ONE– AND TWO–TERM EDGEWORTH EXPANSIONS FOR A FINITE POPULATION SAMPLE MEAN. EXACT RESULTS. I

M. Bloznelis

Abstract. We prove the validity of one– and two–term Edgeworth expansions under optimal conditions (a Cramer type smoothness condition and the minimal moment conditions) and provide precise bounds for the remainders of expansions. The bounds depend explicitly on the ratio $p = N/n$, where N and n denote the sample size and the population size, respectively.

1. Introduction and results

1. Main results. Let $X_1, \dots, X_N$ be a simple random sample drawn without replacement from the set $\mathcal{X} = \{x_1, \dots, x_n\} \subset \mathbb{R}$, where $N < n$, that is, $\mathbf{P}\{\,\{X_1, \dots, X_N\} = \mathcal{B}\} = \binom{n}{N}^{-1}$ for every subset $\mathcal{B} \subset \mathcal{X}$ of size $|\mathcal{B}| = N$. Note that $X_1, \dots, X_N$ are identically distributed and symmetrically dependent. Denote

$$S = X_1 + \dots + X_N, \qquad p = \frac{N}{n}, \qquad q = 1 - p, \qquad \tau^2 = npq.$$

Assume that $\mathbf{E}X_1 = 0$ and $\sigma^2 > 0$, where $\sigma^2 = \mathbf{E}X_1^2$. A simple calculation shows that $\operatorname{Var} S = \sigma^2\tau^2(1 + (n-1)^{-1})$. By the central limit theorem (see Erdős and Rényi [6]), for large N, the distribution of $S/(\sigma\tau)$ can by approximated by the standard normal distribution. We approximate the distribution function

$$F(x) = \mathbf{P}\big\{S \le x\tau\sigma\big\}$$

by the one– and two–term Edgeworth expansions *in powers of* τ^{-1}

$$G_1(x) = \Phi(x) + \tau^{-1}P_1(x), \qquad G_2(x) = \Phi(x) + \tau^{-1}P_1(x) + \tau^{-2}P_2(x) + \tau^{-2}\tilde{P}_2(x),$$

and give *explicit* bounds for the remainders

$$\Delta_j = \sup_{x\in\mathbb{R}} |F(x) - G_j(x)|, \qquad j = 1, 2.$$

1991 *Mathematics Subject Classification.* Primary 62E20; secondary 60F05.
Key words and phrases. Edgeworth expansion, finite population, Cramer condition.

Here $\Phi(x)$ denotes the standard normal distribution function,

$$P_1(x) = -\,\phi(x)H_2(x)\,\frac{q-p}{6}\,\alpha_3,$$
$$P_2(x) = -\,\phi(x)H_3(x)\Big(\,\frac{4pq-1}{8}\,+\,\frac{1-6pq}{24}\,\alpha_4\Big) - \phi(x)H_5(x)\,\frac{1-4pq}{72}\,\alpha_3^2,$$
$$\tilde{P}_2(x) = -\,\phi(x)H_1(x)\,\frac{pq}{2}\,,$$

where $\phi(x) = \Phi^{(1)}(x)$ denotes the standard normal density function, $H_i(x)\phi(x) = (-1)^i\phi^{(i)}(x)$, and where we denote

$$\alpha_i = \sigma^{-i}\mathbf{E}X_1^i, \qquad \beta_i = \sigma^{-i}\mathbf{E}|X_1|^i, \qquad i = 1, 2, \dots. \tag{1.1}$$

Given a random variable Z with $0 < \operatorname{Var} Z < +\infty$, write

$$\rho_Z(A, B) = 1 - \sup\big\{\big|\mathbf{E}\exp\big\{it\,\frac{Z}{\sigma_Z}\,\big\}\big| : A \le |t| \le B\big\}, \qquad 0 < A < B,$$

where $\sigma_Z^2 = \operatorname{Var} Z$. Denote

$$\rho(x) = \rho_{X_1}(b_1/\beta_3, x), \qquad \text{where} \qquad b_1 = 0.001. \tag{1.2}$$

Theorem 1. *There exists an absolute constant $c > 0$ such that, for $n = 3, 4, \dots$ and $1 < N < n$,*

$$\Delta_1 \le c\,\frac{\beta_4}{\tau^2}\,+c\,\exp\big\{-\tau^2\rho(\tau)\big\}\tau\ln\tau, \tag{1.3}$$
$$\Delta_2 \le c\,\frac{\beta_5}{\tau^3}\,+c\,\exp\big\{-\tau^2\rho(\tau^2)\big\}\tau\ln\tau. \tag{1.4}$$

The constant $b_1 = 0.001$ is not the optimal one. No effort was made to find the optimal (the largest) value of b_1, for which (1.3) and (1.4) hold. However, we prefer to give a concrete constant instead of writing “there exists a (small) absolute constant b_1...”.

Using Theorem 1, we easily obtain one– and two–term Edgeworth expansions for the distribution function

$$F_\star(x) = \mathbf{P}\{S \le x(\operatorname{Var} S)^{1/2}\} = F\big(x(1+(n-1)^{-1})^{1/2}\big).$$

Corollary 1. *Theorem 1 remains true if we replace Δ_j by $\Delta_j^\star$, $j = 1, 2$. Here*

$$\Delta_j^\star = \sup_x |F_\star(x) - G_j^\star(x)|, \qquad j = 1, 2,$$
$$G_1^\star(x) = G_1(x), \qquad G_2^\star(x) = \Phi(x) + \tau^{-1}P_1(x) + \tau^{-2}P_2(x).$$

Since the absolute constant in (1.3) and (1.4) is not specified, the results of Theorem 1 and Corollary 1 are purely asymptotic (as N and $n \to \infty$).

To be more precise, consider a sequence of sets $\mathcal{X}_n = \{x_{1,n}, \dots, x_{n,n}\} \subset \mathbb{R}$ and a sequence of integers $N_n \in \mathbb{N}$, with $N_n < n$, for $n = 3, 4, \dots$. Denote

$$\tau_n^2 = n p_n q_n, \qquad p_n = N_n/n, \qquad q_n = 1 - p_n.$$

Given n, let $X_{1,n}, \dots, X_{N_n,n}$ denote a simple random sample of size N_n drawn without replacement from $\mathcal{X}_n$. Assume that $\mathbf{E}X_{1,n} = 0$ and $\sigma_n^2 := \mathbf{E}X_{1,n}^2 > 0$, and let $\alpha_{i,n}$ and $\beta_{i,n}$ denote the i th moments of $X_{1,n}$ corresponding to (1.1). Let $G_{j,n}$, $G_{j,n}^\star$, $\Delta_{j,n}$, $\Delta_{j,n}^\star$ and $\rho_n(\cdot)$ be defined as G_j, $G_j^\star$, Δ_j, $\Delta_j^\star$ and $\rho(\cdot)$ above, but with respect to the sum $S_n = X_{1,n} + \cdots + X_{N_n,n}$ and the random variable $X_{1,n}$.

If $\beta_{4,n}$ respectively $\beta_{5,n}$ remain bounded as $n \to \infty$ and

$$\liminf_n \rho_n(\tau_n) > 0 \qquad \text{respectively} \qquad \liminf_n \rho_n(\tau_n^2) > 0 \tag{1.5}$$

then the results of Theorem 1 and Corollary 1 yield the bounds $O(\tau_n^{-2})$ respectively $O(\tau_n^{-3})$ as $\tau_n \to \infty$.

Observe that (1.5) are Cramer type conditions. Recall that the distribution of a random variable Z is said to satisfy Cramer's condition if, for some $\delta > 0$,

$$1 - \sup_{|t|>\delta} |\mathbf{E}\exp\{itZ\}| > 0. \tag{C}$$

In the case where N remains fixed and $n \to \infty$, the simple random sample model approaches the i.i.d. situation. In this way, Theorem 1 yields the well–known bounds $O(N^{-1})$ and $O(N^{-3/2})$ for the rate of the approximation of the distribution function of a sum of i.i.d. random variables (satisfying (C) and proper moment conditions) by one– and two–term Edgeworth expansions (see., e.g., Petrov [11]).

The goal of our Theorem 1 and Corollary 1 is that the bounds (1.3) and (1.4) depend explicitly (and in an optimal way) on the parameters $p = p_n$ and $q = q_n$. This is important for the applications in the cases where p_n or q_n tend to zero as $n \to \infty$. Another advantage of our results is the use of the minimal smoothness conditions (1.5).

Earlier asymptotic expansions for the distribution function of finite population sample mean were constructed by Robinson [13] and, in the multivariate case, by Babu and Bai [2]).

Let us compare our results with those of Robinson [13] who proved the bound

$$\Delta_{2,n}^\star \le C(p)\beta_{5,n}/n^{3/2}, \tag{1.6}$$

with some constant $C(p)$ depending on p, subject to the condition

(c) *Given* $C_1 > 0$, *there exist* $\varepsilon > 0$, $C_2 > 0$ *and* $\delta > 0$ *not depending on* n *such that, for any fixed* x, *the number of indices* k, *for which*

$$\Big| \frac{x_{k,n}}{\sigma_n} t - x - 2r\pi \Big| > \varepsilon \qquad \textit{for all} \qquad t \in \Big(\frac{C_1}{\beta_{0,n}}, \ \frac{C_2\, n}{\beta_{5,n}} \Big)$$

and all integers r, *is greater than* $n\,\delta$ *for all* n. *Here* $\beta_{0,n} = \sigma_n^{-1} \max_k |x_{k,n}|$.

We show in Sec. 2 that Robinson's condition (c) is more restrictive than $\liminf_n \rho_n(\tau_n^2) > 0$, see (1.5). In particular, in (1.5), we avoid the use of the unpleasant quantity $\beta_{0,n}$. Furthermore, the constant $C(p)$ in (1.6) depends on p and this restricts the area of possible applications of (1.6) to the cases where $p = p_n$ remains unchanged as $n \to \infty$.

Babu and Bai [2]) constructed j-term Edgeworth expansion with the remainder $o(n^{-j/2})$, $j = 1, 2, \dots$, for the distribution function of a sum of R^k valued random observations drawn without replacement from a finite population of size n under the additional condition $\liminf_n \min\{p_n, q_n\} > 0$.

The approach used in our paper differs from those used by Robinson [13] and Babu and Bai [2]) and allows us to prove the validity of j-term Edgeworth expansions for $F(x)$ with the remainders $O(\tau^{-j-1})$ and $o(\tau^{-j})$ for $j = 3, 4, \dots$ as well. We do not consider the case $j \geq 3$ here because of the following reasons: for typical applications, one– and two–term expansions are sufficient; the consideration of the general case $j = 1, 2, 3, \dots$ requires no new ideas, but rather involved calculations.

The closeness between F and the standard normal distribution was studied by Bikelis [3]. The Berry–Essen bound was proved by Höglund [9]. Furthermore, Kokic and Weber [10] and Zhao and Chen [14] investigated the rate of the normal approximation of a nonlinear finite population statistic (U-statistics). Berry-Esseen bounds for the finite population Student's t statistic were constructed by Rao and Zhao [12] and Bloznelis [4]. One–term Edgeworth expansions for U-statistics were established by Kokic and Weber [10] with the remainder $o(N^{-1/2})$, and in Bloznelis and Götze [5] with the remainder $O(N^{-1})$.

2. Remainders of orders $o(\tau)$ and $o(\tau^2)$. Denote

$$\beta^\star_{j,n}(\varepsilon) = \sigma_n^{-j} \mathbf{E} |X_{1,n}|^j \mathbb{I}\{X_{1,n}^2 \geq \varepsilon \sigma_n^2 \tau_n^2\}, \qquad \varepsilon > 0.$$

Theorem 2. *Let* $j = 1, 2$. *The conditions*

$$\limsup_n \beta_{j+2,n} < \infty, \tag{1.7}$$

$$\forall \varepsilon > 0, \qquad \limsup_n \beta^\star_{j+2,n}(\varepsilon) = 0, \tag{1.8}$$

$$\forall \delta > 0, \quad \forall B > 1, \qquad \liminf_n \rho_{X_{1,n}}(\delta, \tau_n^{j-1} B) > 0, \tag{1.9}$$

imply

$$\Delta_{j,n} = o(\tau_n^{-j}), \qquad \Delta^\star_{j,n} = o(\tau_n^{-j}) \qquad \textit{as} \quad \tau_n \to \infty.$$

The conditions of Theorem 2 are quite natural. Note that if we take $j = 0$, (1.8) becomes the Erdős-Rényi condition, the weakest sufficient condition for the asymptotic normality of $\sigma_n^{-1}\tau_n^{-1}(X_{1,n} + \cdots + X_{N_n,n})$. Furthermore, in this case, $\beta_{j+2,n} = 1$ and, therefore, condition (1.7) is automatically satisfied.

Note that the smoothness condition (1.9) is somewhat weaker than the condition

$$\forall B > 1, \qquad \liminf_n \rho_{X_{1,n}}\Big(\frac{b_1}{\beta_{3,n}}, \tau_n^{j-1}B\Big) > 0, \tag{1.10}$$

where b_1 is given in (1.2). However, it is easy to verify that if (1.7) holds, then conditions (1.9) and (1.10) are equivalent.

The remaining part of the paper is organized as follows. In Sec. 2 we study several modifications of Cramer's (C) condition. In Sec. 3 we prove Theorems 1, 2 and Corollary 1. Some technical steps of the proofs are given in Sec. 4. Auxiliary lemmas are collected in Sec. 5.

The paper is divided in two parts. Secs. 1, 2 and the proofs of Theorem 1 and Corollary 1 are given in the first part. The second part containing the proof of Theorem 2 and Secs. 4, 5 will appear in a subsequent issue of this journal under the title "One– and two–term Edgeworth expansions for a finite population sample mean. Exact results. II".

2. Smoothness conditions

Let $Z_{1,n}, \dots, Z_{N_n,n}$, $n = 3, 4, \dots$, be an array of random variables such that, for every n, the random variables $Z_{k,n}$, $k = 1, \dots, N_n$, are identically distributed. Assume that $N_n \to \infty$ and that the sequence of the sums $S_n(Z) = Z_{1,n} + \cdots + Z_{N_n,n}$ is asymptotically standard normal as $n \to \infty$. In order to establish a higher order approximations, i.e., to prove the validity of asymptotic expansions of $\mathbf{P}\{S_n(Z) \le x\}$ with the remainders $o(N_n^{-j/2})$, $j = 1, 2, \dots$, one needs (in addition to the moment conditions) to impose a smoothness condition on the distributions of $Z_{1,n}$, $n = 3, 4, \dots$, like the Cramer condition (C) is imposed in the classical case of a sum of i.i.d. random variables. However, the requirement that, for every $n = 3, 4, \dots$, the random variable $Z_{1,n}$ satisfy the classical Cramer condition (C) is too restrictive in the case of an array of random variables which, in addition, may have discrete distributions. One useful modification of condition (C) was introduced by Albers, Bickel and van Zwet [1] (see also Robinson [13]) and was later used in a number of papers. This condition requires that there exists $\varepsilon > 0$ such that

$$\liminf_n \tau_{Z_{1,n}}(\varepsilon, a_n, b_n) > 0 \tag{2.1}$$

for appropriately chosen sequences $0 < a_n < b_n$. Here, for a random variable Z (with $\operatorname{Var} Z = \sigma_Z^2 > 0$), we write

$$\tau_Z(\varepsilon, a, b) = 1 - \sup_{s\in R,\, a\le|t|\le b} \mathbf{P}\Big\{t\,\frac{Z}{\sigma_Z} \in L^\varepsilon + s\Big\}, \quad L = \{2\,\pi\, r,\ r = 0, \pm 1, \pm 2, \dots\},$$

where $\mathcal{B}^\varepsilon$ denotes the ε–neighborhood of a set $\mathcal{B} \subset R$.

Now Robinson's condition (c) can by formulated as follows: given C_1, there exists $\varepsilon > 0$ and $C_2 > 0$ such that the sequence of random variables $X_{1,n}$ satisfies condition (2.1) with $a_n = C_1/\beta_{0,n}$ and $b_n = C_2\, n/\beta_{5,n}$.

Let a_n and b_n be sequences of positive numbers. We will show below that the condition

$$\exists \quad \varepsilon > 0 \qquad \text{such that} \qquad \liminf_n \tau_{Z_{1,n}}(\varepsilon, a_n, b_n) > 0 \tag{2.2}$$

implies

$$\liminf_n \rho_{Z_{1,n}}(a_n, b_n) > 0. \tag{2.3}$$

Furthermore, (2.3) implies (2.2), by Lemma 2.1 below. That is, conditions (2.2) and (2.3) are equivalent. In particular, Robinson's condition (c) is equivalent to

$$\liminf_n \rho_{X_{1,n}}(a'_n, b'_n) > 0. \tag{2.4}$$

Note that $\beta_{0,n} \geq \beta_{3,n}$. To show that (2.4) is somewhat more restrictive than the second inequality of (1.5), consider the case where $\lim_n \beta_{0,n} = +\infty$ and $\limsup_n \beta_{3,n} < \infty$, simultaneously. In this case, (2.4) fails, but the second inequality of (1.5) still can hold.

In Bloznelis and Götze [5], the following version of (2.1) was introduced. Let $Z'_{1,n}$ denote an independent copy of $Z_{1,n}$. That is, random variables $Z_{1,n}$ and $Z'_{1,n}$ are independent and identically distributed. Let $Z^*_{1,n} = Z_{1,n} - Z'_{1,n}$ denote the symmetrization of $Z_{1,n}$. The condition

$$\exists \quad \varepsilon > 0 \qquad \text{such that} \qquad \liminf_n \tau^*_{Z_{1,n}}(\varepsilon, a_n, b_n) > 0,$$

where

$$\tau^*_Z(\varepsilon, a, b) = 1 - \sup_{a \leq |t| \leq b} \mathbf{P}\big\{ t\, \frac{Z^*}{\sigma_Z} \in L^\varepsilon \big\} > 0,$$

is equivalent to (2.1). This fact is a consequence of Lemma 2.1 below. Lemma 2.1 (without proof) is formulated in Bloznelis and Götze [5]. For convenience, we give the proof here.

Write

$$\delta_Z(a,b) = 1 - \sup\big\{ \mathbf{E}\ \cos\big(t\, \frac{Z}{\sigma_Z} + s\big) :\ s \in R,\ a \leq |t| \leq b \big\}.$$

Lemma 2.1 (Bloznelis and Götze [5]. *Let Z be a random variable with* $\operatorname{Var} Z > 0$. *For $0 < a < b$ and $0 < \varepsilon < \pi$, write*

$$\rho = \rho_Z(a,b), \quad \tau_\varepsilon = \tau_Z(\varepsilon, a, b), \quad \tau^*_\varepsilon = \tau^*_Z(\varepsilon, a, b), \quad u = \pi^{-1} \varepsilon\, \tau^*_\varepsilon \ \text{ and } \ v = \pi^{-1} \varepsilon\, \tau_\varepsilon.$$

The following inequalities hold:

$$\frac{\varepsilon^2\tau_\varepsilon}{\pi^2} \le \rho \le 4\,\tau_\rho, \tag{2.5}$$

$$\frac{\varepsilon^2\tau_\varepsilon^*}{\pi^2} \le \rho \le 4\,\tau_\rho^*, \tag{2.6}$$

$$\tau_v^* \ge \frac{\varepsilon^2\tau_\varepsilon}{2\,\pi^2}\,, \qquad \tau_u \ge \frac{\varepsilon^2\,\tau_\varepsilon^*}{4\,\pi^2}\,, \tag{2.7}$$

$$2\rho \ge \delta_Z(a,b) \ge \rho. \tag{2.8}$$

Proof of Lemma 2.1. We can assume, without loss of generality, that $\operatorname{Var} Z = 1$. For a random variable Y denote $\tau_\delta(Y) = \mathbf{P}\{\cos Y < \cos\delta\}$, $\delta > 0$. Clearly, $\tau_{\delta_1}(Y) \le \tau_{\delta_2}(Y)$ for $0 < \delta_2 < \delta_1 < \pi$. We shall show that, for any $0 < \delta < \pi$ and $0 < d < 1$,

$$\mathbf{E}\,(1 - \cos\, Y) \ge 2\,\pi^{-2}\delta^2\,\tau_\delta(Y), \tag{2.9}$$

$$\tau_{d'}(Y) > \frac{1}{2}\,\big(\mathbf{E}\,(1 - \cos\, Y) - d\big), \quad d' = \sqrt{2\,d}. \tag{2.10}$$

The first inequality follows from Chebyshev's inequality

$$\mathbf{E}\,(1 - \cos\, Y) \ge (1 - \cos\,\delta)\,\mathbf{P}\{1 - \cos\, Y > 1 - \cos\,\delta\},$$

the identity $\tau_\delta(Y) = \mathbf{P}\{1 - \cos\, Y > 1 - \cos\,\delta\}$ and the inequality

$$1 - \cos\, u \ge 2\,\pi^{-2}\,u^2, \qquad 0 \le u \le \pi.$$

To get (2.10) combine the inequalities $\arccos(1 - d) > \sqrt{2\,d}$ and

$$\begin{aligned}\mathbf{E}\,(1 - \cos\, Y) &\le d\,\mathbf{P}\{1 - \cos\, Y \le d\} + 2\,\mathbf{P}\{1 - \cos\, Y > d\}\\ &= d + (2 - d)\,\mathbf{P}\{\cos\, Y < 1 - d\}.\end{aligned}$$

Write $Y_{s,t} = t\,Z + s$. We have

$$\tau_\varepsilon = \inf\{\tau_\varepsilon(Y_{s,t}) :\ s \in R,\ a < |t| < b\} \quad \text{and} \quad \tau_\varepsilon^* = \inf\{\tau_\varepsilon(t\,Z^*) :\ a < |t| < b\}.$$

Recall that $Z^* = Z - Z'$, where Z' denotes a random variable independent of Z and having the same distributions as Z. Denote

$$\varkappa = \sup\big\{\,\mathbf{E}\,\cos Y_{s,t} :\ s \in R,\ a \le |t| \le b\big\}, \quad \varkappa^* = \sup\big\{\,\mathbf{E}\,\cos\big(t\,Z^*\big) :\ a \le |t| \le b\big\}$$

and $\tilde\varkappa = \sup\big\{\big|\mathbf{E}\,\exp\{i\,t\,Z\}\big| :\ a \le |t| \le b\big\}$. Note that $0 \le \varkappa, \varkappa^*, \tilde\varkappa \le 1$. Furthermore,

$$\varkappa^* = \tilde\varkappa^2, \qquad \varkappa^* \le \varkappa, \qquad \varkappa^2 \le \varkappa^*. \tag{2.11}$$

The first inequality is obvious. The second one follows from the inequality

$$\Big(\mathbf{E}\,\cos(t\,Z+s)\Big)^2 \le \mathbf{E}\,\cos\Big((t\,Z+s)-(t\,Z'+s)\Big)$$

which is a consequence of the identity $\cos(x-y)=\cos x\,\cos y+\sin x\,\sin y$ (applied to $x=t\,Z+s$ and $y=t\,Z'+s$) and the independence of x and y.

Let us prove (2.5). Assume that $\tau_\varepsilon>0$. It follows from (2.9) that

$$\mathbf{E}\,\cos Y_{s,t} \le 1-2\,\pi^{-2}\varepsilon^2\,\tau_\varepsilon(Y_{s,t}) \le 1-2\,\pi^{-2}\varepsilon^2\,\tau_\varepsilon.$$

Hence, we have

$$1-\varkappa \ge 2\,\pi^{-2}\varepsilon^2\,\tau_\varepsilon. \tag{2.12}$$

By (2.11), $1-\tilde\varkappa^2 \ge 1-\varkappa$. Therefore, $\rho=1-\tilde\varkappa=(1-\tilde\varkappa^2)/(1+\tilde\varkappa)\ge(1-\varkappa)/2$ and, by (2.12), we obtain

$$\rho \ge (1-\varkappa)/2 \ge \pi^{-2}\varepsilon^2\,\tau_\varepsilon. \tag{2.13}$$

Let us prove the second inequality of (2.5). Assume that $\rho>0$. By (2.11), $\varkappa\le\tilde\varkappa$. Hence, we have

$$\rho=1-\tilde\varkappa\le 1-\varkappa. \tag{2.14}$$

An application of (2.10) gives

$$\tau_{d'}(Y_{s,t}) \ge \frac{1}{2}\Big(1-\mathbf{E}\,\cos Y_{s,t}-d\Big) \ge \frac{1}{2}\,(1-\varkappa-d).$$

Hence, we have

$$\tau_{d'} \ge \frac{1}{2}\,(1-\varkappa-d), \qquad d'=\sqrt{2\,d}. \tag{2.15}$$

Taking $d=\rho/2$ we obtain $\tau_{d'}\ge\rho/4$ and using $d'=\rho^{1/2}\ge\rho$ we get $\tau_\rho\ge\rho/4$.

Let us prove (2.6). Assume that $\tau^*_\varepsilon>0$. Using (2.9) with $Y=t\,Z^*$, we get

$$1-\varkappa^* \ge 2\,\pi^{-2}\,\varepsilon^2\,\tau^*_\varepsilon. \tag{2.16}$$

Since $\varkappa^*=\tilde\varkappa^2$, we have $1-\varkappa^*\le 2\,\rho$. Therefore, $\rho\ge\pi^{-2}\,\varepsilon^2\,\tau^*_\varepsilon$. Now assume that $\rho>0$. We have $\rho\le 1-\tilde\varkappa^2=1-\varkappa^*$. An application of (2.10) with $Y=t\,Z^*$ and $d=\rho/2$ gives $\tau^*_{d'}\ge\rho/4$, where $d'=\rho^{1/2}$. It follows that $\tau^*_\rho\ge\rho/4$.

Let us prove (2.7). Assume that $\tau_\varepsilon>0$. Put $d=\pi^{-2}\,\varepsilon^2\,\tau_\varepsilon$. It follows from (2.12) and the inequality $\varkappa^*\le\varkappa$ (see (2.11)) that

$$1-\varkappa^* \ge 2\,\pi^{-2}\,\varepsilon^2\,\tau_\varepsilon = 2\,d.$$

Invoking (2.10) with $Y = t\,Z^*$ we get

$$2d \le 1 - \varkappa^* \le 1 - \mathbf{E}\,\cos Y \le d + 2\tau_{d'}(Y), \quad d' = \sqrt{2\,d}.$$

This implies $\tau_{d'}(Y) \ge d/2$ and, therefore, $\tau^*_{d'} \ge d/2 = 2^{-1}\pi^{-2}\varepsilon^2\tau_\varepsilon$. Since $d' \ge d^{1/2} \ge v$, we obtain $\tau^*_v \ge 2^{-1}\,\pi^{-2}\varepsilon^2\tau_\varepsilon$. The proof of the second inequality of (2.7) is similar. We use (2.16) and the inequalities $1-\varkappa \ge (1-\varkappa^2)/2 \ge (1-\varkappa^*)/2$ (see (2.11)) to get $1 - \varkappa \ge 2d$, where $d = 2^{-1}\pi^{-2}\varepsilon^2\tau^*_\varepsilon$. Now an application of (2.15) gives $\tau_{d'} \ge d/2$, where $d' = \sqrt{2\,d}$. Since $d' \ge u$, we obtain $\tau_u \ge \tau_{d'} \ge 4^{-1}\pi^{-2}\varepsilon^2\,\tau^*_\varepsilon$.

Finally, (2.8) is a consequence of (2.13) and (2.14): just note that $\delta_Z(a,b) = 1-\varkappa$.

Remark. The lemma remains true if we replace $Z\,\sigma_Z^{-1}$ by Z in the definition of $\rho_Z(a,b)$, $\tau_Z(\varepsilon,a,b)$, $\tau^*_Z(\varepsilon,a,b)$ and $\delta_Z(a,b)$. With this change of notation, the lemma applies as well to random variables without finite second moments.

3. Proofs

The section is organized as follows. We first prove Theorem 1. Then we prove Corollary 1 and Theorem 2.

Assume, without loss of generality, that $\mathbf{E}X_1^2 = \tau^{-2}$. Then

$$F(x) = \mathbf{P}\{S \le x\},$$
$$\sum_{j=1}^n x_j^2 = \frac{1}{pq}\,. \tag{3.1}$$

Write $\Omega_k = \{1, 2, \dots, k\}$, and denote

$$\varkappa_k = \mathbf{E}X_1^k, \qquad \kappa_k = \mathbf{E}|X_1|^k, \qquad k \in \Omega_n.$$

Note that $\alpha_k = \tau^k\varkappa_k$, $\beta_k = \tau^k\kappa_k$. Below we shall use the inequalities

$$1 = \beta_2 \le \beta_3 \le \beta_4 \le \beta_5, \quad \beta_3^2 \le \beta_4, \quad \beta_3\beta_4 \le \beta_5, \quad \beta_3^3 \le \beta_5. \tag{3.2}$$

The last three inequalities are proved in Lemma 5.5 below.

Let us introduce some more notation. By $c, c_1, c_2, \dots$ we denote absolute constants. We will write $A \ll B$ if $A \le cB$. By $c(V_1, V_2, \dots)$ we denote a positive number, which depends only on the quantities $V_1, V_2, \dots$. Given a real function g, we denote $\|g(x)\| = \sup_x |g(x)|$. Let $\theta, \theta_1, \theta_2, \dots$ be independent random variables uniformly distributed in $[0,1]$ and independent of all other random variables considered. For a complex valued smooth function h, we use the Taylor expansion

$$h(x) = h(0) + h^{(1)}(0)x + \dots + h^{(n)}(0)\,\frac{x^n}{n!} + \mathbf{E}_{\theta_1} h^{(n+1)}(\theta_1 x)(1-\theta_1)^n\,\frac{x^{n+1}}{n!}\,.$$

Here $\mathbf{E}_{\theta_1}$ denotes the conditional expectation given all random variables but θ_1, and $h^{(j)}$ denotes the j th derivative of h.

Proof of Theorem 1. We assume, without loss of generality, that $\beta_3/\tau < b_1$. Otherwise, (1.3) and (1.4) follow from the inequalities $\Delta_j \le 1 + \max_x |G_j(x)|$, where $j = 1, 2$, and the inequalities

$$|G_1(x)| \ll 1 + \frac{\beta_3}{\tau}\,, \qquad |G_2(x)| \ll 1 + \frac{\beta_3}{\tau} + \frac{1+\beta_4+\beta_3^2}{\tau^2}$$

and (3.2).

Given a complex–valued function f, denote

$$I_{[a,d]}(f) := \int_{|t|\in(a,d]} \frac{|f(t)|}{|t|}\,dt, \qquad 0 \le a < d.$$

By the Berry – Esseen smoothing lemma (see, e.g., Feller [7], 538 p.) we have

$$\Delta_j \ll I_{[0,T]}\big(\hat F - \hat G_j\big) + T^{-1}\|G_j^{(1)}(x)\|, \quad T>0, \quad j=1,2. \tag{3.3}$$

where $\hat F$ and $\hat G_j$ denote the Fourier-Stieltjes transforms of F and G_j, $j = 1, 2$, respectively. A straightforward calculation shows that

$$\begin{aligned} \hat G_1(t) &= e^{-t^2/2} + e^{-t^2/2}(it)^3\,\frac{q-p}{6}\,\frac{\alpha_3}{\tau} \\ \hat G_2(t) &= \hat G_1(t) + e^{-t^2/2}\,\frac{(it)^4}{\tau^2}\,\Big(\frac{4pq-1}{8} + \frac{1-6pq}{24}\,\alpha_4\Big) \\ &\quad + e^{-t^2/2}\,\frac{(it)^6}{\tau^2}\,\frac{1-4pq}{72}\,\alpha_3^2 + e^{-t^2/2}\,\frac{(it)^2}{\tau^2}\,\frac{pq}{2}\,. \end{aligned} \tag{3.4}$$

In order to estimate the first summand on the right side of (3.3), we write

$$I_{[0,T]}\big(\hat F - \hat G_j\big) \le I_{[0,H]}\big(\hat F - \hat G_j\big) + I_{[H,T]}\big(\hat F\big) + I_{[H,T]}\big(\hat G_j\big), \quad H = \frac{b_1\tau}{\beta_3}\,,$$

thus, obtaining

$$\Delta_j \ll T^{-1}\max_x |G_j^{(1)}(x)| + I_{[0,H]}\big(\hat F - \hat G_j\big) + I_{[H,T]}\big(\hat F\big) + I_{[H,T]}\big(\hat G_j\big). \tag{3.5}$$

Note that the assumption $\beta_3/\tau < b_1$ implies $H > 1$.

Let us prove (1.4). To this aim, we choose $T = \tau^3$ and construct the following bounds for the quantities on the right side of (3.5),

$$T^{-1}\|G_2^{(1)}(x)\| \ll \beta_5\tau^{-3}, \tag{3.6}$$

$$I_{[H,T]}\big(\hat G_2\big) \ll \beta_5\tau^{-3}, \tag{3.7}$$

$$I_{[0,H]}\big(\hat F - \hat G_2\big) \ll \mathcal{R}, \qquad \mathcal{R} = \frac{\beta_5}{\tau^3}\,, \tag{3.8}$$

$$I_{[H,T]}\big(\hat F\big) \ll \tau e^{-\tau^2\rho(\tau^2)}\ln\tau. \tag{3.9}$$

To show (3.6) we bound

$$\|G_2^{(1)}(x)\| \ll 1 + \beta_3 + \beta_3^2 + \beta_4 \ll \beta_4 \ll \beta_5 \tag{3.10}$$

using (3.2).

In order to prove (3.7), we estimate

$$|\hat{G}_2(t)| \ll e^{-t^2/2}\big(1 + \frac{|t|^3\beta_3}{\tau} + \frac{t^4}{\tau^2} + \frac{t^4\beta_4}{\tau^2} + \frac{t^6\beta_3^2}{\tau^2} + \frac{t^2}{\tau^2}\,\big) \tag{3.11}$$

using (3.4), and then apply the bounds

$$\int_{|t|\geq H} |t|^k e^{-t^2/2} dt \leq c(k,j) H^{-j}, \qquad k,j = 0,1,2,\dots, \tag{3.12}$$

combined with (3.2).

For convenience, the remaining part of the proof is divided into two steps.

Step 1. Proof of (3.8). In the proof, we use some ideas and techniques of Höglund [9].

It is convenient to write $\hat{G}_2$ in the following form

$$\begin{aligned}
&\hat{G}_2 = \exp\{-\frac{t^2}{2}\}\big(1 + g_1 + g_2 + g_3\big), \qquad g_i = g_{i,1} + g_{i,2}, \quad i = 1,2,3,\\
&g_{1,1} = (it)^2\,\frac{1-4pq}{4}\,\tau^{-2}, \qquad g_{1,2} = (it)^3\,\frac{q-p}{6}\,\tau^2\varkappa_3,\\
&g_{2,1} = (it)^2\,\frac{6pq-1}{4}\,\tau^{-2}, \qquad g_{2,2} = (it)^4\,\frac{1-6pq}{24}\,\tau^2\varkappa_4,\\
&g_{3,1} = (it)^4\,\frac{4pq-1}{8}\,\tau^{-2}, \qquad g_{3,2} = (it)^6\,\frac{1-4pq}{72}\,\tau^4\varkappa_3^2.
\end{aligned}$$

Let us write the Erdős – Rényi representation for $\mathbf{E}\exp\{itS\}$:

$$\mathbf{E}\exp\{itS\} = \frac{1}{2\pi\lambda}\int_{-\pi}^{\pi}\prod_{j=1}^{n}\beta(s+tx_j)ds, \qquad \lambda = \binom{n}{N}p^N q^{n-N}, \tag{3.13}$$

where λ satisfies $\lambda^{-1} \ll \tau$, by Lemma 1 of Höglund [9]. Here $\beta(u) = pe^{iuq} + qe^{-iup}$, $u \in \mathbb{R}$, denotes the Fourier transform of the distribution of the random variable $\nu - p$, where ν has the Bernoulli distribution with $\mathbf{P}\{\nu = 1\} = 1 - \mathbf{P}\{\nu = 0\} = p$. Denote

$$\gamma(u) = \beta(u)\exp\{\frac{pq}{2}\,u^2\}, \qquad a_j = \gamma(s+tx_j), \qquad A_j = a_1\cdots a_j, \quad 1 \leq j \leq n.$$

It follows from (3.1) that $\sum_{j=1}^{n}(s+tx_j)^2 = ns^2+p^{-1}q^{-1}t^2$. In view of this identity, we can write

$$\prod_{j=1}^{n}\beta(s+tx_j) = A_n \exp\{-\frac{\zeta_0}{2}\}, \qquad \zeta_0 = \tau^2 s^2 + t^2. \tag{3.14}$$

Furthermore, we have

$$\frac{1}{2\pi\lambda}\int_{-\pi}^{\pi}\gamma^n(s)\exp\{-\frac{\tau^2 s^2}{2}\}ds = \frac{1}{2\pi\lambda}\int_{-\pi}^{\pi}\beta^n(s)ds = 1,$$

since $\mathbf{E}\exp\{itS\} = 1$ for $t = 0$. In particular, we can write

$$\hat{G}_2 = \frac{1}{2\pi\lambda}\int_{-\pi}^{\pi}\gamma^n(s)\exp\{-\frac{\zeta_0}{2}\}(1+g_1+g_2+g_3)ds. \tag{3.15}$$

For two complex valued functions f and g, we write $f \sim g$ if

$$\tau\int_{|t|\le H}\frac{dt}{|t|}\int_{|s|\le\pi}|f(s,t)-g(s,t)|\exp\{-\frac{\zeta_0}{2}\}ds \ll \mathcal{R}.$$

In view of (3.13), (3.14), (3.15) and the bound $\lambda^{-1} \ll \tau$, the inequality (3.8) is a consequence of

$$A_n - \gamma^n(s) \sim \gamma^n(s)(g_1+g_2+g_3). \tag{3.16}$$

We prove (3.16) in two steps. We first show (3.17) and thwn we prove (3.31); see below.

Step 1.1. Let us show that

$$A_n - \gamma^n(s) \sim \gamma^{n-1}(s)(S_1+S_2) + \gamma^{n-2}(s)S_3. \tag{3.17}$$

Here we denote

$$S_i = \sum_{j=1}^{n}\hat{v}_{i,j}, \qquad i = 1,2, \qquad \text{and} \qquad S_3 = \sum_{j=2}^{n}\sum_{k=1}^{j-1}\hat{v}_{1,j}\hat{v}_{1,k},$$

$$\hat{v}_{i,j} = tx_j\mathbf{E}_\theta v_i(\xi_j), \qquad v_i(\xi_j) = \gamma^{(i+2)}(0)\,\frac{\xi_j^{i+1}}{(i+1)!}, \qquad \xi_j = s+\theta tx_j, \quad 1 \le j \le n.$$

To prove (3.17) we write

$$A_n - \gamma^n(s) = \gamma^{n-1}(s)(S_1+S_2) + \gamma^{n-2}(s)S_3 + R_1 + R_2 + R_3, \tag{3.18}$$

and show that

$$R_i \sim 0, \qquad i = 1, 2, 3. \tag{3.19}$$

Here

$$R_1 = \sum_{j=1}^{n} \Psi_{1,j} \hat{v}^*_{3,j}, \quad R_2 = \sum_{j=2}^{n} \sum_{k=1}^{j-1} \Psi_{2,k} \big(\hat{v}_{2,j} \hat{v}^*_{1,k} + \hat{v}_{1,j} \hat{v}^*_{2,k} \big),$$

$$R_3 = \sum_{j=3}^{n} \sum_{k=2}^{j-1} \sum_{r=1}^{k-1} \Psi_{3,r} \hat{v}_{1,j} \hat{v}_{1,k} \hat{v}^*_{1,r}, \qquad \Psi_{i,j} = A_{j-1} \gamma^{n-j-i+1}(s),$$

$$\hat{v}^*_{i,j} = t x_j \mathbf{E}_\theta v^*_i(\xi_j), \qquad v^*_i(\xi_j) = \mathbf{E}_{\theta_i} \gamma^{(i+2)}(\theta_i \xi_j) \, \frac{\xi_j^{i+1} (1 - \theta_i)^i}{i!} \, .$$

Let us prove (3.18). For this purpose, we use identities (3.20)-(3.25). An application of the simple identity

$$\prod_{j=1}^{r} s_j - \prod_{j=1}^{r} t_j = \sum_{j=1}^{r} \Big(\prod_{k=1}^{j-1} s_k \Big) \Big(\prod_{k=j+1}^{r} t_k \Big) (s_j - t_j) \tag{3.20}$$

gives

$$A_r - \gamma^r(s) = \sum_{j=1}^{r} A_{j-1} \gamma^{r-j}(s) \big(a_j - \gamma(s) \big), \tag{3.21}$$

where, by the mean value theorem,

$$a_j - \gamma(s) = \gamma(s + t x_j) - \gamma(s) = t x_j \mathbf{E}_\theta \gamma^{(1)}(\xi_j), \quad \xi_j = s + \theta t x_j. \tag{3.22}$$

Expanding $\gamma^{(1)}(\xi)$ in powers of ξ and using the fact that $\gamma^{(1)}(0) = \gamma^{(2)}(0) = 0$ we obtain

$$\gamma^{(1)}(\xi) = v^*_1(\xi), \tag{3.23}$$

$$\gamma^{(1)}(\xi) = v_1(\xi) + v^*_2(\xi), \tag{3.24}$$

$$\gamma^{(1)}(\xi) = v_1(\xi) + v_2(\xi) + v^*_3(\xi). \tag{3.25}$$

Now take $r = n$ in (3.21) and combine (3.22) with (3.25) to get

$$A_n - \gamma^n(s) = S'_1 + S'_2 + R_1, \quad \text{where} \quad S'_i = \sum_{j=1}^{n} A_{j-1} \gamma^{n-j}(s) \hat{v}_{i,j}, \quad i = 1, 2. \tag{3.26}$$

Here we set $A_0 = 1$. Furthermore, we apply (3.21) to $A_{j-1} - \gamma^{j-1}(s)$ and combine (3.22) with (3.23) (respectively with (3.24)) to get (3.27) (respectively (3.28)):

$$A_{j-1} = \gamma^{j-1}(s) + \sum_{k=1}^{j-1} A_{k-1}\gamma^{j-1-k}(s)\hat{v}^*_{1,k}, \qquad j \geq 2, \tag{3.27}$$

$$A_{j-1} = \gamma^{j-1}(s) + \sum_{k=1}^{j-1} A_{k-1}\gamma^{j-1-k}(s)(\hat{v}_{1,k} + \hat{v}^*_{2,k}), \quad j \geq 2. \tag{3.28}$$

Substituting (3.27) (respectively (3.28) into S'_2 (respectively, S'_1) one obtains

$$S'_2 = \gamma^{n-1}(s)S_2 + R_{2,1}, \qquad S'_1 = \gamma^{n-1}(s)S_1 + S^*_1 + R_{2,2}, \tag{3.29}$$

$$S^*_1 = \sum_{j=2}^{n}\sum_{k=1}^{j-1} A_{k-1}\gamma^{n-k-1}\hat{v}_{1,k}\hat{v}_{1,j},$$

$$R_{2,1} = \sum_{j=2}^{n}\sum_{k=1}^{j-1} \Psi_{2,k}\hat{v}_{2,j}\hat{v}^*_{1,k}, \qquad R_{2,2} = \sum_{j=2}^{n}\sum_{k=1}^{j-1} \Psi_{2,k}\hat{v}_{1,j}\hat{v}^*_{2,k}.$$

Substitution of the expression (3.27) into S^*_1 gives

$$S^*_1 = \gamma^{n-2}(s)S_3 + R_3. \tag{3.30}$$

Collecting (3.29) and (3.30) in (3.26) and using the identity $R_{2,1} + R_{2,2} = R_2$, we obtain (3.18).

The proof of (3.19) is postponed to Appendix I.

Step 1.2. Let us prove

$$\gamma^{n-1}(s)(S_1 + S_2) + \gamma^{n-2}(s)S_3 \sim \gamma^n(s)(g_1 + g_2 + g_3). \tag{3.31}$$

We first show that

$$\gamma^{n-1}(s)(S_1 + S_2) + \gamma^{n-2}(s)S_3 \sim \gamma^{n-1}(s)(S^\star_1 + S^\star_2) + \gamma^{n-2}(s)S^\star_3, \tag{3.32}$$

where $S^\star_j = S^\star_{j,1} + S^\star_{j,2}$, for $j = 1, 2, 3$, and

$$S^\star_{1,1} = \frac{\gamma^{(3)}(0)}{2pq}\,t^2 s = i^3\,\frac{q-p}{2}\,t^2 s, \qquad S^\star_{1,2} = \frac{\gamma^{(3)}(0)}{6}\,t^3 n\varkappa_3 = (it)^3\,\frac{q-p}{6}\,\tau^2\varkappa^3,$$

$$S^\star_{2,1} = \frac{\gamma^{(4)}(0)}{4pq}\,s^2t^2 = \frac{1-6pq}{4}\,t^2s^2, \qquad S^\star_{2,2} = \frac{\gamma^{(4)}(0)}{24}\,t^4 n\varkappa_4 = \frac{1-6pq}{24}\,t^4\tau^2\varkappa_4,$$

$$S^\star_{3,k} = \Big(\frac{\gamma^{(3)}(0)}{2}\Big)^2\binom{n}{2}\frac{n}{n-1}\,L_k = \frac{4pq-1}{8}\,\tau^4 L_k, \quad k = 1, 2,$$

$$L_1 = s^2t^4\varkappa_2^2, \quad L_2 = \frac{t^6}{9}\,\varkappa_3^2.$$

Clearly, (3.32) is a consequence of

$$S_1 = S_1^\star, \qquad S_2 = S_2^\star + R_2^\star, \qquad S_3 = S_3^\star + R_3^\star \tag{3.33}$$

and

$$\gamma^{n-1}(s) R_2^\star \sim 0, \qquad \gamma^{n-2}(s) R_3^\star \sim 0. \tag{3.34}$$

Here

$$R_2^\star = \frac{\gamma^{(4)}(0)}{6}\, s t^3 n \varkappa_3, \qquad R_3^\star = \Big(\frac{\gamma^{(3)}(0)}{2}\Big)^2 \binom{n}{2} (R_{3,1}^\star + R_{3,2}^\star + R_{3,3}^\star),$$

with $R_{3,k}^\star$ given in (3.36) and (3.37) below.

Let us prove (3.33). The first two identities follow from (3.1). To prove the last one note that

$$S_3 = \binom{n}{2} \mathbf{E} \hat V_{1,1} \hat V_{1,2}, \qquad \hat V_{1,j} = t X_j \mathbf{E}_\theta v_1(s + \theta t X_j), \quad j = 1, 2. \tag{3.35}$$

A simple calculation shows that

$$\begin{aligned}
&\mathbf{E} \hat V_{1,1} \hat V_{1,2} = \Big(\frac{\gamma^{(3)}(0)}{2}\Big)^2 \Big(\hat L + R_{3,1}^\star + R_{3,2}^\star\Big), \qquad (3.36)\\
&\hat L = s^2 t^4 \mathbf{E} X_1^2 X_2^2 + \frac{1}{9}\, t^6 \mathbf{E} X_1^3 X_2^3, \quad R_{3,1}^\star = \frac{2}{3}\, s t^5 \mathbf{E} X_1^3 X_2^2,\\
&R_{3,2}^\star = s^4 t^2 \mathbf{E} X_1 X_2 + 2 s^3 t^3 \mathbf{E} X_1^2 X_2 + \frac{2}{3}\, s^2 t^4 \mathbf{E} X_1^3 X_2.
\end{aligned}$$

Finally, by Lemma 5.6, we have

$$\hat L = \frac{n}{n-1}\,(L_1 + L_2) + R_{3,3}^\star, \qquad R_{3,3}^\star = -\frac{s^2 t^4}{n-1}\,\varkappa_4 - \frac{t^6}{9}\,\frac{\varkappa_6}{n-1}\,. \tag{3.37}$$

Collecting the expressions (3.36) and (3.37) in (3.35), we obtain (3.33). The proof of (3.34) is given in Appendix I. We arrive at (3.32).

We next show that

$$\gamma^{n-2}(s) S_3^\star \sim \gamma^n(s) S_3^\star, \qquad \gamma^{n-1}(s) S_j^\star \sim \gamma^n(s) S_j^\star, \quad j = 1, 2, \tag{3.38}$$

$$\gamma^n(s) S_{j,k}^\star \sim \gamma^n(s) g_{j,k}, \qquad j = 1, 2, 3, \quad k = 1, 2. \tag{3.39}$$

Clearly, (3.32), (3.38) and (3.39) imply (3.31). The proof of (3.38) is given in Appendix I.

Let us prove (3.39). For $k = 2$, (3.39) is obvious, since $S^{\star}_{j,2} = g_{j,2}$, $j = 1, 2, 3$. In order to prove (3.39), for $k = 1$ and $j = 1, 2, 3$, we show that

$$\gamma^n(s) S^{\star}_{1,1} \sim S^{\star}_{1,1} + S^{*}_{1,1}, \qquad \gamma^n(s) S^{\star}_{j,1} \sim S^{\star}_{j,1}, \qquad j = 2, 3, \tag{3.40}$$

$$S^{\star}_{1,1} \sim 0, \qquad S^{*}_{1,1} \sim g_{1,1}, \qquad S^{\star}_{j,1} \sim g_{j,1}, \qquad j = 2, 3, \tag{3.41}$$

$$g_{j,1} \sim \gamma^n(s) g_{j,1}, \qquad j = 1, 2, 3. \tag{3.42}$$

Here we denote

$$S^{*}_{1,1} = (it)^2 \, \frac{1 - 4pq}{12} \, \tau^2 s^4.$$

The proof of (3.40) and (3.42) is based on the inequalities of Lemma 5.7 and is given in Appendix I. Here we prove (3.41). Before the proof, we introduce some notation. For $k = 0, 1, 2, \ldots$, denote

$$\xi_k = I_k(\mathbb{R}), \qquad \xi^{*}_k = I_k(\{|s| \le \pi\}), \qquad \xi^{\star}_k = I_k(\{|s| > \pi\}),$$

where

$$I_k(A) = \int_A s^k \exp\{-s^2\tau^2/2\} ds.$$

Let η_k denote the k th moment of a standard normal random variable. Clearly, for any $k = 0, 1, 2, \ldots$, we have

$$\xi_k = \xi^{*}_k + \xi^{\star}_k, \qquad \xi_k = \eta_k \tau^{-k-1} (2\pi)^{1/2}, \qquad |\xi^{\star}_k| \le c(k) \tau^{-k-3}, \tag{3.43}$$

where $c(k)$ denotes a number depending only on k.

To prove the first equivalence relation of (3.41), note that

$$\int_{|s| \le \pi} S^{\star}_{1,1} e^{-\zeta_0/2} ds = i^3 t^2 e^{-t^2/2} \, \frac{q - p}{2} \, \xi^{*}_1$$

and use the bound $|\xi^{*}_1| \ll \tau^{-4}$, which follows from (3.43) and the equality $\eta_1 = 0$.

Furthermore, since $\eta_0 = 1$, we obtain from (3.43)

$$\xi^{*}_k = \eta_k \tau^{-k-1} (2\pi)^{1/2} - \xi^{\star}_k, \qquad \tau^{-1} (2\pi)^{1/2} = \xi^{*}_0 + \xi^{\star}_0.$$

Substitution of the second identity into the first one gives

$$\xi^{*}_k = \eta_k \tau^{-k} \xi^{*}_0 + r_k, \qquad r_k = \eta_k \tau^{-k} \xi^{\star}_0 - \xi^{\star}_k, \quad |r_k| \ll \tau^{-k-3}. \tag{3.44}$$

The last inequality follows from bounds (3.43) for $|\xi^{\star}_j|$.

To prove the second equivalence relation of (3.41), note that

$$\int_{|s|\le\pi} S^*_{1,1} e^{-\zeta_0/2} ds = (it)^2 e^{-t^2/2}\, \frac{1-4pq}{12}\, \tau^2 \xi^*_4,$$
$$\int_{|s|\le\pi} g_{1,1} e^{-\zeta_0/2} ds = (it)^2 e^{-t^2/2}\, \frac{1-4pq}{4}\, \tau^{-2} \xi^*_0,$$

and, by (3.44), $\xi^*_4 - \eta_4\tau^{-4}\xi^*_0 = r_4$ with $\eta_4 = 3$. We obtain

$$\int_{|s|\le\pi} (S^*_{1,1} - g_{1,1}) e^{-\zeta_0/2} ds = (it)^2 e^{-t^2/2}\, \frac{1-4pq}{12}\, \tau^2 r_4.$$

Since, by (3.44), $|r_4| \ll \tau^{-7}$, the integral of the right side with respect to the measure $dt/|t|$ is $\ll \tau^{-5}$. This implies $S^*_{1,1} \sim g_{1,1}$.

The proof of the last two equivalence relations of (3.41) is similar: using (3.44) with $k=2$, we replace s^2 in $S^\star_{j,1}$ by $\eta_2\tau^{-2} = \tau^{-2}$ for $j = 2,3$. The proof of (3.41) is complete.

Step 2. Proof of (3.9). In view of (3.13) and the inequalities $\lambda^{-1} \ll \tau$, $I_{[1,T]}(1) \ll \ln T$, it suffices to show that

$$\prod_{j=1}^{n} |\beta(s + tx_j)| \le e^{-\tau^2\rho(\tau^2)}, \qquad \text{for} \quad |s| \le \pi, \quad H \le |t| \le T. \tag{3.45}$$

The identity $|\beta(u)|^2 = 1 - 2pq(1-\cos(u))$ combined with $1 + x \le e^x$ gives

$$\prod_{j=1}^{n} |\beta(s + tx_j)| \le \exp\{-\tau^2 \mathbf{E}\big(1 - \cos(s + tX_1)\big)\}. \tag{3.46}$$

By Lemma 2.1, we have

$$\inf\big\{\mathbf{E}\big(1 - \cos(s+tX_1)\big) : |s| \le \pi, \quad H \le |t| \le \tau^3\big\} \ge \rho(\tau^2).$$

Therefore, (3.45) follows from (3.46). The proof of (1.4) is complete.

Let us prove (1.3). To this aim, we apply (3.5) to G_1 and $T = \tau^2$. Therefore, in order to prove (1.3), it suffices to show that

$$\frac{1}{T}\max_x |G^{(1)}(x)| \ll \frac{\beta_4}{\tau^2}, \quad I_{[H,T]}(\hat G_1) \ll \frac{\beta_4}{\tau^2}, \quad I_{[H,T]}(\hat F) \ll \tau e^{-\tau^2\rho(\tau)} \ln\tau$$

and

$$I_{[0,H]}(\hat F - \hat G_1) \ll \frac{\beta_4}{\tau^2}. \tag{3.47}$$

The first three inequalities are proved in much the same way as (3.6), (3.7) and (3.9), respectively. The proof of (3.47) is almost the same as that of (3.8), but simpler. To avoid the repetition, we do not present the proof of (3.47) here. The theorem is proved.

Proof of Corollary 1. Denote $a_n^2 = 1 + (n-1)^{-1}$. We have

$$\Delta_j^* \le \Delta_j + \tilde{\Delta}_j, \qquad \tilde{\Delta}_j = \sup_x |G_j(x\, a_n) - G_j^\star(x)|, \quad j = 1, 2. \tag{3.48}$$

Expanding

$$G_j(x\, a_n) = G_j(x) + G_j^{(1)}(x) x (a_n - 1) + \dots$$

and using the inequality $|a_n - 1 + n^{-1}/2| \ll n^{-2}$, one obtains

$$\tilde{\Delta}_1 \ll \frac{1}{n} + \frac{\beta_3}{\tau n}, \qquad \tilde{\Delta}_2 \ll \frac{1}{n^2} + \frac{\beta_3}{\tau n} + \frac{1 + \beta_4 + \beta_3^2}{\tau^2 n}.$$

These inequalities combined with (3.2) imply $\tilde{\Delta}_1 \ll \beta_4/\tau^2$ and $\tilde{\Delta}_2 \ll \beta_5/\tau^3$. Collecting the bounds for $\tilde{\Delta}_j$ and Δ_j (Theorem 1) in (3.48) we complete the proof of the corollary.

The proof of Theorem 2 and Secs. 4 and 5 will appear in a subsequent issue of this journal under the title "One– and two–term Edgeworth expansions for a finite population sample mean. Exact results. II".

References

Albers, W., Bickel, P.J. and van Zwet, W.R., *Asymptotic expansions for the power of distribution free tests in the one-sample problem*, Ann. Statist. **4** (1976), 108–156.

Babu, Gutti Jogesh and Bai, Z., D., *Mixtures of global and local Edgeworth expansions and their applications*, J. Multivariate Anal. **59** (1996), 282–307..

Bikelis, A., *On the estimation of the remainder term in the central limit theorem for samples from finite populations*, Studia Sci. Math. Hungar. **4** (1974), 345–354..

Bloznelis, M., *A Berry–Esseen bound for finite population Student's statistic*, Ann. Probab. **27** (1999), 2089–2108.

Bloznelis, M. and Götze, F., *An Edgeworth expansion for finite population U-statistics* (1997), To appear in *Bernoulli*.

Erdös, P. and Rényi, A., *On the central limit theorem for samples from a finite population*, Publ. Math. Inst. Hungar. Acad. Sci. **4** (1959), 49–61.

Feller, W., *An Introduction to Probability Theory and Its Applications. Vol II.* (1971), John Wiley & Sons, New York.

Hoeffding, W., *Probability inequalities for sums of bounded random variables*, J. Amer. Stat. Assoc. **58** (1963), 13–30.

Höglund, T., *Sampling from a finite population. A remainder term estimate*, Scand. J. Statist. **5** (1978), 69–71.

Kokic P.N. and Weber N.C., *An Edgeworth expansion for U-statistics based on samples from finite populations*, Ann. Probab. **18** (1990), 390–404.

Petrov, V.V., *Limit Theorems in Probability Theory. Sequences of Independent Random Variables* (1995), Oxford University Press, New York.

Rao, C.R. and Zhao, L.C., *Berry–Esseen bounds for finite-population t-statistics*, Stat.& Probab. Letters **21** (1994), 409–416.

Robinson J., *An asymptotic expansion for samples from a finite population*, Ann. Statist. **6** (1978), 1005–10011.

Zhao Lin Cheng and Chen Xi Ru, *Normal approximation for finite-population U-statistics*, Acta Mathematicae Applicatae Sinica **6** (1990), 263–272.