j-1} \gamma(s + t x_k), \tag{5.4}$$

$$\gamma^j(s) \le \exp\{\, \frac{jpq}{2}\,(1 - \Theta(b))s^2\} \le \exp\{\, \frac{c_1}{2}\, \zeta_0\}, \tag{5.5}$$

where c_1 is given in (4.1).

Proof of Lemma 5.3. Write

$$\mathbb{I}_k = \mathbb{I}\{x_k \in \mathcal{K}\}, \qquad \mathbb{I}_k^c = \mathbb{I}\{x_k \notin \mathcal{K}\}.$$

Recall that the subset $\mathcal{K} \subset \mathcal{X}$ is introduced in the beginning of Appendix II. Let us show that

$$\prod_{k=1}^{j-1} \gamma(s + tx_k) \leq \exp\{\tfrac{pq}{2}\left(Z_1(1 - \Theta(b)) + Z_2\Theta(b)\right\}, \tag{5.6}$$

$$Z_1 = \sum_{k=1}^{j-1}(s + tx_k)^2, \quad Z_2 = \sum_{k=1}^{j-1}(s + tx_k)^2\mathbb{I}_k.$$

Clearly, $|s + tx_k| \leq \pi + H|x_k| \leq \pi + b$ for $x_k \notin \mathcal{K}$. An application of (5.1) gives

$$|\beta(s+tx_k)|^2 \leq 1 - pq(s+tx_k)^2\Theta(b) \leq \exp\{-pq(s+tx_k)^2\Theta(b)\}, \qquad x_k \notin \mathcal{K}. \tag{5.7}$$

Therefore, for arbitrary $k \in \Omega_n$, we can write

$$|\beta(s + tx_k)| \leq \exp\{-\tfrac{pq}{2}(s + tx_k)^2\Theta(b)\mathbb{I}_k^c\}.$$

This inequality implies (5.6).
Using (5.1) we construct a bound for $\gamma^{n-j}(s)$ as well. As in (5.7), we can write

$$|\beta(s)| \leq \exp\{-2^{-1}pqs^2\Theta(b)\}.$$

This inequality implies (5.5). It follows from (5.6) and (5.5) that

$$|\Psi_{1,j}| \leq \exp\{\tfrac{pq}{2}Z\}, \qquad Z = (s^2(n-j) + Z_1)(1 - \Theta(b)) + Z_2\Theta(b),$$

Clearly, this bound extends to $\Psi_{i,j}$ with $i = 2, 3$ as well. In order to prove (5.4), it remains to show that

$$Z \leq c_1(ns^2 + (pq)^{-1}t^2). \tag{5.8}$$

Höglund [10] showed that $\mathbf{E}X_1 = 0$ and $\mathbf{E}X_1^2 = \tau^{-2}$ imply

$$2\left|\sum_{k=1}^{j} x_k\right| \leq \left(\frac{n}{pq}\right)^{1/2}, \qquad j \in \Omega_n.$$

This inequality, combined with (3.1), gives

$$(n-j)s^2+Z_1=(n-1)s^2+2st\sum_{k=1}^{j-1}x_k+t^2\sum_{k=1}^{j-1}x_k^2\le(n-1)s^2+|st|\Big(\frac{n}{pq}\Big)^{1/2}+\frac{t^2}{pq}\,.$$

Finally, invoking the inequality $|ab|\le(a^2+b^2)/2$, we obtain

$$(n-j)s^2+Z_1\le\frac{3}{2}\,\big(ns^2+\frac{t^2}{pq}\,\big) \tag{5.9}$$

To estimate Z_2, we invoke the bounds

$$|\mathcal{K}|\le n\,\frac{b_1}{b}\qquad\text{and}\qquad\sum_{x_k\in\mathcal{K}}x_k^2\le\frac{b_1}{pqb} \tag{5.10}$$

that follow from Lemma 5.4. The inequality $(a+b)^2\le 2a^2+2b^2$, combined with (5.10), gives

$$Z_2\le 2s^2|\mathcal{K}|+2t^2\sum_{x_k\in\mathcal{K}}x_k^2\le 2\,\frac{b_1}{b}\,\big(ns^2+\frac{t^2}{pq}\,\big).$$

This bound, in combination with (5.9), yields (5.8), thus, completing the proof.

Lemma 5.4. *For subset $\mathcal{K}\subset\mathcal{X}$, we have*

$$|\mathcal{K}|\le n\,\frac{b_1^r}{b^r}\,\frac{\beta_r}{\beta_3^r}\,,\qquad\sum_{x_j\in\mathcal{K}}x_j^2\le\frac{b_1}{b}\,\frac{1}{pq}\,. \tag{5.11}$$

In particular, every $x_k\in\mathcal{K}$ satisfies $x_k^2\le b_1(pqb)^{-1}$. Recall that the subset $\mathcal{K}\subset\mathcal{X}$ is introduced in the beginning of Appendix II.

Proof of Lemma 5.4. To prove the first inequality, we write

$$|\mathcal{K}|\le\sum_{x_j\in\mathcal{K}}\Big(\frac{H|x_j|}{b}\Big)^r\le\frac{H^r}{b^r}\,n\mathbf{E}|X_1|^r=n\,\frac{b_1^r}{b^r}\,\frac{\beta_r}{\beta_3^r}\,.$$

To prove the second inequality, we apply Hölder's inequality:

$$\sum_{x_j\in\mathcal{K}}x_j^2\le\Big(\sum_{x_j\in\mathcal{K}}|x_j|^3\Big)^{2/3}|\mathcal{K}|^{1/3}\le\big(n\mathbf{E}|X_1|^3\big)^{2/3}|\mathcal{K}|^{1/3}.$$

Furthermore, writing $n\mathbf{E}|X_1|^3=n\beta_3\tau^{-3}$ and substituting the bound (5.11) for $|\mathcal{K}|$ with $r=3$, we see that the right-hand side is bounded by $b_1(pqb)^{-1}$.

Lemma 5.5. *Let $Z \geq 0$ be a random variable satisfying $\mathbf{E}Z^2 = 1$. Then*

$$(\mathbf{E}Z^3)^2 \leq \mathbf{E}Z^4, \qquad \big(\mathbf{E}Z^3\big)^3 \leq \mathbf{E}Z^5, \qquad \mathbf{E}Z^4\,\mathbf{E}Z^3 \leq \mathbf{E}Z^5. \tag{5.12}$$

Proof of Lemma 5.5. The first inequality follows from the Cauchy–Schwartz inequality:

$$\mathbf{E}Z^3 = \mathbf{E}Z\,Z^2 \leq (\mathbf{E}Z^2)^{1/2}(\mathbf{E}Z^4)^{1/2}.$$

This implies $(\mathbf{E}Z^3)^3 \leq \mathbf{E}Z^4\mathbf{E}Z^3$. It remains to prove the last inequality of (5.12). By Hölder's inequality, we have

$$\begin{aligned}
\mathbf{E}Z^4 &= \mathbf{E}Z^{2/3}Z^{10/3} \leq (\mathbf{E}Z^2)^{1/3}(\mathbf{E}Z^5)^{2/3} = (\mathbf{E}Z^5)^{2/3},\\
\mathbf{E}Z^3 &= \mathbf{E}Z^{4/3}Z^{5/3} \leq (\mathbf{E}Z^2)^{2/3}(\mathbf{E}Z^5)^{1/3}.
\end{aligned}$$

Clearly, these inequalities yield (5.12).

Lemma 5.6. *Let k and l be positive integers. We have*

$$\mathbf{E}X_1^kX_2^l = \frac{n}{n-1}\,\varkappa_k\varkappa_l - \frac{1}{n-1}\,\varkappa_{k+l}.$$

Proof of Lemma 5.6. The lemma follows from the identities

$$\mathbf{E}X_1^kX_2^l = \mathbf{E}X_1^k\mathbf{E}(X_2^l|X_1), \qquad \mathbf{E}(X_2^l|X_1) = \frac{n}{n-1}\,\varkappa_l - \frac{1}{n-1}\,X_1^l.$$

Lemma 5.7. *For all $k \in \Omega_n$ and $|s| \leq \pi$, we have*

$$\begin{aligned}
&|\gamma^k(s) - 1| \ll kpq|s|^3\delta_{k-1} \ll r_1^*, \qquad r_1^* = \tau^2|s|^3\exp\{\tfrac{c_1}{2}\,\zeta_0\}, && (5.13)\\
&\Big|\gamma^k(s) - 1 - k\,\frac{\gamma^{(3)}(0)}{6}\,s^3\Big| \ll kpqs^4(1+kpqs^2)\delta_{k-1} \ll r_2^* && (5.14)\\
&\delta_k = \exp\{\tfrac{kpq}{2}\,s^2(1-\Theta(b))\}, \qquad r_2^* = \tau^2s^4(1+\tau^2s^2)\exp\{-\tfrac{c_1}{2}\,\zeta_0\}.
\end{aligned}$$

where c_1 is given in (4.1).

Proof of Lemma 5.7. By (3.20), we have

$$\gamma^k(s) - 1 = (\gamma(s)-1)\sum_{j=1}^{k}\gamma^{j-1}(s). \tag{5.15}$$

Expanding $\gamma(s)$ in powers of s and using the fact that $\gamma(0) = 1$ and $\gamma'(0) = \gamma''(0) = 0$ (see Lemma 5.1), we get

$$\gamma(s) - 1 = r_1 \qquad \gamma(s) - 1 = \frac{\gamma^{(3)}(0)}{6}\, s^3 + r_2, \qquad r_i = \mathbf{E}_\theta \gamma^{(i+2)}(\theta s) s^{i+2}\, \frac{(1-\theta)^{i+1}}{(i+1)!}\,,$$

for $i = 1, 2$. Invoking the inequality $|\gamma^{(i+2)}(\theta s)| \ll pq$ (see (5.2)), we obtain

$$|r_i| \ll pq|s|^{i+2}, \qquad i = 1, 2. \tag{5.16}$$

To prove the first inequality of (5.13), we construct a bound for the right-hand side of (5.15). Using (5.16) with $i = 1$, we bound $|\gamma(s) - 1|$. To bound $|\gamma^{j-1}(s)|$, $j = 2, \ldots, r$, we use (5.5). The second inequality of (5.13) is trivial.

To prove (5.14) we replace the right-hand side of (5.15) by $k\gamma^{(3)}(0)s^3/6$ and show that the error of this replacement does not exceed $kpqs^4(1 + kpq)\delta_{k-1}$. In the first step, we replace $(\gamma(s) - 1)$ by $\gamma^{(3)}(0)s^3/6$ in (5.15). The error of this replacement

$$|r_2| \sum_{j=1}^{k} |\gamma^{j-1}(s)| \ll kpqs^4\delta_{k-1}, \tag{5.17}$$

by (5.5) and (5.16). In the second step, using (5.13), we replace $\gamma^{j-1}(s)$ in the right-hand side of (5.15) by 1, for $j = 2, \ldots, k$. The error of the second replacement

$$\begin{aligned} \Big|\, \frac{\gamma^{(3)}(0)s^3}{6} \,\Big| \sum_{j=2}^{k} |\gamma^{j-1}(s) - 1| &\ll \Big|\, \frac{\gamma^{(3)}(0)s^3}{6} \,\Big| (k-1) \max_{2 \le j \le k} |\gamma^{j-1}(s) - 1| \\ &\ll p^2 q^2 k^2 s^6 \delta_{k-1}. \qquad (5.18) \end{aligned}$$

The first inequality of (5.14) follows from (5.17) and (5.18). The second inequality of (5.13) is trivial.

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