AN EDGEWORTH EXPANSION FOR FINITE POPULATION U-STATISTICS

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Abstract. Suppose that U is a U-statistic of degree two based on N random observations drawn without replacement from a finite population. For the distribution of a standardized version of U we construct an Edgeworth expansion with remainder $O(N^{-1})$ provided that the linear part of the statistic satisfies a Cramér type condition.

1. Introduction and results

Let $\mathcal{A} = \{a_1, \dots, a_n\}$ denote a population of size n and let $\mathcal{H} : \mathcal{A} \times \mathcal{A} \to R$ denote symmetric function of its two arguments. By $X_1, \dots, X_N$, $N \le n$, we denote random variables with values in $\mathcal{A}$ such that $X = \{X_1, \dots, X_N\}$ represents a random sample from $\mathcal{A}$ of size N drawn without replacement, i.e., $\mathbf{P}\{X = B\} = \binom{n}{N}^{-1}$ for any subset $B \subset \mathcal{A}$ of size N. We shall investigate the second-order asymptotics of the distribution of the statistic

$$U = \sum_{1 \le i < j \le N} \mathcal{H}(X_i, X_j).$$

We assume that the statistic is centered. Write

$$U = L + Q, \tag{1.1}$$

where

$$L = \sum_{i=1}^{N} g_1(X_i) \qquad \text{respectively} \qquad Q = \sum_{1 \le i < j \le N} g_2(X_i, X_j)$$

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is the linear, respectively the quadratic part of the statistic. Here

$$g_1(x) = (N-1)\,t(x), \qquad t(x) = \frac{n-1}{n-2}\,\mathbf{E}\big(\mathcal{H}(X_1,\,X_2)\,|\,X_1 = x\big)$$

and

$$g_2(x_1,\ x_2) = \mathcal{H}(x_1,\,x_2) - t(x_1) - t(x_2).$$

Since

$$\mathbf{E}\big(g_2(X_1,\,X_2)\,|\,X_1 = x\big) = 0, \qquad \text{for all} \quad x \in \mathcal{A}, \tag{1.2}$$

the random variables $g_1(X_i)$ and $g_2(X_j,\,X_k)$, $1 \le i \le N$, $1 \le j < k \le N$, (and thus the parts L and Q) are uncorrelated.

If the linear part L dominates the statistic, for large N, the distribution of U can be approximated by a Gaussian distribution using the Central Limit Theorem (CLT).

The asymptotic normality of linear statistics based on samples drawn without replacement from finite populations has been studied by a number of authors. Erdös and Rényi (1959) proved the CLT under very mild conditions. The rate of convergence in the CLT was first studied by Bikelis (1969). Berry–Esseen bounds of order $O(N^{-1/2})$ were obtained by Höglund (1978). Robinson (1978) proved the validity of an Edgeworth expansion with a remainder of order $O(N^{-3/2})$, see also Bickel and van Zwet (1978).

Nandi and Sen (1963) studied the asymptotic behavior of finite population U-statistics and showed that under proper regularity conditions the sequence of distributions of normalized U-statistics converges to the standard normal distribution. The rate of this convergence was investigated by Zhao and Chen (1987, 1990), Kokic and Weber (1990, 1991) and, as a particular case of the rate of convergence of general multivariate sampling statistics, by Bolthausen and Götze (1993). In the case of independent and identically distributed observations the second order asymptotic theory has been developed for U-statistics, see Bickel (1974), Götze (1979), Callaert, Janssen and Veraverbeke (1980), Bickel, Götze and van Zwet (1986), and for more general asymptotically normal symmetric statistics, see Bentkus, Götze and van Zwet (1997) ([BGZ] for short). In contrast to the independent case, there are only a few results concerned with higher order asymptotics of non-linear finite population statistics. Babu and Singh (1985) proved the validity of an Edgeworth expansion with a remainder $o(N^{-1/2})$ for finite population multivariate sample mean and applied this result to establish expansions for statistics that can be represented as smooth functions of multivariate sample means, e.g. Student's t. Kokic and Weber (1990) established one term Edgeworth expansion with the remainder $o(N^{-1/2})$ for finite population U-statistics of degree 2.

In comparison to the results described above we shall provide an *explicit* remainder term of order $O(N^{-1})$ for finite population U-statistics which is optimal assuming a Cramér condition on the linear term only. The proof is based on a finite population variant of Hoeffding's decomposition as well as the Erdös-Rényi representation and some ideas of [BGZ] like the data dependent smoothing.

Assume that

$$\sigma^2 = N\,\mathbf{E} g_1^2(X_1) > 0.$$

The distribution function of the standardized statistic, $F(x) = \mathbf{P}\{U \le x\,\sigma\}$, will be approximated by the one term Edgeworth expansion,

$$G(x) = \Phi\big(\frac{x}{\sqrt{q}}\big) - \frac{(q-p)\,q^{-1/2}\,\alpha + 3\,q^{1/2}\,\kappa}{6\,\sigma^3\,N^{1/2}}\,\Phi'''\big(\frac{x}{\sqrt{q}}\big). \tag{1.3}$$

Here $\Phi(x)$ is the standard normal distribution function,

$$p = N/n, \qquad q = 1 - p$$

and

$$\alpha = N^{3/2}\,\mathbf{E} g_1^3(X_1), \qquad \kappa = N^{5/2}\,\mathbf{E} g_2(X_1, X_2)\,g_1(X_1)\,g_1(X_2). \tag{1.4}$$

We shall derive bounds for the remainder

$$\Delta = \sup_{x \in R} \big| F(x) - G(x) \big|.$$

To prove the validity of an Edgeworth expansion, i.e., to establish bounds for Δ, in addition to moment conditions one needs to impose a smoothness condition, cf. Bickel and Robinson (1982). For instance, in the classical case of standardized sums $S = (Y_1 + \cdots + Y_N)/\sqrt{N}$ of independent and identically distributed (i.i.d.) random variables $Y_1, \ldots, Y_N$ such that $\mathbf{E}\,Y_1 = 0$, $\mathbf{E}\,Y_1^2 = 1$ and $\mathbf{E}\,Y_1^4 < \infty$ asymptotic expansions for the distribution F_S of S with the remainder $O(N^{-1})$ are obtained assuming Cramér's condition (C),

$$\sup_{|t| > a} \big| \,\mathbf{E}\,\exp\{i\,t\,Y_1\}\, \big| < 1. \tag{C}$$

Bentkus, Götze and van Zwet (1997) introduced a local version of Cramér's condition (C), namely,

$$\rho_{Y_1}(a, b) := 1 - \sup_{a \le |t| \le b} \big| \mathbf{E} \exp\{i\,t\,Y_1\} \big| > 0. \tag{C'}$$

Condition (C') (with $a = 1/\mathbf{E}\,|Y_1|^3$ and $b = N^{1/2}$) is somewhat weaker than (C) but still sufficient to prove the validity of Edgeworth expansions for F_S up to an order $O(N^{-1})$. This modification is useful in more general situations, where Y_1 depends on N in an implicit way, see [BGZ].

For a sufficiently small absolute constant b_1 like, e.g., $b_1 = 0.00144$, we shall assume that the distribution of the random variable $Z = \sqrt{N}\, g_1(X_1)/\sigma$ satisfies condition (C') with $a' = b_1/\mathbf{E}\,|Z|^3$ and $b' = N^{1/2}$, i.e.,

$$\rho = \rho_Z(a',\, b') > 0. \tag{1.5}$$

Write

$$\beta_r = \mathbf{E}\,\bigl|N^{1/2}\, g_1(X_1)\bigr|^r \quad \text{and} \quad \gamma_r = \mathbf{E}\,\bigl|N^{3/2}\, g_2(X_1, X_2)\bigr|^r, \quad r = 1, 2, \dots . \tag{1.6}$$

Then the following estimate for the remainder Δ holds.

Theorem 1.1. *There exists an absolute constant $A > 0$ such that*

$$\sup_{x\in R}\bigl|F(x) - G(x)\bigr| \le \frac{A}{N}\,\frac{\beta_4 + \gamma_4}{\rho^2\, q^2\, \sigma^4}\,.$$

For linear statistics we obtain the following result.

Theorem 1.2. *There exists an absolute constant $B > 0$ such that*

$$\Bigl|\mathbf{P}\{L \le x\} - \Phi\bigl(\tfrac{x}{\sqrt{q}}\bigr) + \frac{(q-p)\, q^{-1/2}\, \alpha}{6\, \sigma^3\, N^{1/2}}\, \Phi'''\bigl(\tfrac{x}{\sqrt{q}}\bigr)\Bigr| \le \frac{B}{N}\,\frac{\beta_4}{\rho^2\, q\, \sigma^4}\,.$$

The estimates in Theorems 1.1 and 1.2 hold for any fixed sample size N, population size n and functions $\mathcal{H}$. If β_4/σ^4 and γ_4/σ^4 are bounded and q and ρ are bounded away from 0 as $N \to \infty$ and $n \to \infty$, then these results establish Edgeworth expansions with the remainder $O(N^{-1})$.

The case where $n \to \infty$ and N is fixed corresponds to the i.i.d. situation. By the law of large numbers we obtain a corollary for independent observations:

Let $\mathcal{E}$ denote a measurable space and let $\mathcal{X}_1,\, \mathcal{X}_2, \dots$ be i.i.d. random variables with values in $\mathcal{E}$. Write

$$\tilde{U} = \sum_{1\le i<j\le N} \mathcal{H}(\mathcal{X}_i,\, \mathcal{X}_j).$$

Here $\mathcal{H} : \mathcal{E}\times\mathcal{E} \to R$ denotes a measurable function symmetric in its two arguments such that $\mathbf{E}\,\mathcal{H}^2(\mathcal{X}_1, \mathcal{X}_2) < \infty$. We assume that $\mathbf{E}\,\tilde{U} = 0$ and decompose

$$\tilde{U} = \sum_{i=1}^{N} \tilde{g}_1(\mathcal{X}_i) + \sum_{1\le i<j\le N} \tilde{g}_2(\mathcal{X}_i, \mathcal{X}_j).$$

Here $\tilde{g}_1$ and $\tilde{g}_2$ are defined in the same way as g_1 and g_2, but using $\tilde{t}(x) = \mathbf{E}\bigl(\mathcal{H}(\mathcal{X}_1,\, \mathcal{X}_2)\,|\,\mathcal{X}_1 = x\bigr)$ instead of $t(x)$. Let $\tilde{\sigma}$, $\tilde{\alpha}$, $\tilde{\beta}_k$, $\tilde{\gamma}_k$, $k = 2, 3, 4$, and $\tilde{\kappa}$ denote the moments of $\tilde{g}_1(\mathcal{X}_1)$ and $\tilde{g}_2(\mathcal{X}_1, \mathcal{X}_2)$ corresponding to σ, α, β_k, γ_k and κ. We shall assume that

$$\tilde{\rho} = \rho_{\mathcal{Z}}\bigl(\tilde{a}, \tilde{b}\bigr) > 0, \qquad \text{where} \quad \mathcal{Z} = \sqrt{N}\, \tilde{g}_1(\mathcal{X}_1)/\tilde{\sigma},$$

and where $\tilde{a} = b_1/\mathbf{E}\,|\mathcal{Z}|^3$ and $\tilde{b} = \sqrt{N}$. Then we have

Corollary 1.3. *There exists an absolute constant $A > 0$ such that*

$$\Bigl|\mathbf{P}\{\tilde{U} \le \tilde{\sigma} x\} - \Phi(x) + \frac{\tilde{\alpha} + 3\tilde{\kappa}}{6\,\tilde{\sigma}^3\,N^{1/2}}\,\Phi'''(x)\Bigr| \le \frac{A}{N}\;\frac{\tilde{\beta}_4 + \tilde{\gamma}_4}{\tilde{\rho}^2\,\tilde{\sigma}^4}\,.$$

Hence, Theorem 1.1 which yields this result as a special case may be regarded as a partial extension of the result of [BGZ] to a simple random sampling model. They proved the validity of an Edgeworth expansion with the remainder $O(N^{-1})$ for general symmetric asymptotically normal statistics based on i.i.d. observations. In the case of U-statistics of degree two their result yields the estimate as in Corollary 1.3 but with a lower moment $\tilde{\gamma}_3/\tilde{\sigma}^3$ instead of $\tilde{\gamma}_4/\tilde{\sigma}^4$ in the remainder.

An example given in Theorem 1.4 in [BGZ] shows that a Cramér type condition on the linear part and the existence of moments of arbitrarily high order of the linear and quadratic parts of the statistic (based on i.i.d. observations) are not sufficient to obtain a higher order approximations (those with remainders $o(N^{-1})$) to the distribution function of U. Hence, in this sense Corollary 1.3 and thus Theorem 1.1 are the best possible. To prove the validity of an Edgeworth expansion with remainder $o(N^{-1})$ one needs in addition to impose a smoothness condition on the distribution of the quadratic part, see, e.g., Bickel, Götze and van Zwet (1986).

Let us compare our results with those of Robinson (1978) and Kokic and Weber (1990). Robinson (1978) proved the validity of a two term Edgeworth expansion with the remainder $O(N^{-3/2})$ for linear statistics like L in (1.1) assuming the following Cramér type condition. This condition, first used in Albers, Bickel and van Zwet (1976), requires for a random variable Z that there exists an $\varepsilon > 0$ such that

$$\tau_Z(\varepsilon, a, b) = 1 - \sup_{s\in R,\, a\le|t|\le b} \mathbf{P}\bigl\{t\,Z \in \mathcal{L}^{\varepsilon} + s\bigr\} > 0. \qquad (c)$$

Here $\mathcal{L} = \{2\,\pi\,r,\ r = 0, \pm1,\ \pm2, \dots\}$ and $\mathcal{B}^{\varepsilon}$ denotes the ε neighborhood of a set $\mathcal{B} \subset R$. Notice that $\varepsilon_1 \le \varepsilon_2$ implies $\tau_Z(\varepsilon_1, a, b) \ge \tau_Z(\varepsilon_2, a, b)$. Robinson assumed that given $C' > 0$

$$\text{there exist}\quad \varepsilon,\, \delta > 0 \quad \text{and} \quad C > 0 \quad \text{such that} \quad \tau_Z(\varepsilon, a, b) > \delta, \qquad (1.7)$$

for

$$Z = \sqrt{N}\,g_1(X_1)/\sigma, \quad a^{-1} = \max_{1\le i\le n} |z_i|/C' \quad \text{and} \quad b^{-1} = p\,\mathbf{E}\,|Z|^5/(C\,N).$$

Here $\{z_1, \dots, z_n\}$ denotes the set of values of the r.v. Z. Note that $\max_i |z_i| = \max_i |z_i| \mathbf{E}Z^2 \ge \mathbf{E}|Z|^3$, because of $\mathbf{E}Z^2 = 1$. For a sequence of finite population linear statistics, say (L_n), Robinson's (1978) Theorem establishes an Edgeworth expansion with the remainder $O(N^{-3/2})$ provided that β_5/σ^5 is bounded, p and q

are bounded away from 0 and (1.7) holds with ε, δ and C not depending on n as $n \to \infty$. Robinson's (1978) result was used by Kokic and Weber (1990) to show $\Delta = o(N^{-1/2})$. The bounds for the remainders in these papers involve constants which implicitly depend on p.

In Section 2 we compare conditions (c) and (1.5). Proofs of the Theorems 1.1, 1.2 and the Corollary 1.3 are given in Sections 3 and 4. Auxiliary results are collected in Section 5.

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2. Smoothness conditions

Modifications of Cramér's condition (C) that ensure the validity of Edgeworth expansions for sums of random variables assuming a finite number of values only were considered by Albers, Bickel and van Zwet (1976), van Zwet (1982), Does (1983), Schneller (1989), see also Bickel and Robinson (1982). In this section we show that a Cramér type condition used in Albers, Bickel and van Zwet (1976) and Robinson (1978) is equivalent to that introduced in [BGZ], namely, that the conditions (1.5) and (c) are equivalent. More specifically, given a random variable Z and numbers $0 < a < b$, (1.5) implies $\tau_Z(\rho, a, b) > \rho/4$. Furthermore, if (c) holds for some $\varepsilon > 0$, then $\rho_Z(a, b) > \varepsilon^2 \tau_Z(\varepsilon, a, b)/\pi^2$, see Lemma 2.1 below.

In order to check condition (c) one needs to maximize a bivariate function over the set $(s, t) \in [-\pi, \pi] \times \{a \le |t| \le b\}$. Such a (maximization) problem can be difficult to solve numerically. A symmetrization argument suggests a version of condition (c) which is easier to check. Let Z' denote an independent copy of Z and let $Z^* = Z - Z'$ denote a symmetrization of Z. The condition

$$\text{there exists} \quad \varepsilon > 0 \quad \text{such that} \quad \tau_Z^*(\varepsilon, a, b) = 1 - \sup_{a \le |t| \le b} \mathbf{P}\{t\, Z^* \in \mathcal{L}^\varepsilon\} > 0 \quad (c^*)$$

requires the estimation of the maximum of an univariate function only. Condition (c*) was proposed by V.Bentkus. Notice that $\varepsilon_1 < \varepsilon_2$ implies $\tau_Z^*(\varepsilon_2, a, b) \le \tau_Z^*(\varepsilon_1, a, b)$. The following Lemma 2.1 shows that conditions (c^*) and (c) are equivalent.

Write

$$\delta_Z(a, b) = 1 - \sup\{\mathbf{E}\, \cos(t\, Z + s) : \ s \in R, \ a \le |t| \le b\}. \tag{2.1}$$

Lemma 2.1. *Let Z be a random variable. For $0 < a < b$ and $0 < \varepsilon < \pi$ write*

$$\rho = \rho_Z(a, b), \quad \tau_\varepsilon = \tau_Z(\varepsilon, a, b), \quad \tau_\varepsilon^* = \tau_Z^*(\varepsilon, a, b), \quad u = \pi^{-1} \varepsilon\, \tau_\varepsilon^* \quad \text{and} \quad v = \pi^{-1} \varepsilon\, \tau_\varepsilon.$$

The following inequalities hold:

$$\frac{\varepsilon^2 \tau_\varepsilon}{\pi^2} \le \rho \le 4\, \tau_\rho, \quad \frac{\varepsilon^2 \tau_\varepsilon^*}{\pi^2} \le \rho \le 4\, \tau_\rho^*, \quad \tau_v^* \ge \frac{\varepsilon^2 \tau_\varepsilon}{2\, \pi^2}, \quad \tau_u \ge \frac{\varepsilon^2\, \tau_\varepsilon^*}{4\, \pi^2}, \quad \delta_Z(a, b) \ge \rho.$$

The proof of Lemma 2.1 is elementary, see Bloznelis and Götze (1997).

3. Proofs

Throughout the proof we shall assume without loss of generality that $\beta_2 = 1$.

Since the proof of our main result, Theorem 1.1, is rather complex and involved we shall first outline the various steps. In the first step, choosing $m \approx \ln N$ we replace the statistic U by

$$U_1 = L' + U', \qquad U' = g_1(X_{m+1}) + \cdots + g_1(X_N) + \sum_{m+1 \le i < j \le N} g_2(X_i, X_j), \tag{3.1}$$

where

$$L' = l(X_1) + \cdots + l(X_m), \quad \text{with} \quad l(x) = g_1(x) + l_0(x), \quad l_0(x) = \sum_{j=m+1}^{N} g_2(x, X_j).$$

is a conditionally linear statistic given $X_{m+1}, \dots, X_N$. Write

$$F_X(x) = \mathbf{P}\{U_1 \le x | X_{m+1}, \dots, X_N\}, \qquad f_1(t) = \mathbf{E}\big(\exp\{itU_1\} | X_{m+1}, \dots, X_N\big).$$

In the second step we construct upper/lower bounds for conditional probabilities

$$F_X(x+) \le \frac{1}{2} + \text{V.P.} \int_R \mathrm{e}\{-x\,t\} \, \frac{1}{H} \, K\big(\frac{t}{H}\big) \, f_1(t) \, dt,$$
$$F_X(x-) \ge \frac{1}{2} - \text{V.P.} \int_R \mathrm{e}\{-x\,t\} \, \frac{1}{H} \, K\big(\frac{-t}{H}\big) \, f_1(t) \, dt,$$

where $F(x+) = \lim_{z \downarrow x} F(z)$, $F(x-) = \lim_{z \uparrow x} F(z)$ and V.P. denotes Cauchy's Principal Value (Prawitz (1972) smoothing lemma). The bounded weight function $K(t/H)$, vanishing for $|t| > H$, and the cuttoff $H = O(N)$ are specified below. Taking expectations of the left and right hand sides respectively we obtain upper and lower bounds for the distribution function $F_1(x) = \mathbf{P}\{U_1 \le x\}$, see (3.7) and (3.8) below.

In the third step we construct a bound for the integral of $f_1(t)K(t/H)$ over the region $c\,N^{1/2} \le |t| \le H$. In the classical case of linear statistic the bounds for characteristic function for large values of t, like $c\,N^{1/2} \le |t| \le C\,N$, are implied by Cramer's condition. We write

$$|f_1(t)| \le |\mathbf{E}(\exp\{it(l(X_1) + \cdots + l(X_m))\} | X_{m+1}, \dots X_N)|$$

and show that the Cramer condition $|\mathbf{E} \exp\{itg_1(X_1)\}| < 1 - \rho$, (we do not require $|\mathbf{E} \exp\{itl(X_1)\}| < 1 - \rho$)! in combination with a suitable choice of the cuttoff $H = H(X_{m+1}, \dots, X_N)$ implies a bound like $|f_1| \le (1 - c\rho)^m$, for some $0 <$

$c < 1$. The techniques are somewhat complicated by the fact that $X_1, \dots, X_m$ are exchangeable only and we get the independence via the Erdös-Rényi decomposition for (conditional and unconditional) characteristic functions.

In the next step we interchange the conditional characteristic function with the unconditional one by changing the order of integration with respect to Lebesgue measure and with respect to the distribution of $X_{m+1}, \dots, X_N$, for $|t| \le C N^{1/2}$. Finally, by means of expansions we estimate the difference between the Fourier-Stieltjes transforms of F and G.

Our proofs may be considered as an extension to the case of finite population statistics of techniques used by Bentkus, Götze and van Zwet (1997) in the i.i.d. case. We remark that the approach developed in the present paper applies as well to more general nonlinear symmetric statistics based on samples drawn without replacement from finite populations. These results will appear elsewhere.

3.1. Notation. By $C, C_0, C_1, \dots$ and $c, c_0, c_1, \dots$ we denote generic absolute constants. We shall write $A \ll B$ if $A < C\,B$. The expression $\exp\{i\,x\}$ will be abbreviated by $\mathrm{e}\{x\}$. Write

$$\Theta(t) = \Big(\frac{2}{\pi}\ \frac{\pi - t}{\pi + t}\Big)^2, \quad \mathcal{K} = \big\{a \in \mathcal{A}:\ H_1 |g_1(a)| < b_2\big\}, \qquad H_1 = \frac{b_1 N^{1/2}}{\beta_3}. \tag{3.2}$$

Here b_1 is the same constant as in (1.5) and b_2 denotes a sufficiently small absolute constant like, e.g., $b_2 = 0.075$.

Let $\nu = \{\nu_1, \dots, \nu_n\}$ be a sequence of independent Bernoulli random variables with probabilities $\mathbf{P}\{\nu_i = 1\} = p$ and $\mathbf{P}\{\nu_i = 0\} = q$, for $i = 1, 2, \dots, n$. Write

$$\beta(t) = \mathbf{E}\,\mathrm{e}\{(\nu_1 - p)t\}, \qquad \tau = \sqrt{n\,p\,q}, \qquad \delta = \delta(b_1/\beta_3,\, N^{1/2}),$$

where $\delta(\cdot,\cdot)$ is defined by (2.1). Let $\overline{A} = \big(A_1, A_2, \dots, A_n\big)$ denote random permutation which is uniformly distributed on the permutations of the ordered set $(a_1, \dots, a_n)$ of elements of $\mathcal{A}$, independent of ν. By $\mathbf{E}^*$ we denote the conditional expectation given $\overline{A}$, i.e., $\mathbf{E}^*(\cdots) = \mathbf{E}(\cdots|\overline{A})$. For $k = 1, 2, \dots,$ write $\Omega_k = \{1, \dots, k\}$ and $D_k = \Omega_N \setminus \Omega_k$. Given $D = \{i_1, i_2, \dots, i_k\} \subset \Omega_n$, $\mathbf{E}^{i_1, \dots, i_k}$ and $\mathbf{E}^D$ denote the conditional expectation given $A_{i_1}, \dots, A_{i_k}$.

3.2. Proof of Theorem 1.1

We may and shall assume, that for sufficiently small $c_0 > 0$,

$$\frac{\beta_4}{q\,N} < c_0, \quad \frac{\ln N}{\delta\,N} < c_0, \quad \frac{\gamma_2}{\delta^2 q^2 N} < c_0, \quad \frac{\ln N}{\delta\,q\,n} < c_0. \tag{3.3}$$

Indeed, if (3.3) fails, then the bound of Theorem 1.1 follows from the inequalities $F(x) \le 1$ and $|G(x)| \ll 1 + q^{-1/2}\beta_4^{1/2}/N^{1/2} + q^{1/2}\gamma_2/N^{1/2}$ and $\rho \le \delta$, see Lemma 2.1.

Step 1. Fix an integer $m \approx C_0 \delta^{-1} \ln N$, with sufficiently large C_0, and write

$$\Lambda_m = \sum_{1 \le k < l \le m} g_2(X_k, X_l). \tag{3.4}$$

Note that $U = \Lambda_m + U_1$, where U_1 is given by (3.1). Let F_1 denote the probability distribution function of U_1 and $\Delta_1 = \sup_x \big|F_1(x) - G(x)\big|$. We have

$$\Delta \le \Delta_1 + \mathbf{P}\big\{|\Lambda_m| \ge N^{-1}\delta^{-3/2}\big\} + \delta^{-3/2} N^{-1} \max_x |G'(x)|.$$

By Chebyshev's inequality and the inequality $\mathbf{E}|\Lambda_m|^3 \ll m^6 \mathbf{E}\,|g_2(X_1, X_2)|^3$,

$$\mathbf{P}\{|\Lambda_m| \ge N^{-1}\delta^{-3/2}\} \le \delta^{9/2} N^3 \mathbf{E}\,|\Lambda_m|^3 \ll \delta^{-3/2}\, \gamma_3\, N^{-3/2} \ln^6 N.$$

Finally, using the bound $|G'(x)| \ll \beta_4/q + \gamma_2$ we obtain

$$\Delta \ll \Delta_1 + N^{-1}\, \delta^{-3/2} \big(\beta_4/q + \gamma_2 + \gamma_3\big). \tag{3.5}$$

Therefore, in order to prove the theorem it suffices to bound Δ_1.

Smoothing. Let k be an integer approximately equal to $(N+m)/2$. Put $\mathcal{I}_0 = \{m+1, \dots, N\}$, $\mathcal{J}_0 = \Omega_n \setminus \mathcal{I}_0$, $\mathcal{J}_1 = \mathcal{J}_0 \cup \big\{m+1, \dots, k\big\}$ and $\mathcal{J}_2 = \mathcal{J}_0 \cup \big\{k+1, \dots, N\big\}$. Given $\overline{A}$ define (random) subpopulations $\mathcal{A}_i = \{A_k,\, k \in \mathcal{J}_i\}$, $i = 0, 1, 2$ and let A_i^* be random variables uniformly distributed in $\mathcal{A}_i$, $i = 0, 1, 2$, independent of ν. Write

$$v_1(a) = \sum_{j=k+1}^{N} g_2(a, A_j), \qquad v_2(a) = \sum_{j=m+1}^{k} g_2(a, A_j), \tag{3.6}$$
$$H = N\,\delta / \big(32\, q^{-1} N\, (\Theta_1 + \Theta_2) + 1\big), \qquad \Theta_i = \mathbf{E}^* |v_i(A_i^*)|, \quad i = 1, 2.$$

Notice, that Θ_1 is a function of the r.v. $A_{k+1}, \dots, A_N$, and Θ_2 is a function of r.v. $A_{m+1}, \dots, A_k$.

Step 2. Split the sample as follows. Put $X_j = A_j$, for $m < j \le N$. The rest of the sample, $X_1, \dots, X_m$, is obtained by simple random sampling without replacement from the (random) subpopulation $\mathcal{A}_0$.

An application of Prawitz's (1972) smoothing lemma conditionally, given $X_{m+1}, \dots, X_m$, or equivalently, given $A_{m+1}, \dots, A_N$, gives

$$F_1(x+) \le \frac{1}{2} + \mathbf{E}\,\text{V.P.} \int_R \mathrm{e}\{-x\,t\}\, \frac{1}{H}\, K\Big(\frac{t}{H}\Big)\, f_1(t)\, dt, \tag{3.7}$$

$$F_1(x-) \ge \frac{1}{2} - \mathbf{E}\,\text{V.P.} \int_R \mathrm{e}\{-x\,t\}\, \frac{1}{H}\, K\Big(\frac{-t}{H}\Big)\, f_1(t)\, dt, \tag{3.8}$$

where $2\,K(s) = K_1(s) + i\,K_2(s)/(\pi\, s)$, see, e.g., [BGZ]. Here

$K_1(s) = \mathbb{I}\{|s| \le 1\}\,(1-|s|)$ and $K_2(s) = \mathbb{I}\{|s| \le 1\}\,((1-|s|)\,\pi\, s\,\cot(\pi\, s) + |s|)$.

Combining (3.7) and the inversion formula,

$$G(x) = \frac{1}{2} + \frac{i}{2\,\pi} \lim_{M\to\infty} \text{V.P.} \int_{|t|\le M} \mathrm{e}\{-t\,x\}\,\hat{G}(t)\,\frac{dt}{t}\,, \tag{3.9}$$

we get, see e.g. [BGZ],

$$\begin{aligned} &F_1(x+) - G(x) \le \mathbf{E} I_1 + \mathbf{E} I_2 + \mathbf{E} I_3, \qquad (3.10)\\ &I_1 = \frac{1}{2}\,H^{-1} \int_R \mathrm{e}\{-x\,t\}\,K_1\big(\frac{t}{H}\big)\,f_1(t)\,dt,\\ &I_2 = \frac{i}{2\,\pi}\ \text{V.P.} \int_R \mathrm{e}\{-x\,t\}K_2\big(\frac{t}{H}\big)\big(f_1(t) - \hat{G}(t)\big)\,\frac{dt}{t}\,,\\ &I_3 = \frac{i}{2\,\pi}\ \text{V.P.} \int_R \mathrm{e}\{-x\,t\}\Big(K_2\big(\frac{t}{H}\big) - 1\Big)\hat{G}(t)\,\frac{dt}{t}\,, \end{aligned}$$

where V.P. means also that one should take $\lim_{M\to\infty}$, if it is necessary.

Combining (3.8) and (3.9) we obtain a bound for $G(x) - F_1(-x)$ similar to (3.10). We shall bound $F_1(x+) - G(x)$ only. To this aim we prove that

$$|\mathbf{E} I_1| + |\mathbf{E}(I_2 + I_3)| \ll N^{-1}\big(\beta_4/q + \delta^{-1}(\delta^{-1} + q^{-1}) + \delta^{-2} q^{-2}(\gamma_2^{1/2} + \gamma_2) + \gamma_4\big). \tag{3.11}$$

The analogous bound for $G(x) - F_1(x-)$ can be derived in the same way. Using these bounds, (3.5) and the inequality $\delta \ge \rho$, see Lemma 2.1, we obtain the estimate of the theorem. In the remaining part of the proof we verify (3.11).

Step 3. **Estimate for $|\mathbf{E} I_1|$.** We shall replace the random bound H in the integral I_1 by a non–random one and $K_1(t/H)$ by 1. We have $|\mathbf{E} I_1| \le |\mathbf{E} I_4| + \mathbf{E} I_5$ where,

$$I_4 = H^{-1} \int_{\mathcal{Z}} \mathrm{e}\{-t\,x\}K_1\big(\frac{t}{H}\big)f_1(t)dt, \qquad \mathcal{Z} = \{t \in R:\ |t| \le H_1\},$$
$$I_5 = H^{-1} \int_{H_1 \le |t|} K_1\big(\frac{t}{H}\big)\big|f_1(t)\big|dt \le H^{-1} \int_{H_1 \le |t| \le H} \big|f_1(t)\big|dt.$$

Next we construct bounds for $\mathbf{E} I_5$ and $|\mathbf{E} I_4|$, see (3.12) and (3.19) below. It follows from these bounds that $|\mathbf{E} I_1|$ does not exceed the right-hand side of (3.11).

Let us show

$$\mathbf{E} I_5 \ll N^{-2}\,\beta_3. \tag{3.12}$$

For this purpose we represent $f_1(t)$ in Erdös-Rényi (1959) form conditionally, given $A_{m+1}, \dots, A_N$. Let $\nu^* = \{\nu_1^*, \dots, \nu_n^*\}$ be a sequence of independent Bernoulli random variables independent of $\overline{\mathcal{A}}$ and with probabilities

$$\mathbf{P}\{\nu_i^* = 1\} = p^*, \qquad \mathbf{P}\{\nu_i^* = 0\} = q^*, \qquad p^* = \frac{m}{n-(N-m)}, \qquad q^* = 1 - p^*.$$

Write $S_* = \sum_{k\in\mathcal{J}_0}(\nu_k^* - p^*)$ and $L_* = \sum_{k\in\mathcal{J}_0} l(A_k)\nu_k^*$. We have

$$f_1(t) = \mathbf{P}^{-1}\{S_* = 0\}\,\frac{1}{2\pi}\int_{-\pi}^{\pi} W ds, \qquad W = \mathbf{E}^*\, \mathrm{e}\{t(L_* + U') + sS_*\}. \tag{3.13}$$

We shall construct an upper bound for $|W|$. We have

$$|W| = \prod_{k\in\mathcal{J}_0} |\beta_*(z(A_k) + tv(A_k))|, \qquad \text{where} \quad \beta_*(x) = \mathbf{E}\,\mathrm{e}\{(\nu_1^* - p^*)x\}.$$

Here we denote

$$z(a) = tg_1(a) + s \qquad \text{and} \qquad v(a) = v_1(a) + v_2(a),$$

with $v_i(a)$ given by (3.6). Then we apply the identity $|\beta_*(x)|^2 = 1 - 2p^*q^*(1-\cos x)$ to $x = z(a) + tv(a)$ and expand the cosine function in powers of $tv(a)$ to get

$$\begin{aligned} &|\beta_*(z(a) + tv(a))|^2 \le u_1(a) + u_2(a), && (3.14)\\ &u_1(a) = 1 - 2p^*q^*\big(1 - \cos(z(a))\big), && u_2(a) = 2\,p^*\,q^*\,|t\,v(a)|. \end{aligned}$$

Furthermore, we may assume that $p^* \le 8^{-1}$ (it is a consequence of the last inequality of (3.3) provided that c_0 is small enough). This inequality implies $u_1(a) \ge 1/2$ and therefore,

$$u_1(a) + u_2(a) \le u_1(a)(1 + 2u_2(a)). \tag{3.15}$$

Combining (3.14) and (3.15) we obtain

$$|W|^2 \le W_1W_2, \quad W_1 = \prod_{k\in\mathcal{J}_0} u_1(A_k), \quad W_2 = \prod_{k\in\mathcal{J}_0}(1 + 2u_2(A_k)). \tag{3.16}$$

To estimate W_2 we apply the arithmetic-geometric mean inequality,

$$W_2 \le \Big(\frac{1}{|\mathcal{J}_0|}\sum_{k\in\mathcal{J}_0}(1 + 2u_2(A_k))\Big)^{|\mathcal{J}_0|} = \big(\mathbf{E}^*(1 + 2u_2(A_0^*))\big)^{n-N+m} \tag{3.17}$$

and use (5.2) to bound $\mathbf{E}^*|v(A_0^*)| \le q^{-1}(\Theta_1+\Theta_2)$. Thus, for $|t|\le H$, we get

$$\mathbf{E}^*(1+2u_2(A_0^*)) \le 1+4p^*q^*q^{-1}H(\Theta_1+\Theta_2) \le 1+p^*q^*\,\frac{\delta}{8}\,\le \exp\{p^*q^*\,\frac{\delta}{8}\,\}.$$

This inequality in combination with (3.17) implies $W_2^{1/2}\le \exp\{mq^*\delta/16\}$. Now in view of (3.16) and (3.13) we obtain, for $|t|\le H$,

$$|f_1(t)| \ll W_3W_1^{1/2}, \qquad W_3 = m^{1/2}\exp\{m\,q^*\delta/16\}.$$

Here we estimated $P^{-1}\{S_*=0\}\ll m^{1/2}$, see (5.16). We have

$$\mathbf{E}I_5 \le \frac{1}{H_1}\,\mathbf{E}\int_{H_1\le|t|\le H}|f_1(t)|dt \le \frac{W_3}{H_1}\int_{H_1\le|t|\le N}\mathbf{E}W_1^{1/2}dt. \tag{3.18}$$

To bound $\mathbf{E}W_1^{1/2}$ we apply Hölder's inequality and Theorem 4 of Hoeffding (1963), see Section 5 below,

$$(\mathbf{E}W_1^{1/2})^2 \le \mathbf{E}W_1 \le \big(\mathbf{E}u_1(A_1)\big)^{|\mathcal{J}_0|}.$$

Note that $\mathbf{E}u_1(A_1)\le 1-2p^*q^*\delta$, for $H_1\le|t|\le N$, by the choice of δ. Therefore,

$$\mathbf{E}W_1^{1/2} \le (1-2p^*q^*\delta)^{(n-N+m)/2} \le \exp\{-p^*q^*\delta(n-N+m)\} = \exp\{-mq^*\delta\}.$$

Combining this bound with (3.18) and using the inequality $q^*=1-p^*\ge 7/8$ we obtain (3.12), provided that the constant C_0 (in the definition of m) is sufficiently large.

It remains to bound $\mathbf{E}\,I_4$. We shall show

$$|\mathbf{E}I_4| \ll \mathcal{R}_0, \quad \mathcal{R}_0 = N^{-1}\,\delta^{-2}\,(1+q^{-2}\,\gamma_2) + N^{-1}\,\delta^{-1}\,q^{-1}(1+q^{-1}\,\gamma_2^{1/2}). \tag{3.19}$$

It follows from the inequality $\big|K_1(u)-1\big|\le|u|$ that

$$I_4 = I_6+R, \qquad I_6 = H^{-1}\int_{\mathcal{Z}}\mathrm{e}\{-t\,x\}f_1(t)dt, \tag{3.20}$$
$$\mathbf{E}|R| \le \mathbf{E}H^{-1}\int_{\mathcal{Z}}|t|H^{-1}dt = H_1^2\mathbf{E}H^{-2}\ll\mathcal{R}_0,$$

where in the last step we applied (5.1). Recall that $U=U_1+\Lambda_m$. Now, using the inequality $|\,\mathrm{e}\{t\Lambda_m\}-1|\le|t\Lambda_m|$ we obtain

$$I_6 = I_7+R, \qquad I_7 = H^{-1}\int_{\mathcal{Z}}\mathrm{e}\{-tx\}f_2(t)dt, \quad f_2(t)=\mathbf{E}^{D_m}\,\mathrm{e}\{tU\}, \tag{3.21}$$
$$\mathbf{E}|R| \le \mathbf{E}H^{-1}\int_{\mathcal{Z}}\mathbf{E}^{D_m}|t\Lambda_m|dt \le H_1^2\mathbf{E}H^{-1}|\Lambda_m| \ll \mathcal{R}_0,$$

where in the last step we used the inequality $|\Lambda_m H^{-1}| \le \Lambda_m^2 + H^{-2}$ and moment inequalities (5.1) and (5.3). Next we replace I_7 by

$$I_8 = H^{-1} \int_{\mathcal{Z}_0} \mathrm{e}\{-tx\} f_2(t) dt, \qquad \mathcal{Z}_0 = \{C_1 q^{-1} \le |t| \le H_1\}, \tag{3.22}$$

where C_1 is a sufficiently large constant. We have $I_7 = I_8 + R$ with $|R| \le 2C_1 q^{-1} H^{-1}$. Hölder's inequality in combination with (5.1) gives $\mathbf{E}|R| \ll \mathcal{R}_0$.

It remains to estimate $\mathbf{E} I_8$. Write $I_8 = 32 q^{-1} \delta^{-1}(J_1 + J_2) + \delta^{-1} J_3$, where

$$J_i = \int_{\mathcal{Z}_0} \mathrm{e}\{-tx\} f_2(t) \Theta_i dt, \quad i = 1, 2, \quad \text{and} \quad J_3 = N^{-1} \int_{\mathcal{Z}_0} \mathrm{e}\{-tx\} f_2(t) dt.$$

In order to complete the proof of (3.19) we shall show

$$\mathbf{E} J_i \ll N^{-1}(1 + \gamma_2), \qquad i = 1, 2, 3. \tag{3.23}$$

Let us prove (3.23) for $i = 1, 2$. By the symmetry, it suffices to consider the case where $i = 1$. Recall that the random variable Θ_1 is a function of $X_{k+1}, \dots, X_N$. In view of the inequality $k \approx (N + m)/2 > m$ we can write

$$\mathbf{E}\Theta_1 f_2(t) = \mathbf{E}\Theta_1 f_3(t), \qquad \text{where} \qquad f_3(t) = \mathbf{E}\big(\mathrm{e}\{tU\} \,|\, X_{k+1}, \dots, X_N\big). \tag{3.24}$$

Given $t \in \mathcal{Z}_0$ choose an integer $m_1 = C_2 N t^{-2} \ln |t|$. Here C_2 is a sufficiently large constant to be specified latter. Given C_2 we may choose C_1 in (3.22) large enough so that $m_1 < 10^{-1} q N < k$, for $t \in \mathcal{Z}_0$. Write $\mathcal{J}_3 = \Omega_{m_1} \cup (\Omega_n \setminus \Omega_N)$. We shall represent our sample $X_1, \dots, X_N$ as follows. For $m_1 + 1 \le j \le N$, put $X_j = A_j$. The remaining part of the sample (the observations $X_1, \dots, X_{m_1}$) represents a simple random sample drawn without replacement form the set $\mathcal{A}_3 = \{A_k,\, k \in \mathcal{J}_3\}$. Let A_3^* be a random variable uniformly distributed in $\mathcal{A}_3$. Put

$$v_3(a) = \sum_{k=m_1+1}^{N} g_2(a, A_j) \qquad \text{and} \qquad \Theta_3 = \mathbf{E}^* |v_3(A_3^*)|. \tag{3.25}$$

Notice that the random variable Θ_3 is a function of $A_{m_1+1}, \dots, A_N$.

Write $U = U_1^\star + \Lambda_{m_1}$, where $U_1^\star = L_\star' + U_\star'$, with

$$L_\star' = l_\star(X_1) + \dots + l_\star(X_{m_1}), \qquad l_\star(x) = g_1(x) + l_0^\star(x), \quad l_0^\star(x) = \sum_{j=m_1+1}^{N} g_2(x, X_j).$$

and with $U_\star'$ defined by (3.1), but with m replaced by m_1. Furthermore, Λ_{m_1} is given by (3.4).

Using the inequality $|\,\mathrm{e}\{t\Lambda_{m_1}\}-1|\le|t\Lambda_{m_1}|$ we obtain

$$\mathbf{E}\Theta_1 f_3(t)=\mathbf{E}\Theta_1 f_4(t)+R_1,\quad f_4(t)=\mathbf{E}\big(\,\mathrm{e}\{tU_1^\star\}|\,X_{k+1},\dots,X_N\big),\tag{3.26}$$

where $|R_1|\le\mathbf{E}|t\Lambda_{m_1}|\Theta_1$. Furthermore, combining (3.24) and (3.26) we get

$$\mathbf{E}J_1=\mathbf{E}J_4+R,\qquad J_4=\int_{\mathcal{Z}_0}\mathrm{e}\{-tx\}f_4(t)\Theta_1 dt,\tag{3.27}$$

$$|R|\le\int_{\mathcal{Z}_0}\mathbf{E}|t\Lambda_{m_1}|\Theta_1 dt\ll N^{-1}\gamma_2.$$

In the last step we invoke (5.1), (5.3) and apply Hölder's inequality to get

$$\mathbf{E}|\Lambda_{m_1}|\Theta_1\le(\mathbf{E}\Lambda_{m_1}^2)^{1/2}(\mathbf{E}\Theta_1^2)^{1/2}\ll m_1N^{-5/2}\gamma_2\ll t^{-2}\ln|t|N^{-3/2}\gamma_2$$

and bound the integral of the function $|t|^{-1}\ln|t|$ over the region $\mathcal{Z}_0$ by $\ln^2 N$.

To estimate $\mathbf{E}J_4$ observe that, by the inequality $m_1<k$,

$$\mathbf{E}\Theta_1 f_4=\mathbf{E}\Theta_1 f_5,\qquad f_5=\mathbf{E}\big(\mathrm{e}\{tU'_\star\}|X_{m_1+1},\dots,X_N\big).$$

Therefore, $\mathbf{E}J_4=\mathbf{E}J_5$, where J_5 is defined in the same way as J_4, see (3.27), but with f_4 replaced by f_5. Furthermore,

$$\mathbf{E}J_5=\mathbf{E}J_6+R,\qquad J_6=\int_{\mathcal{Z}_0}\mathrm{e}\{-tx\}f_5(t)\Theta_1 I_\Theta dt,\tag{3.28}$$

$$I_\Theta=I\{N\Theta_3\le c_1|t|\},\qquad |R|\le\int_{\mathcal{Z}_0}\mathbf{E}\Theta_1 I\{N\Theta_3>c_1|t|\}dt\ll N\mathbf{E}\Theta_1\Theta_3.$$

Here c_1 denotes a small positive constant to be determined below. Combining (5.3) and Hölder's inequality we get $|R|\ll N^{-1}\gamma_2$.

In order to bound $\mathbf{E}J_6$ we represent f_5 in the Erdös-Rényi (1959) form, see (3.29). Let $\nu^\star=\{\nu_1^\star,\dots,\nu_n^\star\}$ be a sequence of independent Bernoulli random variables independent of $\overline{\mathcal{A}}$ and with probabilities

$$\mathbf{P}\{\nu_i^\star=1\}=p^\star,\qquad\mathbf{P}\{\nu_i^\star=0\}=q^\star,\qquad p^\star=\frac{m_1}{n-(N-m_1)},\qquad q^\star=1-p^\star.$$

Write $S_\star=\sum_{k\in\mathcal{J}_3}(\nu_k^\star-p^\star)$ and $L_\star=\sum_{k\in\mathcal{J}_3}l_\star(A_k)\nu_k^\star$ and $\tau_\star^2=m_1q^\star$. We have

$$f_5(t)=\lambda_\star\int_{-\pi\tau_\star}^{\pi\tau_\star}W_\star ds,\qquad W_\star=\mathbf{E}^*\,\mathrm{e}\{t(L_\star+U'_\star)+\frac{s}{\tau_\star}S_\star\},\tag{3.29}$$

with $\lambda_\star^{-1}=2\pi\tau_\star\mathbf{P}\{S_\star=0\}$ satisfying $\lambda_\star\ll 1$, by (5.16).

Combining (3.28) and (3.29) we get

$$\mathbf{E}J_6 \ll \int_{\mathcal{Z}_0} dt \int_{-\pi\tau_\star}^{\pi\tau_\star} \mathbf{E}\Theta_1 I_\Theta |W_\star| ds. \tag{3.30}$$

In the next step we construct an upper bound for $\mathbf{E}\Theta_1 I_\Theta |W_\star|$. Note that the inequality $m_1 < 10^{-1}qN$ implies $p^\star \le 10^{-1}$. The same argument as above, see (3.16), gives

$$|W_\star|^2 \le W_1^\star W_2^\star \qquad W_1^\star = \prod_{k\in\mathcal{J}_3} u_1^\star(A_k), \quad W_2^\star = \prod_{k\in\mathcal{J}_3} (1+2u_2^\star(A_k)), \tag{3.31}$$

where $u_1^\star$ and $u_2^\star$ are given by (3.14), but with p^*, q^*, $z(a)$ and $v(a)$ replaced by $p^\star$, $q^\star$, $z_\star(a) := tg_1(a) + s/\tau_\star$ and $v_3(a)$ (defined in (3.25)) respectively.

To bound $W_2^\star$ we proceed as in (3.17) and obtain

$$W_2^\star \le \big(1 + 2\mathbf{E}^* u_2^\star(A_3^*)\big)^{n-N+m_1} = \big(1 + 4p^\star q^\star |t|\Theta_3\big)^{n-N+m_1} \le \exp\{4m_1 q^\star |t|\Theta_3\}.$$

Furthermore, by our choice of m_1, $I_\Theta (W_2^\star)^{1/2} \le \exp\{2q^\star C_2 c_1 \ln|t|\}$. Therefore, in view of (3.31),

$$\mathbf{E}\Theta_1 I_\Theta |W_\star| \le \exp\{2q^\star C_2 c_1 \ln|t|\} \mathbf{E}\Theta_1 (W_1^\star)^{1/2}. \tag{3.32}$$

Now we apply Hölder's inequality and invoke (5.1) to get,

$$E\Theta_1 (W_1^\star)^{1/2} \le (\mathbf{E}\Theta_1^2)^{1/2} (\mathbf{E}W_1^\star)^{1/2} \ll N^{-1}\gamma_2^{1/2} (\mathbf{E}W_1^\star)^{1/2}. \tag{3.33}$$

To bound $\mathbf{E}W_1^\star$ we apply Theorem 4 of Hoeffding (1963) and obtain

$$\mathbf{E}W_1^\star \le \big(\mathbf{E}u_1^\star(A_1)\big)^{|\mathcal{J}_3|} = \big(1 - 2p^\star q^\star M\big)^{n-N+m_1} \le \exp\{-2m_1 q^\star M\}, \tag{3.34}$$

where $M = \mathbf{E}\big(1 - \cos z_\star(A_1)\big)$. Combining the inequalities

$$M \ge \mathbf{E}\big(1 - \cos z_\star(A_1)\big) I_{\mathcal{K}}(A_1), \quad I_{\mathcal{K}}(a) = I\{a \in \mathcal{K}\},$$
$$1 - \cos z_\star(a) \ge 2^{-1}\Theta(b_2) z_\star^2(a), \quad a \in \mathcal{K},$$

see (5.15), we get $M \ge 2^{-1}\Theta(b_2)\mathbf{E}z_\star^2(A_1) I_{\mathcal{K}}(A_1)$. Now by Lemma 5.3,

$$M \ge b_3\big(t^2 N^{-1} + s^2 \tau_\star^{-2}\big), \qquad \text{where} \qquad b_3 = 2^{-1}\Theta(b_2)(1 - 2b_1 b_2^{-1})$$

is a positive constant (because of our choice of $0 < 2\,b_1 < b_2$ in (1.5) and (3.2)). Substituting this inequality in (3.34) and using $q^\star = 1 - p^\star \geq 9/10$, we obtain

$$\mathbf{E}W_1^\star \leq \exp\{-2b_3 m_1 q^\star (t^2 N^{-1} + s^2 \tau_\star^{-2})\} \leq \exp\{-2b_3(\,\frac{9}{10}\,C_2 \ln|t| + s^2)\}. \quad (3.35)$$

Finally, collecting the inequalities (3.32), (3.33) and (3.35) in (3.30) we get

$$\mathbf{E}J_6 \ll N^{-1}\gamma_2^{1/2} \int_{\mathcal{Z}_0} dt \int_{-\pi\tau_\star}^{\pi\tau_\star} \exp\{C_2(2c_1 - \frac{9}{10}\,b_3)\ln|t| - b_3 s^2\} ds. \quad (3.36)$$

Choosing $c_1 = b_3/4$ and $C_2 = 4/b_3$ we obtain bounded integrals in (3.36) and thus, $\mathbf{E}J_6 \ll N^{-1}\gamma_2^{1/2} \leq N^{-1}(1+\gamma_2)$. This inequality together with (3.27) and (3.28) completes the proof of (3.23) in the case where $i = 1$.

The proof of (3.23) in the case where $i = 3$ is similar, but simpler: just write N^{-1} instead of Θ_1 in the proof above.

Collecting the bounds (3.20), (3.21), (3.22) and (3.23) we obtain (3.19).

Step 4. **Estimate for $|\mathbf{E}(I_2 + I_3)|$.** Write $I_2 + I_3 = i\,(2\,\pi)^{-1}(I_9 + I_{10} - I_{11} + I_{12})$,

$$I_9 = \int_{|t|\leq H_1} \mathrm{e}\{-t\,x\}\,\frac{f_1(t) - \hat{G}(t)}{t}\,dt, \quad I_{10} = \int_{H_1\leq|t|\leq H} \mathrm{e}\{-t\,x\}K_2\big(\,\frac{t}{H}\,\big)f_1(t)\,\frac{dt}{t}\,,$$

$$I_{11} = \int_{|t|>H_1} \mathrm{e}\{-t\,x\}\,\hat{G}(t)\,\frac{dt}{t}\,, \qquad I_{12} = \int_{|t|\leq H_1} \mathrm{e}\{-t\,x\}\,\big(K_2\big(\,\frac{t}{H}\,\big) - 1\big)f_1(t)\,\frac{dt}{t}\,.$$

Using (3.3) it is easy to show that $|\mathbf{E}I_{11}| \ll q^{-1}\beta_4/N + \gamma_2/N$.

Using the inequality $|K_2(s) - 1| \leq c\,s^2$, and invoking (5.1), we get

$$|\mathbf{E}I_{12}| \ll \mathbf{E}H^{-2}\,H_1^2 \ll \delta^{-2}\,N^{-1}\,(1 + q^{-2}\gamma_2).$$

To bound $|\mathbf{E}I_{10}|$ write

$$|\mathbf{E}I_{10}| \leq \mathbf{E}I_{13}, \qquad I_{13} = \int_{H_1\leq|t|\leq H} |f_1(t)|\,\frac{dt}{|t|}\,.$$

The bound $\mathbf{E}I_{13} \ll N^{-1}\beta_3$ is obtained in a similar way as (3.12) above.

Collecting these inequalities we get

$$\big|\mathbf{E}\,(I_2 + I_3)\big| \ll |\mathbf{E}I_9| + N^{-1}q^{-1}\beta_4 + N^{-1}\delta^{-2}(1 + q^{-2}\gamma_2). \quad (3.37)$$

In order to complete the proof of (3.11) we shall show

$$|\mathbf{E}I_9| \ll \delta^{-2}N^{-1}(1+\gamma_2) + N^{-1}(q^{-1}\beta_4 + \gamma_4). \quad (3.38)$$

We have

$$\mathbf{E}\, I_9 = \int_{|t|\le H_1} \mathrm{e}\{-t\, x\} \left(\mathbf{E}\ \mathrm{e}\{t\, U_1\} - \hat{G}(t)\right) \frac{dt}{t}\,.$$

Recall that $U_1 = U - \Lambda_m$. Write $\mathrm{e}\{tU_1\} = \mathrm{e}\{tU\}\,\mathrm{e}\{-t\Lambda_m\}$ and expand $\mathrm{e}\{-t\Lambda_m\}$ in powers of $-it\Lambda_m$ to get $\mathbf{E}I_9 = I_{14} - i\, I_{15} + R$, where

$$I_{14} = \int_{-H_1}^{H_1} \mathrm{e}\{-tx\}\, \frac{\hat{F}(t) - \hat{G}(t)}{t}\, dt, \qquad I_{15} = \int_{-H_1}^{H_1} \mathrm{e}\{-tx\}\mathbf{E}\Lambda_m\, \mathrm{e}\{tU\}dt,$$

and where $|R| \le H_1^2\, \mathbf{E}\, \Lambda_m^2 \ll \delta^{-2}\, N^{-1}\gamma_2$, by (5.3). By the symmetry, $\mathbf{E}I_{15} = \binom{m}{2}\mathbf{E}I_{16}$, where I_{16} is defined in the same way as I_{15}, but with Λ_m replaced by $g_2(X_{N-1}, X_N)$. The bound $\mathbf{E}I_{16} \ll N^{-3/2}(1+\gamma_2)$ is obtained in a similar way as (3.23): just take $\tilde{f}_3 = \mathbf{E}\,(\mathrm{e}\{t\, U\}\big| X_{N-1}, X_N)$ instead of f_3 and $g_2(X_{N-1}, X_N)$ instead of Θ_1 in the proof of (3.23) (for $i = 1$). We obtain

$$|\mathbf{E}I_9 - I_{14}| \ll \delta^{-2}\, N^{-1}\,(1+\gamma_2).$$

In the next section on expansions, see (4.1) below, we shall show $|I_{14}| \ll N^{-1}\,(\beta_4/q + \gamma_4)$ thus completing the proof of (3.38).

Proof of Theorem 1.2. The bound of the theorem follows from (3.11). Just note that for a linear statistic we have $g_2(x, y) = 0$, for any $x,\, y \in \mathcal{A}$. In particular, we do not need to assume that the last two inequalities of (3.3) hold.

Proof of Corollary 1.3. The corollary follows from Theorem 1.1, by the law of large numbers (LLN) for U- statistics, see, e.g., Serfling (1980). Given N, the function $\mathcal{H}$ and a sequence of i.i.d. observations $\mathcal{X}_1, \mathcal{X}_2, \dots$, introduce the sequence of finite populations $\mathcal{A}_n = \{\mathcal{X}_1, \dots, \mathcal{X}_n\}$ and the corresponding sequence of U-statistics, (U_n). Given $x \in R$, apply the bound of Theorem 1.1 to the sequence of probabilities $P_n\{x\} = \mathbf{P}\{U_n \le x\}$. By LLN, we obtain $\lim_n P_n\{x\} = \mathbf{P}\{\tilde{U} \le x\}$. Furthermore, the moments of the linear and quadratic parts of U_n in the expansion and in the remainder (in the estimate of Theorem 1.1) converge to the corresponding moments of the statistics $\tilde{U}$ thus proving Corollary 1.3.

4. Expansions

Throughout the section we assume that $\beta_2 = 1$ and that the inequalities (3.3) hold. With H_1 given in (3.2) we shall prove the inequality

$$\int_{|t|\le H_1} |t|^{-1}\big|\hat{F}(t) - \hat{G}(t)\big|dt \ll \mathcal{R}, \quad \mathcal{R} := \frac{1}{N}\,\big(\,\frac{\beta_4}{q} + \gamma_4\big). \tag{4.1}$$

Introduce some notation. Let $\theta_1, \theta_2, \dots$ denote independent random variables uniformly distributed in $[0,1]$ and independent of all other random variables considered. For a vector valued smooth function H we use the Taylor expansion

$$H(x) = H(0) + H'(0)\,x + \cdots + H^{(n)}(0)\,\frac{x^n}{n!} + \mathbf{E}_{\theta_1} H^{(n+1)}(\theta_1\,x)\,(1-\theta_1)^n\,\frac{x^{n+1}}{n!}\,.$$

Here $\mathbf{E}_{\theta_1}$ denotes the conditional expectation given all the random variables but θ_1. In particular, we have the mean value formula, $H(x) - H(0) = \mathbf{E}_{\theta_1} H'(\theta_1\,x)\,x$.

Given a sum $S = s_1 + \cdots + s_k$ denote $S^{(i)} = S - s_i$ and, similarly, $S^{(i,j)} = S - s_i - s_j$.

Using the fact that the distribution of U coincides with the conditional distribution of

$$\begin{aligned} U_0 :&= \sum_{1\le i<j\le n} h(A_i, A_j)\nu_i\nu_j \\ &= \sum_{i=1}^n g_1(A_i)(\nu_i - p) + \sum_{1\le i<j\le n} g_2(A_i, A_j)(\nu_i - p)(\nu_j - p), \end{aligned}$$

conditioned on the event $\mathbb{B} := \{S_0 = N\}$, where $S_0 = \sum_{i=1}^n \nu_i$, we obtain

$$\hat{F}(t) = \frac{1}{2\pi\,\mathbf{P}\{\mathbb{B}\}} \int_{-\pi}^{\pi} \mathbf{E}\,\mathrm{e}\{t\,U_0 + s(S_0 - N)\}ds,$$

see Erdös and Rényi (1959). Write

$$T = \sum_{i=1}^n T_i, \quad T_i = z_i(\nu_i - p), \quad z_i = t\,x_i + s\tau^{-1}, \quad x_i = g_1(A_i), \quad \tau = (n\,p\,q)^{1/2},$$

$$Q = \sum_{1\le i<j\le n} Q_{i,j}, \qquad Q_{i,j} = t\,y_{i,j}(\nu_i - p)(\nu_j - p), \qquad y_{i,j} = g_2(A_i, A_j).$$

We have $T + Q = t\,U_0 + s\tau^{-1}(S_0 - N)$ and, therefore,

$$\hat{F}(t) = \lambda \int_{-\pi\tau}^{\pi\tau} \mathbf{E}\,\mathrm{e}\{T + Q\}ds, \qquad \lambda^{-1} = 2\,\pi\,\tau\,\mathbf{P}\{\mathbb{B}\}.$$

Höglund (1978) showed that $2^{-1/2}\pi \le \lambda^{-1} \le (2\pi)^{1/2}$, see (5.16).

We shall approximate the integrand $\mathbf{E}\,\mathrm{e}\{T + Q\}$ by the sum $h_1 + h_2$, where

$$h_1 = \mathbf{E}\;\mathrm{e}\{T\}, \qquad h_2 = i^3 \binom{n}{2} \mathbf{E}\,\mathrm{e}\{T^{(1,2)}\}V, \qquad V = Q_{1,2}\,T_1\,T_2.$$

Proof of (4.1). Clearly, it suffices to prove the inequalities

$$\int_{|t|\le H_1} \big|\lambda \int_{|s|\le \pi\,\tau} (h_1+h_2)ds - \hat{G}(t)\big| \,\frac{dt}{|t|} \,\ll \mathcal{R}, \tag{4.2}$$

$$I := \int_{|t|\le H_1} \lambda \int_{|s|\le \pi\,\tau} \big|\mathbf{E}\ \mathrm{e}\{T+Q\} - (h_1+h_2)\big| ds \,\frac{dt}{|t|} \,\ll \mathcal{R}. \tag{4.3}$$

Note that in the i.i.d. case the inequality corresponding to (4.2) is proved in [BGZ], Lemma 6.1. We prove (4.2) by combining the proof of this lemma with the proof of the Berry–Esseen bound for finite population sample mean given in Höglund (1978). For details we refer to Bloznelis and Götze (1997), Lemma 4.3.

To prove (4.3), we expand $\mathrm{e}\{T+Q\}$ in powers of T_i and $Q_{i,j}$. In order to ensure the integrability (with respect to the measure $ds\,dt/|t|$) of the remainders of these expansions we split $\mathbf{E}\ \mathrm{e}\{T+Q\}$ into a product of two functions (different for different values of s and t) so that the first one is the characteristic function of a sum of conditionally independent random variables and vanishes sufficiently fast as s and t tends to infinity. This type of approach has been used earlier by Helmers and van Zwet (1982), van Zwet (1984), Götze and van Zwet (1991), [BGZ] in the i.i.d. situation.

Introduce the set $\mathcal{Z} = \{(s,t) : |s| \le \pi\tau,\ |t| \le H_1\}$. For technical reasons it is convenient to split the integral I in two parts $I = I_1 + I_2$ according to the regions $\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2$,

$$Z_1 = \mathcal{Z} \cap \{|t| \le C_3 q^{-1}\} \quad \text{and} \quad Z_2 = \mathcal{Z} \cap \{C_3 q^{-1} < |t| \le H_1\}. \tag{4.4}$$

Here C_3 denotes a sufficiently large absolute constant. We choose $C_3 = 600\Theta^{-1}(1)$. In Lemma 4.1 we prove the bound $I_2 \ll \mathcal{R}$. The proof of the bound $I_1 \ll \mathcal{R}$ is similar but simpler. We skip it and refer to Bloznelis and Götze (1997), Lemma 4.2, for details. It remains to prove Lemma 4.1.

Note that, for any $i, j, i_1, \ldots, i_k \in \Omega_n$ such that $\{i,j\} \cap \{i_1,\ldots,i_k\} = \emptyset$ we have

$$\mathbf{E}^{i_1,\ldots,i_k}|y_{i,j}|^r \le c(k,r)\mathbf{E}|y_{i,j}|^r, \quad \mathbf{E}^{i_1,\ldots,i_k}|x_j|^r \le c(k,r)\mathbf{E}|x_j|^r, \quad r \ge 0. \tag{4.5}$$

We need to introduce some more notation. Given $D = \{i,j,\ldots,k\} \subset \Omega_n$, let $\mathbf{E}_{\{D\}} = \mathbf{E}_{\{i,j,\ldots,k\}}$ (respectively $\mathbf{E}_{[D]} = \mathbf{E}_{[i,j,\ldots,k]}$) denote the conditional expectation given all the random variables but $\{\nu_j,\ j \in D\}$ (respectively the conditional expectation given $\{\nu_j,\ A_j,\ j \in D\}$). Given $1 \le m \le n$, introduce the random variables

$$\xi_i = t\,(\nu_i - p)\zeta_m(A_i), \qquad \zeta_m(a) = \sum_{j=m+1}^{n} g_2(a, A_j)(\nu_j - p). \tag{4.6}$$

Here $i \in \Omega_m$ and $a \in \mathcal{A} \setminus \{A_{m+1}, \dots, A_n\}$. Given $B \subset \Omega_m$, denote

$$Y_B = \big|\mathbf{E}_{\{B\}}\, \mathrm{e}\{\sum_{i \in B} T_i\}\big|, \qquad Z_B = \big|\mathbf{E}_{\{B\}}\, \mathrm{e}\{\sum_{i \in B}(T_i + \xi_i)\}\big|.$$

Furthermore, given A_i, $i \in B$, let A_B^* denote the random variable uniformly distributed in the set $\{A_i,\, i \in B\}$ and let $\mathbf{E}_B^*$ denote the conditional expectation given all the random variables, but A_B^*. Introduce the random variables

$$\Psi_B = g_B(t) \prod_{k \in B} u_{[1]}^{1/2}(z_k), \qquad \varkappa_B = \alpha N \mathbf{E}_B^* \zeta_m^2(A_B^*), \qquad I_B = I\{\varkappa_B > \delta\}, \quad (4.7)$$

where $\alpha = 2\pi(4\Theta^{-1}(1) + 1)$ and $\delta = \Theta(1)/40$, are constants,

$$g_B(t) = \exp\{pq\, \frac{\delta}{2}\, \frac{|B|}{N}\, t^2\}, \quad u_{[d]}(x) = 1 - \frac{pq}{2}\, \Theta(d) x^2 I\{|x| < d + \pi\}, \ \ d > 0. \quad (4.8)$$

In Lemma 5.4 below, for $|t| \le H_1$ and $|s| \le \pi\tau$, we prove the inequalities

$$Z_B \ll I_B + \Psi_B, \qquad Y_B \ll \Psi_B, \qquad \mathbf{E}^{i_1,\dots,i_4} \Psi_B^r \ll F_B^r, \quad r = 1, 2, \qquad (4.9)$$

where $i_1, \dots, i_4 \in \Omega_n \setminus B$. Here we denote

$$F_B = \exp\{-8\,\delta\, p\, q\, |B|\, N^{-1}(t^2 + s^2/q)\}.$$

We often take $|B| \ge m/4$, with m given by (4.13). In this case we have

$$F_B \le (t^2 + s^2/q)^{-10}. \qquad (4.10)$$

Lemma 4.1. *Assume that $\beta_2 = 1$ and that (3.3) holds. Then*

$$I_2 = \lambda \int_{\mathcal{Z}_2} \frac{\mathbf{E}\, \mathrm{e}\{T + Q\} - (h_1 + h_2)}{|t|}\, ds dt \ll \mathcal{R}, \qquad (4.11)$$

where $\mathcal{Z}_2$ is given by (4.4).

Proof of Lemma 4.1. Given a positive number L and a complex valued function $f(s,t)$ we write $f \prec L$, if

$$\int_{\mathcal{Z}_2} |f(s,t)|\, |t|^{-1} ds\, dt \ll L.$$

Furthermore, for two complex valued functions f, g we write $f \sim g$ if $f - g \prec \mathcal{R}$. In view of the inequality $\lambda \le 2^{1/2}\pi^{-1}$, (4.11) can be written in short as follows,

$$\mathbf{E}\, \mathrm{e}\{T + Q\} \sim h_1 + h_2. \qquad (4.12)$$

Given $(s,t) \in \mathcal{Z}_2$ write $u = t^2 + s^2/q$ and let

$$m = m(s,t) > C_4 q^{-1} n\, u^{-1} \ln u, \qquad C_4 = 300\Theta^{-1}(1), \tag{4.13}$$

denote the smallest integer which is greater than $C_4 q^{-1} n\, u^{-1} \ln u$. A simple calculation shows that $C_4 \le m(s,t) \le C_4 C_3^{-1} n$, for $(s,t) \in \mathcal{Z}_2$. Since $C_4 = C_3/2$ we have $10 \le m(s,t) \le n/2$.

Write $\mu := mpqN^{-1} = C_4 u^{-1} \ln u$. We shall often use the following fact,

$$(t^2)^\alpha (s^2)^\beta \mu^\gamma \prec q^{\beta+1/2} c(\alpha,\beta,\gamma), \qquad \text{for} \quad \gamma > \alpha+\beta+1/2, \quad \alpha,\beta \ge 0.$$

In what follows B always denote the set $\{4,\dots,m\}$. Letters $R, R_1, R_2 \dots$ will denote random variables (remainders) which may be different in different places. This will not cause any misunderstanding if we assume that $R, R_1, R_2, \dots$ always take the latest prescribed values.

Let us prove (4.12). Split $Q = Q_A + Q_D + \xi$ and $T = T_A + T_D$, where

$$Q_A = \sum_{1\le i<j\le m} Q_{i,j}, \quad Q_D = \sum_{m<i<j\le N} Q_{i,j}, \quad \xi = \sum_{1\le i\le m} \xi_i,$$
$$T_A = \sum_{1\le i\le m} T_i, \qquad T_D = \sum_{m<i\le N} T_i,$$

and where ξ_i are given by (4.6). Furthermore, write $W = T_D + Q_D$. We have $T + Q = T_A + Q_A + \xi + W$ and $\mathrm{e}\{T+Q\} = v\,\mathrm{e}\{Q_A\}$, with $v = \mathrm{e}\{W + T_A + \xi\}$.

Expanding in powers of iQ_A and using symmetry we obtain

$$\mathbf{E}\,\mathrm{e}\{T+Q\} = f_1^* + f_2^* + R, \quad f_1^* = \mathbf{E}v, \quad f_2^* = i\binom{m}{2}\mathbf{E}vQ_{1,2}, \tag{4.14}$$

with $|R| \le \mathbf{E}Q_A^2$. By symmetry, we have

$$\mathbf{E}Q_A^2 = \binom{m}{2} p^2 q^2 t^2 \mathbf{E}y_{1,2}^2 \le \mu^2 t^2 N^{-1}\gamma_2 \prec \mathcal{R}.$$

Now (4.14) implies $\mathrm{e}\{T+Q\} \sim f_1^* + f_2^*$.

The rest of the proof consists of two steps. In the first step we show that

$$f_2^* \sim h_3, \qquad h_3 = i^3 \binom{m}{2} \mathbf{E}\,\mathrm{e}\{T^{(1,2)}\} V. \tag{4.15}$$

In the second step we prove

$$f_1^* \sim h_1 + h_4, \qquad \text{where} \qquad h_4 = h_2 - h_3. \tag{4.16}$$

Step 1. We start by showing

$$f_2^* \sim f_3^*, \qquad f_3^* = i\binom{m}{2}\mathbf{E}v_1 Q_{1,2}, \qquad v_1 = \mathrm{e}\{W + T_A + \xi^{(1,2)}\}. \tag{4.17}$$

Write $v = v_1\,\mathrm{e}\{\xi_1 + \xi_2\}$. Expanding the exponent in powers of $(\xi_1 + \xi_2)$ we obtain

$$f_2^* = f_3^* + f_4^* + f_5^* + f_6^*, \qquad f_j^* = i^2\binom{m}{2}\mathbf{E}v_1 Q_{1,2} l_j, \quad j = 4,5,6,$$
$$l_4 = \xi_1 + \xi_2, \quad l_5 = (\xi_1^2 + \xi_2^2)v_2, \quad l_6 = 2\,\xi_1\xi_2 v_2, \quad v_2 = i^2\,\mathrm{e}\{\theta_1(\xi_1 + \xi_2)\}(1-\theta).$$

In order to prove (4.17) we shall show $f_i^* \sim 0$, for $i = 4,5,6$.

To show $f_6^* \sim 0$ we bound $|v_1 v_2| \le 1$ and obtain

$$|f_6^*| \le m^2\mathbf{E}|Q_{1,2}\xi_1\xi_2| = m^2p^2q^2|t|^3\mathbf{E}|y_{1,2}\zeta_m(A_1)\zeta_m(A_2)|$$

Combining the inequalities $|\zeta_m(A_1)\zeta_m(A_2)| \le \zeta_m^2(A_1) + \zeta_m^2(A_2)$ and

$$\mathbf{E}|y_{1,2}|\zeta_m^2(A_i) = pq(n-m)\mathbf{E}|y_{1,2}|y_{i,n}^2 \le qN^{-7/2}\gamma_3, \qquad i = 1,2,$$

and the bound $|t| \le N^{1/2}$ we get $|f_6| \ll \mu^2t^2\gamma_3N^{-1} \prec \mathcal{R}$.

Let us show $f_5^* \sim 0$. By symmetry, it suffices to show $m^2\mathbf{E}v_1v_2Q_{1,2}\xi_1^2 \sim 0$. Expanding the exponent in v_2 in powers of $i\theta\xi_2$ and then the exponent in v_1 in powers of iT_2 we obtain

$$|\mathbf{E}v_1v_2Q_{1,2}\xi_1^2| \le R_1 + R_2, \qquad R_1 = \mathbf{E}|Q_{1,2}\xi_2|\xi_1^2, \quad R_2 = \mathbf{E}|Q_{1,2}T_2|\xi_1^2.$$

Invoking (4.45) and the inequality $|t| \le N^{1/2}$, we get

$$m^2R_1 \ll \mu^2t^4N^{-5/2}\gamma_4 \prec \mathcal{R}, \qquad m^2R_2 \ll \mu^2t^2N^{-1}(1+\gamma_4) \prec \mathcal{R},$$

thus completing the proof of $f_5^* \sim 0$.

Let us show $f_4^* \sim 0$. By symmetry, it suffices to show $m^2R \sim 0$ with $R = \mathbf{E}v_1Q_{1,2}\xi_1$. Expanding v_1 in powers of iT_2 we can replace v_1 by iT_2v_3, with $v_3 = \mathrm{e}\{W + T_A^{(2)} + \xi^{(1,2)} + \theta_1T_2\}$. Now, using the simple bound $|\mathbf{E}_{\{B\}}v_3| \le Z_B$ we obtain

$$|R| \le \mathbf{E}R_1R_2, \quad R_1 = |Q_{1,2}T_2|, \quad R_2 = \mathbf{E}_{[1,2]}|\xi_1|\,Z_B. \tag{4.18}$$

At first we bound R_2. By Hölder's inequality,

$$R_2 \le R_3R_4, \qquad R_3^2 = \mathbf{E}_{[1,2]}\xi_1^2, \qquad R_4^2 = \mathbf{E}_{[1,2]}Z_B^2. \tag{4.19}$$

Furthermore, by (4.9), $R_4^2 \le 2R_5^2 + 2R_6^2$, where

$$R_5^2 = \mathbf{E}_{[1,2]} I_B^2 \le \delta^{-1} \mathbf{E}_{[1,2]} \varkappa_B \ll \mathbf{E}_{[1,2]} \varkappa_B, \quad R_6^2 = \mathbf{E}_{[1,2]} \Psi_B^2 \le F_B^2. \tag{4.20}$$

Combining (4.20) with the relations (which follow from symmetry and (4.5))

$$\mathbf{E}_{[1,2]} \xi_1^2 = pq(n-m)(\nu_1 - p)^2 t^2 \mathbf{E}^{1,2} y_{1,n}^2 = q(\nu_1 - p)^2 t^2 N \mathbf{E}^{1,2} y_{1,n}^2, \tag{4.21}$$
$$\mathbf{E}_{[1,2]} \varkappa_B = \alpha N \mathbf{E}_{[1,2]} \zeta_m (A_4)^2 = \alpha N (n-m) pq \mathbf{E}^{1,2} y_{4,n}^2 \ll q N^{-1} \gamma_2, \tag{4.22}$$

we obtain

$$R_3 R_5 \ll q |\nu_1 - p| |t| \gamma_2^{1/2} \tilde{\gamma}_2^{1/2} \quad \text{and} \quad R_3 R_6 \ll N^{1/2} q^{1/2} |\nu_1 - p| |t| \tilde{\gamma}_2^{1/2} F_B. \tag{4.23}$$

Here we denote $\tilde{\gamma}_2 = \mathbf{E}^{1,2} y_{1,n}^2$. Using the first inequality of (4.23) we obtain

$$m^2 \mathbf{E} R_1 R_3 R_5 \ll m^2 p^2 q^3 t^2 \gamma_2^{1/2} \mathbf{E} |y_{1,2}| \tilde{\gamma}_2^{1/2} |z_2|$$

and invoking the second inequality of (4.47) we get

$$m^2 \mathbf{E} R_1 R_3 R_5 \ll \mu^2 t^2 N^{-1} \gamma_2^{3/2} \prec \mathcal{R}. \tag{4.24}$$

Using the second inequality of (4.23) we obtain

$$m^2 \mathbf{E} R_1 R_3 R_6 \ll m^2 p^2 q^{5/2} N^{1/2} F_B t^2 \mathbf{E} |y_{1,2}| \tilde{\gamma}_2^{1/2} |z_2|,$$

and invoking the first inequality of (4.47) and (4.10) we get

$$m^2 \mathbf{E} R_1 R_3 R_6 \ll \mu^2 N^{-1} F_B t^2 (|t| q^{1/2} + |s|) \gamma_2 \prec \mathcal{R}. \tag{4.25}$$

Since, by (4.18) and (4.19), $|R| \ll \mathbf{E} R_1 R_3 R_5 + \mathbf{E} R_1 R_3 R_6$, it follows from (4.24) and (4.25) that $m^2 R \sim 0$.

In the next step we show that

$$f_3^* \sim f_7^*, \qquad f_7^* = i^3 \binom{m}{2} \mathbf{E} v_4 V, \quad v_4 = \mathrm{e}\{W + T_A^{(1,2)} + \xi^{(1,2)}\}. \tag{4.26}$$

Substitute $v_1 = v_4 \, \mathrm{e}\{T_1 + T_2\}$ in f_3^*. Furthermore, using the expansion

$$\begin{aligned} \mathrm{e}\{T_1 + T_2\} &= \big(1 + T_2 + T_2^2 \, \mathrm{e}\, \theta_1 T_2\}(1 - \theta_1)\big) \, \mathrm{e}\{T_1\} \\ &= \mathrm{e}\{T_1\} + T_2 \big(1 + T_1 + T_1^2 \, \mathrm{e}\{\theta_2 T_1\}(1 - \theta_2)\big) \\ &+ T_2^2 \, \mathrm{e}\{\theta_1 T_2\}(1 - \theta_1)(1 + T_1 \, \mathrm{e}\{\theta_3 T_1\}), \end{aligned} \tag{4.27}$$

we obtain $\mathbf{E}v_1 Q_{1,2} = \mathbf{E}v_4 V + R_1 + R_2$, with $|R_i| \le \mathbf{E} Z_B |VT_i|$, $i = 1, 2$. Therefore, in order to prove (4.26) it remains to show $m^2 R_i \sim 0$, for $i = 1, 2$. By symmetry, it suffices to show $m^2 R_1 \sim 0$.

It follows from (4.9) and (4.22) that

$$|R_1| \le \mathbf{E}|VT_1|\varkappa_B + \mathbf{E}|VT_1|\Psi_B \ll (N^{-1} q\gamma_2 + F_B)\mathbf{E}|VT_1|. \tag{4.28}$$

Combining (4.46) and the inequalities $|t| \le N^{1/2}$ and $|s| \le (Nq)^{1/2}$ we obtain

$$\mathbf{E}|V\,T_1| \ll p^2 q^2 |t|(|t| + |s| q^{-1/2}) N^{-2} (\beta_4 + \gamma_4)^{1/2}.$$

Therefore,

$$m^2 N^{-1} q\gamma_2 \mathbf{E}|VT_1| \ll \mu^2 (t^2 + s^2) N^{-1} (\beta_4 + \gamma_4) \prec \mathcal{R}.$$

Finally, (4.46) in combination with (4.10) yields $m^2 F_B \mathbf{E}|VT_1| \prec \mathcal{R}$ and this inequality in view of (4.28) completes the proof of (4.26).

Now we show

$$f_7^* \sim f_8^*, \qquad f_8^* = i^3 \binom{m}{2} \mathbf{E} v_5 V, \qquad v_5 = \mathrm{e}\{W + T_A^{(1,2)}\}, \tag{4.29}$$

Expanding v_4 in powers of $i\xi^{(1,2)}$ we obtain

$$\mathbf{E}v_4 V = \mathbf{E}v_5 V + i\mathbf{E}v_5 V \xi^{(1,2)} + R, \qquad \text{with} \quad |R| \le \mathbf{E}\big(\xi^{(1,2)}\big)^2 |V|. \tag{4.30}$$

Write $|R| \le \mathbf{E}|V| \mathbf{E}_{[1,2]}\big(\xi^{(1,2)}\big)^2$. By symmetry and (4.5),

$$\mathbf{E}_{[1,2]}\big(\xi^{(1,2)}\big)^2 = p^2 q^2 (m-2)(n-m) t^2 \mathbf{E}^{1,2} y_{3,n}^2 \ll \mu q t^2 N^{-1} \gamma_2.$$

Now invoking (4.31), see below, and the bound $|t| \le N^{1/2}$ we get

$$m^2 |R| \ll \mu^3 t^2 (t^2 + s^2) N^{-1} \gamma_2 (\gamma_2 + 1) \prec \mathcal{R}.$$

This inequality together with (4.30) implies $f_7^* \sim f_8^* + f_9^*$, where

$$f_9^* = i^4 \binom{m}{2} \mathbf{E} v_5 V \xi^{(1,2)} = i^4 \binom{m}{2} (m-2) \mathbf{E} v_5 V \xi_3,$$

by symmetry. In order to prove (4.29) it remains to show $f_9^* \sim 0$.

Let us prove $f_9^* \sim 0$. Expanding v_5 in powers of iT_3 we obtain $|\mathbf{E}v_5 V\xi_3| \le \mathbf{E}Y_B|V\xi_3 T_3|$. Now, using (4.9) we get

$$|f_9^*| \le m^3\mathbf{E}|v_5 V\xi_3 T_3| \le m^3 F_B\mathbf{E}|V\xi_3 T_3| \le m^3 F_B \mathbf{E}|V|\mathbf{E}^{1,2}|\xi_3 T_3|.$$

Finally, invoking (4.10) and the following bounds (which follow from symmetry and (4.5))

$$\mathbf{E}|V| \le (\mathbf{E}Q_{1,2}^2)^{1/2}(\mathbf{E}T_1^2T_2^2)^{1/2} \le p^2q^2|t|(t^2+s^2/q)N^{-5/2}\gamma_2^{1/2}, \tag{4.31}$$
$$\mathbf{E}^{1,2}|\xi_3 T_3| \le (\mathbf{E}^{1,2}\xi_3^2)^{1/2}(\mathbf{E}^{1,2}T_3^2)^{1/2} \le pq|t|(|t|+|s|)N^{-3/2}\gamma_2^{1/2},$$

we obtain $f_9^* \prec \mathcal{R}$.

Let us show that $f_8^* \sim h_3$. Expanding v_5 in powers of iQ_D we get

$$\mathbf{E}v_5 V = \mathbf{E}v_6 V + i\mathbf{E}v_6 V Q_D + R, \qquad v_6 = \mathrm{e}\{T^{(1,2)}\}, \tag{4.32}$$

with $|R| \le \mathbf{E}Y_B|V|Q_D^2$. Note, that by the symmetry,

$$\mathbf{E}Y_B|V|Q_D^2 = \binom{n-m}{2} t^2p^2q^2\mathbf{E}Y_B|V|y_{n-1,n}^2. \tag{4.33}$$

Invoking (4.9) and then using (4.5) we get

$$\mathbf{E}Y_B|V|y_{n-1,n}^2 \le F_B\mathbf{E}|V|y_{n-1,n}^2 \le F_B N^{-3}\gamma_2\mathbf{E}|V|. \tag{4.34}$$

Combining (4.33) and (4.34) and then invoking (4.31) and (4.10) we obtain $m^2 R \prec \mathcal{R}$. Now it follows from (4.32) that $f_8^* \sim h_3 + f_{10}^*$, where

$$f_{10}^* = i^4\binom{m}{2}\mathbf{E}v_6 V Q_D = i^4\binom{m}{2}(n-m)\mathbf{E}v_6 V\, Q_{n-1,n},$$

by symmetry.

We complete the proof of (4.15) by showing that $f_{10}^* \sim 0$. Expanding v_6 in powers of iT_{n-1} and iT_n we get $\left|\mathbf{E}v_6 V\, Q_{n-1,n}\right| \ll \mathbf{E}|V^*V|Y_B$, where we denote $V^* = T_{n-1}T_nQ_{n-1,n}$. Furthermore, using (4.9) and then invoking the simple inequality $\mathbf{E}^{1,2}|V^*| \ll \mathbf{E}|V^*|$ we obtain

$$\mathbf{E}|V^*V|Y_B \ll F_B\mathbf{E}|V^*V| \ll F_B\mathbf{E}|V^*|\mathbf{E}|V| = F_B(\mathbf{E}|V|)^2.$$

Therefore, $|f_{10}^*| \le m^2(n-m)F_B(\mathbf{E}|V|)^2$. Finally, an application of (4.31), and (4.10) yields $f_{10}^* \prec \mathcal{R}$ thus completing the proof of (4.15).

Step 2. In order to prove (4.16) it suffices to show

$$f_1^* \sim f_{11}^* + f_{12}^*, \qquad f_{11}^* = \mathbf{E} v_7, \quad f_{12}^* = i \mathbf{E} v_7 \xi, \quad v_7 = \mathrm{e}\{T + Q_D\}, \tag{4.35}$$

$$f_{11}^* \sim h_1 + f_{13}^*, \qquad f_{13}^* = \binom{n-m}{2} i^3 \mathbf{E}\, \mathrm{e}\{T^{(1,2)}\} V, \tag{4.36}$$

$$f_{12}^* \sim f_{14}^*, \qquad f_{14}^* = m(n-m) i^3 \mathbf{E}\, \mathrm{e}\{T^{(1,2)}\} V. \tag{4.37}$$

Let us prove (4.35). Expanding v in powers of $i\xi$ we obtain

$$f_1^* = f_{11}^* + f_{12}^* + f_{15}^*, \qquad \text{with} \quad f_{15}^* = i^2 \mathbf{E} v_7 \xi^2\, \mathrm{e}\{\theta_1 \xi\}(1-\theta_1).$$

In order to prove (4.35) we shall show $f_{15}^* \sim 0$. Split

$$\Omega_m = S_1 \cup S_2 \cup S_3 \cup S_4, \quad \text{with } S_i \cap S_j = \emptyset, \ i \neq j, \quad \text{and} \quad |S_j| \approx m/4, \ 1 \le j \le 4.$$

Split $\xi = \delta_1 + \dots + \delta_4$, where $\delta_j = \sum_{i \in S_j} \xi_i$. We have

$$f_{15}^* = \sum_{1 \le j,k \le 4} r_{j,k}, \qquad r_{j,k} = i^2 \mathbf{E} v_7 \delta_j \delta_k\, \mathrm{e}\{\theta_1 \xi\}(1-\theta_1).$$

We shall show $r_{j,k} \sim 0$, for every $1 \le j,k \le 4$. By symmetry, it suffices to prove $r_{1,1} \sim 0$ and $r_{1,2} \sim 0$.

Let us show $r_{1,1} \sim 0$. Expanding in powers of $i\theta_1\delta_2$ we obtain

$$\mathrm{e}\{\theta_1 \xi\} = v_8 + i\delta_2 v_8 \tilde{v}, \quad v_8 = \mathrm{e}\{\theta_1(\delta_1 + \delta_3 + \delta_4)\}, \quad \tilde{v} = \theta_1 \mathbf{E}_{\theta_2}\, \mathrm{e}\{\theta_1 \theta_2 \delta_2\}.$$

Substitution of this formula gives

$$r_{1,1} = R_1 + R_2, \quad R_1 = i^2 \mathbf{E} v_7 v_8 \delta_1^2 (1-\theta_1), \quad R_2 = i^3 \mathbf{E} v_7 v_8 \tilde{v} \delta_1^2 \delta_2 (1-\theta_1).$$

Similarly, expanding v_8 in powers of $i\theta_1\delta_3$ we get $R_2 = R_3 + R_4$, where

$$R_3 = i^3 \mathbf{E} v_7 v_9 \delta_1^2 \delta_2 \tilde{v} (1-\theta_1), \qquad v_9 = \mathrm{e}\{\theta_1(\delta_1 + \delta_4)\}, \qquad |R_4| \le \mathbf{E} \delta_1^2 |\delta_2 \delta_3|.$$

Therefore, $|r_{1,1}| \le |R_1| + |R_2| + |R_3|$. Furthermore, invoking the inequalities $|\mathbf{E}_{\{S_2\}} v_7 v_8| \le Y_{S_2}$ and $|\mathbf{E}_{\{S_3\}} v_7 v_9 \tilde{v}| \le Y_{S_3}$ we obtain

$$|r_{1,1}| \le r_1 + r_2 + r_3, \quad r_1 = \mathbf{E} \delta_1^2 Y_{S_2}, \quad r_2 = \mathbf{E} \delta_1^2 |\delta_2| Y_{S_3}, \quad r_3 = \mathbf{E} \delta_1^2 |\delta_2 \delta_3|. \tag{4.38}$$

Now we show $r_i \sim 0$, for $i = 1, 2, 3$. Denote for brevity $m_i = |S_i|$, $1 \le |i| \le 4$.

Let us show $r_2 \sim 0$. By symmetry,

$$\mathbf{E}_{\{S_1\}}\delta_1^2 = m_1 pqt^2\zeta_m^2(A_1), \qquad \mathbf{E}_{\{S_2\}}\delta_2^2 = m_2 pqt^2\zeta_m^2(A_{i_0}), \tag{4.39}$$

with $i_0 \in S_2$. Combining (4.39) and the inequality $\mathbf{E}_{\{S_2\}}|\delta_2| \le (\mathbf{E}_{\{S_2\}}\delta_2^2)^{1/2}$ and using symmetry again we get

$$\begin{aligned} r_2 = \mathbf{E}Y_{S_3}(\mathbf{E}_{\{S_1\}}\delta_1^2)\mathbf{E}_{\{S_2\}}|\delta_2| &\le m_1 m_2^{1/2}(pq)^{3/2}|t|^3\mathbf{E}\zeta_m^2(A_1)|\zeta_m(A_{i_0})|Y_{S_3} \\ &\ll m^{3/2}(pq)^{3/2}|t|^3\mathbf{E}|\zeta_m(A_1)|^3 Y_{S_3}. \end{aligned} \tag{4.40}$$

In the last step we applied (4.44) and used again symmetry. Furthermore, invoking (5.4) and using symmetry and (4.9), we obtain

$$\mathbf{E}|\zeta_m(A_1)|^3 Y_{S_3} \ll N^{1/2}pq(n-m)\mathbf{E}|y_{1,n}|^3 Y_{S_3} \le N^{-3}F_{S_3}\gamma_3.$$

This inequality in combination with (4.40) and (4.10) implies $r_2 \sim 0$.

To show $r_1 \sim 0$ we use the symmetry, and apply (4.9) and (4.10),

$$r_1 = m_1 t^2 p^2 q^2 (n-m)\mathbf{E}y_{1,n}^2 Y_{S_2} \ll t^2 F_{S_2} N^{-1}\gamma_2 \prec \mathcal{R} \tag{4.41}$$

To show $r_3 \sim 0$ we first use (4.44) to get $r_3 \le \mathbf{E}\delta_1^2\delta_2^2 + \mathbf{E}\delta_1^2\delta_3^2$ and then apply (4.48). Finally, collecting the bounds $r_i \prec \mathcal{R}$, $i = 1,2,3$ in (4.38) we get $r_{1,1} \sim 0$.

Let us show $r_{1,2} \sim 0$. Expanding in powers of $i\theta_1\delta_3$ and $i\theta_1\delta_4$ we get

$$\begin{aligned} &\mathrm{e}\{\theta_1\xi\} = v_{10} + v_{10}v_{11}i\theta_1\delta_3, \quad v_{10} = \mathrm{e}\{\theta_1(\delta_1+\delta_2+\delta_4)\}, \quad v_{11} = \mathbf{E}_{\theta_2}\,\mathrm{e}\{\theta_1\theta_2\delta_3\}, \\ &v_{10} = v_{12} + v_{12}v_{13}i\theta_1\delta_4, \quad v_{12} = \mathrm{e}\{\theta_1(\delta_1+\delta_2)\}, \quad v_{13} = \mathbf{E}_{\theta_3}\,\mathrm{e}\{\theta_1\theta_3\delta_4\}. \end{aligned}$$

Combining these expansions we obtain

$$\mathrm{e}\{\theta_1\xi\} = v_{10} + v_{11}v_{12}i\theta_1\delta_3 + v_{11}v_{12}v_{13}i^2\theta_1^2\delta_3\delta_4.$$

The last identity in combination with the bounds $|\mathbf{E}_{\{S_3\}}v_7 v_{10}| \le Y_{S_3}$ and $|\mathbf{E}_{\{S_4\}}v_7 v_{11}v_{12}| \le Y_{S_4}$ implies

$$\begin{aligned} r_{1,2} &\le \mathbf{E}|\delta_1\delta_2|Y_{S_3} + \mathbf{E}|\delta_1\delta_2\delta_3|Y_{S_4} + \mathbf{E}|\delta_1\delta_2\delta_3\delta_4| \\ &\le \mathbf{E}\delta_1^2 Y_{S_3} + \mathbf{E}\delta_2^2 Y_{S_3} + \mathbf{E}\delta_1^2|\delta_2|Y_{S_4} + \mathbf{E}\delta_3^2|\delta_2|Y_{S_4} + \mathbf{E}\delta_1^2\delta_2^2 + \mathbf{E}\delta_3^2\delta_4^2. \end{aligned} \tag{4.42}$$

In the last step we used the simple inequality $ab \le a^2 + b^2$ several times. Note, that the quantities in (4.42) can be bounded in the same way as r_1, r_2 and r_3

above in the proof of $r_{1,1} \sim 0$. Hence, $r_{1,2} \sim 0$ and this completes the proof of (4.35).

Let us prove (4.36). Expanding v_7 in powers of iQ_D we get

$$f_{11}^* = h_1 + f_{16}^* + R, \qquad f_{16}^* = i\mathbf{E}\,\mathrm{e}\{T\}Q_D, \quad \text{with} \quad |R| \le \mathbf{E}Y_B Q_D^2.$$

Furthermore, by symmetry,

$$f_{16}^* = \binom{n-m}{2} i\mathbf{E}\,\mathrm{e}\{T\}Q_{1,2} \quad \text{and} \quad \mathbf{E}Y_B Q_D^2 = \binom{n-m}{2} p^2 q^2 t^2 \mathbf{E}Y_B y_{n-1,n}^2.$$

Combining (4.9) and (4.10) we obtain $R \prec \mathcal{R}$ and, therefore, $f_{11}^* \sim h_1 + f_{16}^*$.

Let us show $f_{16}^* \sim f_{13}^*$. Write $\mathrm{e}\{T\} = \mathrm{e}\{T^{(1,2)}\}\,\mathrm{e}\{T_1 + T_2\}$ and use (4.27) to get

$$\mathbf{E}\,\mathrm{e}\{T\}Q_{1,2} = i^2\mathbf{E}\,\mathrm{e}\{T^{(1,2)}\}V + R_1 + R_2, \quad \text{with} \quad |R_i| \ll \mathbf{E}|VT_i|Y_B. \qquad (4.43)$$

By (4.9), $|R_i| \ll F_B\mathbf{E}|VT_i|$. Furthermore, invoking (4.46) and (z.a5) we obtain $n^2 R_i \prec \mathcal{R}$, $i = 1, 2$. These bounds together with (4.43) imply $f_{16}^* \sim f_{13}^*$, thus completing the proof of (4.36).

Let us prove (4.37). By symmetry, $f_{12}^* = mi\mathbf{E}v_7\xi_1$. Expanding v_7 in powers of iT_1 we get

$$f_{12}^* = f_{17}^* + R_1, \qquad f_{17}^* = mi^2\mathbf{E}\,\mathrm{e}\{T^{(1)} + Q_D\}\xi_1 T_1, \quad |R_1| \le m\mathbf{E}Y_B|\xi_1|T_1^2.$$

Furthermore, expanding the exponent in powers of iQ_D we obtain

$$f_{17}^* = f_{18}^* + R_2, \quad f_{18}^* = mi^2\mathbf{E}\,\mathrm{e}\{T^{(1)}\}\xi_1 T_1, \quad |R_2| \le m\mathbf{E}Y_B|\xi_1 T_1 Q_D|.$$

Note, that by symmetry, $f_{18}^* = m(n-m)i^2\mathbf{E}\,\mathrm{e}\{T^{(1)}\}Q_{1,2}T_1$. Finally, expanding the exponent in powers of iT_2 we get

$$f_{18}^* = f_{14}^* + R_3, \qquad \text{with} \quad |R_3| \le n(n-m)\mathbf{E}Y_B|VT_2|.$$

Therefore in order to prove (4.37) it remains to show $R_i \prec \mathcal{R}$, for $i = 1, 2, 3$.

To show $R_1 \prec \mathcal{R}$ use the inequality $|\xi_1|T_1^2 \le \xi_1^2 + T_1^4$. We get $|R_1| \le R_{1,1} + R_{1,2}$, with $R_{1,1} = m\mathbf{E}Y_B T_1^4$ and $R_{1,2} = m\mathbf{E}Y_B\xi_1^2$. By (4.9) and (4.10), $R_{1,1} \prec \mathcal{R}$. Furthermore, the bound $R_{1,2} \prec \mathcal{R}$ is obtained in the same way as (4.41).

To show $R_2 \prec \mathcal{R}$ use the inequality $|\xi_1 T_1 Q_D| \le \xi_1^2 + T_1^2 Q_D^2$. We get $|R_2| \le R_{2,1} + R_{2,2}$, with $R_{2,1} = m\mathbf{E}Y_B\xi_1^2 \prec \mathcal{R}$, cf. (4.41), and with

$$R_{2.2} = m\mathbf{E}Y_B T_1^2 Q_D^2 = m\binom{n-m}{2}\mathbf{E}Y_B T_1^2 Q_{n-1,n}^2 \le mn^2 F_B \mathbf{E}T_1^2 Q_{n-1,n}^2,$$

by symmetry and (4.9). Now, combining (4.10) and the inequality

$$\mathbf{E}T_1^2 Q_{n-1,n}^2 = p^3 q^3 t^2 \mathbf{E} z_1^2 \mathbf{E}^1 y_{n-1,n}^2 \ll p^3 q^3 t^2 (t^2 + s^2/q) N^{-4} \gamma_2,$$

(here we use (4.5)) we obtain $R_{2,2} \prec \mathcal{R}$.

To show $R_3 \prec \mathcal{R}$ we apply (4.9) to get $R_3 \le nmF_B \mathbf{E}|VT_2|$. Then combining (4.46) and (4.10) we obtain $R_3 \prec \mathcal{R}$.

We arrive at (4.37), thus completing the proof of the lemma.

In the next lemma we collect some auxiliary inequalities used in Lemma 4.1.

We shall frequently use the inequalities

$$ab \le a^2 + b^2, \qquad a^2 b \le a^3 + b^3. \tag{4.44}$$

Lemma 4.2. *We have*

$$\mathbf{E}|Q_{1,2}\xi_2|\xi_1^2 \ll p^2 q^3 t^4 N^{-9/2} \gamma_4, \quad \mathbf{E}|Q_{1,2}T_2|\xi_1^2 \ll p^2 q^3 |t|^3 N^{-7/2}(1+\gamma_4), \tag{4.45}$$

$$\mathbf{E}|Q_{1,2}T_2T_1^2| \ll p^2 q^2 |t| \big(|t|^3 + |s|^3 q^{-3/2}\big) N^{-3} (\beta_4 + \gamma_4)^{1/2}, \tag{4.46}$$

$$\mathbf{E}|y_{1,2}x_2|\tilde\gamma_2^{1/2} \ll N^{-7/2}\gamma_2, \quad \mathbf{E}|y_{1,2}z_2|\tilde\gamma_2^{1/2} \ll N^{-3}\gamma_2, \quad \tilde\gamma_2 = \mathbf{E}^{1,2} y_{1,n}^2, \tag{4.47}$$

$$\mathbf{E}\delta_K^2 \delta_M^2 \ll p^2 q^2 t^2 m^2 N^{-3} \gamma_4, \qquad \textit{for any} \quad K, M \subset \Omega_m, \quad K \cap M = \emptyset, \tag{4.48}$$

with $|K|, |M| > 0$. *Here* $\delta_K = \sum_{i\in K} \xi_i$.

Proof. Let us prove (4.45). We have

$$\mathbf{E}|Q_{1,2}\xi_2\xi_1^2| \ll p^2 q^2 t^4 \mathbf{E}|y_{1,2}\zeta_m(A_2)|\zeta_m^2(A_1) \le 2p^2 q^2 t^4 \mathbf{E}|y_{1,2}\zeta_m^3(A_1)|,$$

where in the last step we apply (4.44) and use symmetry. Furthermore, writing $\mathbf{E}|y_{1,2}\zeta_m^3(A_1)| = \mathbf{E}|y_{1,2}|\mathbf{E}^{1,2}|\zeta_m^3(A_1)|$ and invoking (5.4) we obtain the first inequality in (4.45). To prove the second inequality we apply (4.21),

$$\mathbf{E}|Q_{1,2}T_2|\xi_1^2 = \mathbf{E}|Q_{1,2}T_2|\mathbf{E}_{[1,2]}\xi_1^2 \le p^2 q^3 N |t|^3 \mathbf{E} y_{1,n}^2 |y_{1,2}z_2|,$$

and use the inequality $\mathbf{E}y_{1,n}^2|y_{1,2}z_2| \le N^{-9/2}(1+\gamma_4)$. To prove this inequality combine (4.5), Hölder's inequality and the bounds

$$|y_{1,2}z_2| \le \sqrt{N}|y_{1,2}x_2| + |y_{1,2}|, \qquad |y_{1,2}x_2| \le N y_{1,2}^2 + N^{-1} x_2^2.$$

Let us prove (4.46). We have $\mathbf{E}|Q_{1,2}T_2|T_1^2 \le p^2 q^2 |t| \mathbf{E}|y_{1,2}z_2|z_1^2$. Now (4.46) follows from the bound

$$\mathbf{E}|y_{1,2}z_2|z_1^2 \le (|t|^3 + t^2|s|q^{-1/2} + |t|s^2/q + |s|^3 q^{-3/2}) N^{-3} (\beta_4 + \gamma_4)^{1/2}$$

which is a consequence of the following inequalities

$$
\begin{aligned}
&z_1^2|z_2| \le (t^2 x_1^2 + s^2/\tau^2)|z_2|, \qquad \mathbf{E}|y_{1,2}x_2| \le (\mathbf{E}y_{1,2}^2)^{1/2}(\mathbf{E}x_2^2)^{1/2} \le N^{-2}\gamma_2^{1/2},\\
&\mathbf{E}|y_{1,2}x_2|x_1^2 \le (\mathbf{E}y_{1,2}^2x_1^2)^{1/2}(\mathbf{E}x_1^2x_2^2)^{1/2} \le N^{-1}(\mathbf{E}y_{1,2}^4\mathbf{E}x_1^4)^{1/4} \le N^{-3}(\beta_4+\gamma_4)^{1/2},\\
&\mathbf{E}|y_{1,2}|x_1^2 \le (\mathbf{E}y_{1,2}^2x_1^2)^{1/2}(\mathbf{E}x_1^2)^{1/2} \le N^{-1/2}(\mathbf{E}y_{1,2}^4\mathbf{E}x_1^4)^{1/2} \le N^{-5/2}(\beta_4+\gamma_4)^{1/2}.
\end{aligned}
$$

Let us prove (4.47). By (4.44), $|y_{1,2}x_2|\tilde\gamma_2^{1/2} \le N^{-1/2}y_{1,2}^2 + N^{1/2}x_2^2\tilde\gamma_2$. Now invoking (4.5) we obtain the first inequality of (4.47). To prove the second one, write

$$\mathbf{E}|y_{1,2}z_1|\tilde\gamma_2^{1/2} \le N^{1/2}\mathbf{E}|y_{1,2}x_2|\tilde\gamma_2^{1/2} + \mathbf{E}|y_{1,2}|\tilde\gamma_2^{1/2},$$

(here we used the bounds $|t| \le N^{1/2}$ and $|s| \le \tau$) and apply Hölder's inequality to the second summand.

Let us prove (4.48). By symmetry,

$$\mathbf{E}\delta_K^2\delta_M^2 = |K|\,|M|\,p^2q^2t^4\mathbf{E}\zeta_m^2(A_1)\zeta_m^2(A_2). \tag{4.49}$$

A simple calculation yields

$$\mathbf{E}\zeta_m^2(A_1)\zeta_m^2(A_2) \le pq(n-m)N^{-6}\gamma_4 + p^2q^2(n-m)(n-m-1)N^{-6}\gamma_2^2 \ll N^{-4}\gamma_4.$$

Substituting this bound in (4.49) and using the inequalities $|K|, |M| < m$ and $t^2 \le N$ we obtain (4.48).

5. Auxiliary results

Lemma 5.1. *For the random variables v_i and Θ_i, $i = 1,2,3$, defined in (3.6) and (3.25) above, the following inequalities hold*

$$\mathbf{E}\Theta_i^2 \le N^{-2}\,\gamma_2, \qquad i = 1,2,3, \tag{5.1}$$

$$\mathbf{E}^*|v_i(A_0^*)| \le q^{-1}\,\Theta_i, \qquad i = 1,2. \tag{5.2}$$

For $\Lambda_m = \sum_{1\le i<j\le m} g_2(A_i,A_j)$, with $3 \le m \le n$, we have

$$\mathbf{E}\,\Lambda_m^2 = \frac{m\,(m-1)}{2\,N^3}\,(1-c_\Lambda)\,\gamma_2, \quad c_\Lambda = \frac{2\,(m-2)}{n-2} - \frac{(m-2)(m-3)}{(n-2)(n-3)}\,. \tag{5.3}$$

For the random variable $\zeta_m(A_k)$ defined in (4.6) the following inequality holds

$$\mathbf{E}_{\{m+1,\dots,n\}}|\zeta_m(A_k)|^3 \ll pq(npq)^{1/2}\sum_{j=m+1}^{n}|g_2(A_k,A_j)|^3, \quad k \le m \le n, \tag{5.4}$$

recall the definition of $\mathbf{E}_{\{m+1,\dots,n\}}$ *just before (4.6).*

Proof of Lemma 5.1. We shall prove (5.1) in the case $i=1$ only. For $i=2,3$ the proof is similar. By Hölder's inequality,

$$\Theta_1^2 \le \mathbf{E}^* v_1^2(A_1^*) = \sum_{k+1\le i,j\le N} \mathbf{E}^* g_2(A_1^*, A_i)\, g_2(A_1^*, A_j).$$

By symmetry,

$$\mathbf{E}\,\mathbf{E}^* g_2(A_1^*, A_i)\, g_2(A_1^*, A_j) = \mathbf{E} g_2(A_1, A_i)\, g_2(A_1, A_j),$$

and therefore,

$$\mathbf{E}\Theta_1^2 \le (N-k)\,\mathbf{E} g_2^2(A_1,A_2) + (N-k)\,(N-k-1)\,\mathbf{E} g_2(A_1,A_2)\, g_2(A_1,A_3).$$

Now, invoking the identity

$$\mathbf{E} g_2(A_1,A_2)\, g_2(A_1,A_3) = -(n-2)^{-1}\,\mathbf{E} g_2^2(A_1,A_2), \tag{5.5}$$

(use (1.2)) we complete the proof of $\mathbf{E}\Theta_1^2 \le N^{-2}\gamma_2$.

To prove (5.2) we combine the obvious inequality $|\mathcal{J}_i|/|\mathcal{J}_0| \le q^{-1}$ and the inequalities

$$\mathbf{E}^*\big|v_i(A_0^*)\big| = |\mathcal{J}_0|^{-1}\sum_{k\in\mathcal{J}_0} |v_i(A_k)| \le |\mathcal{J}_0|^{-1}\sum_{k\in\mathcal{J}_i} |v_i(A_k)| \le \Theta_i\,\frac{|\mathcal{J}_i|}{|\mathcal{J}_0|}\,, \quad i=1,2.$$

Let us prove (5.3). By symmetry, $\mathbf{E}\Lambda_m^2 = 2^{-1}(m-1)m\,\mathbf{E}\, g_2(A_1,A_2)\Lambda_m$. Furthermore,

$$\mathbf{E} g_2(A_1,A_2)\,\Lambda_m = \mathbf{E} g_2^2(A_1,A_2) + 2\,(m-2)\,\mathbf{E} g_2(A_1,A_2)\, g_2(A_1,A_3) \\ + 2^{-1}\,(m-2)\,(m-3)\,\mathbf{E} g_2(A_1,A_2)\, g_2(A_3,A_4).$$

Now, invoking (5.5) and the identity

$$\mathbf{E} g_2(A_1,A_2)\, g_2(A_3,A_4) = 2(n-2)^{-1}(n-3)^{-1}\mathbf{E}\, g_2^2(A_1,A_2),$$

(use (1.2)) we obtain (5.3).

In order to prove (5.4) we apply Rosenthal's inequality,

$$\mathbf{E}\big|Z_1+\dots+Z_j\big|^r \le c(r)\sum_{l=1}^{j}\mathbf{E}\,|Z_l|^r + c(r)\Big(\sum_{l=1}^{j}\mathbf{E}\,Z_l^2\Big)^{r/2}, \quad r\ge 2,$$

where $Z_1, \dots, Z_j$ are independent and centered random variables. We apply this inequality to the sum $\zeta_m(A_k)$, cf. (4.15), conditionally given $\overline{A}$,

$$\mathbf{E}_{\{m+1,\dots,n\}}|\zeta_m(A_k)|^3 \ll pq \sum_{l=m+1}^{n} |g_2(A_k, A_l)|^3 + \Big(pq \sum_{l=m+1}^{n} g_2^2(A_k, A_l)\Big)^{3/2}.$$

Finally, using Hölder's inequality, we bound the second sum above by

$$\Big(\sum_{l=m+1}^{n} g_2^2(a, A_l)\Big)^{3/2} \le (n-m)^{1/2} \sum_{l=m+1}^{n} |g_2(a, A_l)|^3, \qquad a \in \mathcal{A},$$

thus arriving at (5.4).

Lemma 5.2. *For each $0 < d < \pi$ and $x,\ y \in R$, and $\beta(x)$ defined in Section 3.1, we have*

$$\big|\,\beta(x+y)\big|^2 \le u_{[d]}(x) v_{[d]}(y), \quad \textit{where} \quad v_{[d]}(y) = 1 + pq\,\frac{2\pi}{d}\,(\,\frac{4}{\Theta(d)}\,+1)y^2,$$

and where the function $u_{[d]}$ is defined in (4.8).

Proof of Lemma 5.2. In the case where $|x| \ge \pi + d$, we have $u_{[d]}(x) = 1$ and the desired inequality follows from the simple bound $|\beta(x+y)| \le 1$.
In the case where $|x| < \pi + d$ we apply the mean value theorem to get

$$\begin{aligned}\big|\cos(x+y) - \cos(x)\big| &\le \big|\mathbf{E}\,\sin(x + \theta_1\, y)\, y\big| \\ &\le \big(|x| + |y|\big)\,|y| \le c\,x^2 + \big(c^{-1}+1\big)\,y^2. \end{aligned} \tag{5.6}$$

In the last step we applied the inequality $|x\,y| \le c\,x^2 + c^{-1}\,y^2$, with $c > 0$. Combining (5.6) and the identity $|\beta(x+y)|^2 = 1 - 2pq(1-\cos(x+y))$ we get

$$|\beta(x+y)|^2 \le 1 - 2\,p\,q\,\big(1 - \cos(x) - c\,x^2 - \big(c^{-1}+1\big)\,y^2\big).$$

Now invoking (5.15) we obtain

$$|\beta(x+y)|^2 \le w_1 + w_2, \quad w_1 = 1 - p\,q\,\big(\Theta(d) - 2\,c\big)\,x^2, \quad w_2 = \big(c^{-1}+1\big)\,2\,p\,q\,y^2.$$

But $1 - p\,q\,x^2\,\Theta(d) \ge d/\pi$, for $|x| \le \pi + d$. Hence, $w_1 > d/\pi$ and therefore $w_1 + w_2 \le w_1\,\big(1 + \pi\,d^{-1}\,w_2\big)$. Choosing $c = \Theta(d)/4$ completes the proof of the lemma.

Lemma 5.3. *Assume that $\beta_2 = 1$. For every $s, t \in R$ and $0 < d < \pi$, we have*

$$\mathbf{E} Z^2(A_1) I_{[d]}(A_1) \geq \Big(\frac{t^2}{N} + s^2 \Big)(1 - 2c_{[d]}), \qquad c_{[d]} = \max\Big\{ \frac{b_1}{d} \, ; \, \frac{b_1^2}{d^2} \Big\}.$$

Here $Z(a) = t g_1(a) + s$ and $I_{[d]}(a) = I\{H_1 |g_1(a)| < d\}$, for $a \in \mathcal{A}$.

Note, that a similar inequality was used already by Höglund (1978), where the constant (corresponding to $c_{[d]}$) was not specified. For our purposes the dependence of c_d on the parameters b_1 and d is important and thus we include the proof.

Proof of Lemma 5.3. Denote $\mathcal{K}_{[d]} = \{k : I_{[d]}(a_k) = 0\}$. Clearly, for $r > 0$,

$$|\mathcal{K}_{[d]}| = \sum_{k \in \mathcal{K}_{[d]}} 1 \leq \sum_{k \in \mathcal{K}_{[d]}} \big| g_1(a_k) \, H_1 / d \big|^r \leq n \, \beta_r \, \beta_3^{-r} \, b_1^r d^{-r}. \tag{5.7}$$

Furthermore, since $\mathbf{E} Z^2(A_1) = t^2 N^{-1} + s^2$, we have

$$\mathbf{E} Z^2(A_1)^2 I_{[d]}(A_1) = \frac{t^2}{N} + s^2 - W n^{-1}, \quad W = \sum_{k \in \mathcal{K}_{[d]}} Z^2(a_k). \tag{5.8}$$

The inequality $(a+b)^2 \leq 2a^2 + 2b^2$ implies $W \leq 2W_1 + 2W_2$, where

$$W_1 = s^2 |K_{[d]}|, \qquad W_2 = t^2 \sum_{k \in \mathcal{K}_{[d]}} g_1^2(a_k) \leq \frac{t^2}{N} n^{2/3} \beta_3^{2/3} |\mathcal{K}_{[d]}|^{1/3}.$$

In the last step we applied Hölder's inequality to get

$$\sum_{k \in \mathcal{K}_{[d]}} g_1^2(a_k) \leq \Big(\sum_{k \in \mathcal{K}_{[d]}} |g_1^3(a_k)| \Big)^{2/3} |\mathcal{K}_{[d]}|^{1/3}.$$

Now, (5.7) (with $r = 2$) implies $W_1 \leq s^2 n c_d$. Furthermore, (5.7) (with $r = 3$) implies $W_2 \leq t^2 N^{-1} n c_d$. These inequalities combined with (5.8) complete the proof of the lemma.

Lemma 5.4. *Assume that $\beta_2 = 1$ and that (3.3) holds. For $|t| \leq H_1$ and $|s| \leq \pi\tau$, the inequalities (4.9) hold true.*

Proof of Lemma 5.4. Throughout the proof we use the notation introduced in Section 4. Fix $B \subset \Omega_m$. By Lemma 5.2,

$$Z_B = \prod_{k \in B} |\beta(z_k + t\zeta_m(A_k))| \leq \eta_1 \eta_2, \quad \eta_1^2 = \prod_{k \in B} u_{[1]}(z_k), \quad \eta_2^2 = \prod_{k \in B} v_{[1]}(t\zeta_m(A_k)),$$

Using the inequality $1 + x \le \exp\{x\}$, we obtain $\eta_2^2 \le \exp\{pqt^2 \varkappa_B |B|/N\}$ and therefore, $(1 - I_B)\eta_2 \le g_B(t)$. Finally, combining the inequalities $Z_B \le 1$ and $Z_B \le \eta_1\eta_2$, we get

$$Z_B = I_B Z_B + (1 - I_B) Z_B \le I_B + (1 - I_B)\eta_1\eta_2 \le I_B + \Psi_B,$$

thus proving the first inequality in (4.9). Here the random variables $\Psi_B = \eta_1 g_B(t)$, $\varkappa_B$ and I_B are defined in (4.7) and the function $g_B(t)$ is given by (4.8).

Clearly, Lemma 5.2 (with $x = z_k$ and $y = 0$) implies the second inequality of (4.9). To prove the last one observe that by Hölder's inequality we have $\mathbf{E}^{i_1,\dots,i_4}\Psi_B \ll (E^{i_1,\dots,i_4}\Psi_B^2)^{1/2}$ and thus, it suffices to show

$$E^{i_1,\dots,i_4}\Psi_B^2 \le F_B^2, \qquad \text{for every} \quad i_1,\dots,i_4 \in \Omega_n \setminus B. \tag{5.9}$$

To prove (5.9) note that the inequalities $|t| \le H_1$ and $|s| \le \pi\tau$ imply

$$u_{[1]}(z_k) \le w(A_k), \qquad w(A_k) = 1 - \frac{pq}{2}\,\Theta(1) z_k^2 I_{[1]}(A_k), \quad k \in \Omega_n.$$

where we denote $I_{[1]}(a) = I\{H_1|g_1(a)| < 1\}$. Therefore,

$$\Psi_B^2 = g_B^2(t)\eta_1^2 \le g_B^2(t)\eta, \qquad \text{where} \quad \eta = \prod_{k\in B} w(A_k). \tag{5.10}$$

Denote $D_1 = \{i_1,\dots,i_4\}$ and $D_2 = \Omega_n \setminus D_1$. By Theorem 4 of Hoeffding (1963),

$$\mathbf{E}^{i_1,\dots,i_4}\eta \le w_*^{|B|}, \quad \text{where} \;\; w_* = 1 - \frac{pq}{2}\,\Theta(1)\Gamma_*, \;\; \Gamma_* = \frac{1}{|D_2|}\sum_{k\in D_2} z_k^2 I_{[1]}(A_k). \tag{5.11}$$

Below we construct the following lower bound for Γ_*,

$$\Gamma_* \ge \frac{9}{10}\,(t^2 + s^2/q)\,\frac{1}{N}\,. \tag{5.12}$$

Combining (5.11), (5.12) and the inequality $1 + x \le \exp\{x\}$ we get $\eta \le \exp\{-0.45pq\Theta(1)(t^2 + s^2/q)|B|N^{-1}\}$. Now (5.9) follows from (5.10).

Let us prove (5.12). We have

$$\Gamma_* = \frac{n}{n-4}\,\mathbf{E} z_1^2 I_{[1]}(A_1) - \frac{1}{n-4}\,M, \qquad M = \sum_{k\in D_1} z_k^2 I_{[1]}(A_k). \tag{5.13}$$

The simple inequality $(a+b)^2 \le 2a^2 + 2b^2$ gives

$$M \le 8s^2/\tau^2 + 2t^2 M_1, \qquad M_1 = \sum_{k\in D_1} g_1^2(A_k). \tag{5.14}$$

By Hölder's inequality and (3.3),

$$M_1 \le 4^{1/3} \Big(\sum_{k\in D_1} |g_1(A_k)|^3 \Big)^{2/3} \le 4^{1/3} \beta_3^{2/3} n^{2/3} N^{-1} \le (4c_o)^{1/3} \frac{n}{N}.$$

Here we estimated $\beta_3^2/n \le \beta_4/n \le c_0$, see (3.3). This inequality in combination with (5.14) implies $M \le 100^{-1} n(t^2 + s^2/q)N^{-1}$, provided that c_0 in (3.3) is sufficiently small. Substituting this bound in (5.13) and invoking Lemma 5.3 we obtain (5.12) thus completing the proof of the lemma.

Theorem 4 of Hoeffding (1963) and Höglund's inequalities. Consider a population $\mathcal{P}$ of n numbers $p_1, \dots, p_n$. Let $\mathcal{X}_1, \dots, \mathcal{X}_N$ denote a random sample *without* replacement from $\mathcal{P}$ and let $\mathcal{Y}_1, \dots, \mathcal{Y}_N$ denote a random sample *with* replacement from $\mathcal{P}$. In particular, $\mathcal{Y}_1, \dots, \mathcal{Y}_N$ are independent random variables.

Theorem (Hoeffding (1963)). *If the function $f(x)$ is continuous and convex then*

$$\mathbf{E}\, f\Big(\sum_{k=1}^N \mathcal{X}_k\Big) \le \mathbf{E}\, f\Big(\sum_{k=1}^N \mathcal{Y}_k\Big).$$

The following inequalities are proved in Höglund (1978),

$$1 - \cos v \ge \frac{1}{2}\, v^2\, \Theta(u), \quad \text{for} \quad |v| \le \pi + u \quad \text{and} \quad 0 \le u \le \pi, \tag{5.15}$$

$$\frac{\pi^{1/2}}{2} \le \binom{n}{N} s^N (1-s)^{n-N} \big(2\,\pi\, s(1-s)n\big)^{1/2} \le 1, \quad \text{with} \quad s = \frac{N}{n}, \tag{5.16}$$

where $1 \le N \le n$.

References

Albers, W., Bickel, P.J. and van Zwet, W.R., *Asymptotic expansions for the power of distribution free tests in the one-sample problem*, Ann. Statist. **4** (1976), 108–156.

Babu, G.J. and Singh, K., *Edgeworth expansions for sampling without replacement from finite populations*, J. Multivar. Analysis. **17** (1985), 261–278.

Bentkus, V., Götze, F. and van Zwet, W. R., *An Edgeworth expansion for symmetric statistics,*, Ann. Statist. **25** (1997), 851–896.

Bickel, P.J., *Edgeworth expansions in non parametric statistics*, Ann. Statist. **2** (1974), 1–20.

Bickel, P.J. and Robinson, J., *Edgeworth expansions and smoothness*, Ann. Probab. **10** (1982), 500–503.

Bickel, P.J., Götze, F. and van Zwet, W. R., *The Edgeworth expansion for U-statistics of degree two*, Ann. Statist. **14** (1986), 1463–1484.

Bickel, P.J. and van Zwet, W. R., *Asymptotic expansions for the power of distribution free tests in the two-sample problem*, Ann. Statist. **6** (1978), 937–1004.

Bikelis, A., *On the estimation of the remainder term in the central limit theorem for samples from finite populations.*, Studia Sci. Math. Hungar. **4** (1969), 345–354. (In Russian).

Bloznelis, M. and Götze, F., *An Edgeworth expansion for finite population U-statistics* (1997), Preprint 97 - 012, SFB 343, Universität Bielefeld., Publication available at the internet: **http://www.mathematik.uni-bielefeld.de/sfb343/Welcome.html**.

Bolthausen, E. and Götze, F., *The rate of convergence for multivariate sampling statistics*, Ann. Statist. **21** (1993), 1692–1710.

Callaert, H., Janssen, P. and Veraverbeke, N., *An Edgeworth expansion for U-statistics*, Ann. Statist. **8** (1980), 299–312.

Does, R.J.M.M., *An Edgeworth expansion for simple linear rank statistics under the null-hypothesis*, Ann. Statist. **11** (1983), 607–624.

Erdös, P. and Rényi, A., *On the central limit theorem for samples from a finite population*, Publ. Math. Inst. Hungar. Acad. Sci. **4** (1959), 49–61.

Götze, F., *Asymptotic expansions for bivariate von Mises functionals*, Z. Wahrsch. verw. Gebiete **50** (1979), 333–355.

Götze, F. and van Zwet, W.R., *Edgeworth expansions for asymptotically linear statistics* (1991), Preprint 91 - 034, SFB 343, Universität Bielefeld.

Helmers, R., and van Zwet, W. R., *The Berry–Esseen bound for U-statistics.* **1** (1982), Statistical Decision Theory and Related Topics,III. Vol. 1. (S.S. Gupta and J.O. Berger, eds.), Academic Press, New York, 497–512.

Hoeffding, W., *Probability inequalities for sums of bounded random variables*, J. Amer. Stat. Assoc. **58** (1963), 13–30.

Höglund, T., *Sampling from a finite population: a remainder term estimate*, Scand. J. Stat. **5** (1978), 69–71.

Kokic, P.N. and Weber, N.C., *An Edgeworth expansion for U-statistics based on samples from finite populations*, Ann. Probab. **18** (1990), 390–404.

Kokic, P.N. and Weber, N.C., *Rates of strong convergence for U-statistics in finite populations*, J. Austral. Math. Soc. Ser. A **50** (1991), 468–480.

Nandi, H. K. and Sen, P.K., *On the properties of U-statistics when the observations are not independent.II. Unbiased estimation of the parameters of a finite population.*, Calcutta Statist. Assoc. Bull. **12** (1963), 124–148.

Prawitz, H., *Limits for a distribution, if the characteristic function is given in a finite domain*, Scand. AktuarTidskr. (1972), 138–154.

Robinson, J., *An asymptotic expansion for samples from a finite population*, Ann. Statist. **6** (1978), 1005–1011.

Serfling, R. J., *Approximation Theorems of Mathematical Statistics*, Wiley, New York, 1980.

Schneller, W., *Edgeworth expansions for linear rank statistics*, Ann. Statist. **17** (1989), 1103–1123.

Zhao, L. C. and Chen, X. R., *Berry–Esseen bounds for finite-population U-statistics*, Sci.Sinica Ser. A **30** (1987), 113–127.

Zhao, L. C. and Chen, X. R., *Normal approximation for finite-population U-statistics*, Acta Math. Appl. Sinica **6** (1990), 263–272.

van Zwet, W.R., *A Berry-Esseen bound for symmetric statistics*, Z. Wahrsch. verw. Gebiete **66** (1984), 425–440.

van Zwet, W.R., *On the Edgeworth expansion for the simple linear rank statistic*, Coll. Math. Soc. J. Bolyai **32** (1982), B.V. Gnedenko, M.L.Puri, I.Vincze (eds)., North-Holland, Amsterdam, 889–909.