A BERRY–ESSEEN BOUND FOR FINITE POPULATION STUDENT'S STATISTIC

M. Bloznelis

Abstract. A general and precise Berry–Esseen bound is proved for the Studentized mean based on N random observations drawn without replacement from a finite population. The bound yields the optimal rate $O(N^{-1/2})$ under minimal conditions. If Erdős–Rényi condition holds this bound implies the asymptotic normality of Student's statistic and the selfnormalized sum.

1. Introduction and results

Let $\{x\}$ denote a sequence of real numbers $x_1, \dots, x_n$ and let $X_1, \dots, X_N$, $N < n$, denote random variables with values in $\{x\}$ such that $\mathbb{X} = \{X_1, \dots, X_N\}$ represents a simple random sample of size N drawn without replacement from $\{x\}$. We shall assume that $\mathbf{E}\, X_1 = 0$ and $\sigma^2 = \mathbf{E}\, X_1^2 > 0$.

Let

$$\mathbf{t} = \mathbf{t}(\mathbb{X}) = \overline{X}/\hat{\sigma}$$

denote the Student statistic, where

$$\overline{X} = N^{-1}(X_1 + \dots + X_N) \qquad \text{and} \qquad \hat{\sigma}^2 = \hat{\sigma}^2(\mathbb{X}) = N^{-1} \sum_{i=1}^{N} (X_i - \overline{X})^2.$$

Put $\mathbf{t} = 0$ if $\hat{\sigma} = 0$. By the finite population central limit theorem (CLT), see Erdős and Rényi (1959), for large N, the distribution of $\sqrt{N}\,\mathbf{t}$ can be approximated by a normal distribution. In this paper we estimate the rate of the normal approximation. We construct a bound for

$$\delta_N = \sup_x \Big| \mathbf{P}\big\{ \sqrt{N/q}\, \mathbf{t}(\mathbb{X}) < x \big\} - \Phi(x) \Big|,$$

where $\Phi(x)$ denotes the standard normal distribution function,

$$p = N/n \qquad \text{and} \qquad q = 1 - p.$$

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Theorem 1.1. *There exists an absolute constant $c > 0$ such that*

$$\delta_N \le \frac{c}{\sqrt{q}} \; \frac{\beta_3}{\sqrt{N}\,\sigma^3}\,, \qquad \beta_3 := \mathbf{E}\,|X_1|^3. \tag{1.1}$$

A similar Berry–Esseen bound but for the finite population sample mean was proved by Höglund (1978). The estimate of Theorem 1.1 holds for any fixed sample size N and population size n. If β_3/σ^3 is bounded and q is bounded away from 0 as $N \to \infty$ and $n \to \infty$, then (1.1) establishes a Berry–Esseen bound of order $O(N^{-1/2})$. Note that the factor $1/\sqrt{q}$ in the right hand side of (1.1) can not be removed or replaced by q^α with $\alpha > -1/2$, cf. one term Edgeworth expansion for $\mathbf{P}\{\sqrt{N/q}\,\mathbf{t}(\mathbb{X}) < x\}$ given in Babu and Singh (1985).

Write $w = \sqrt{n\,p\,q}$.

Theorem 1.2. *There exists an absolute constant $c > 0$ such that*

$$\delta_N \le \frac{c}{\sigma^2}\,\mathbf{E}\,X_1^2\,\mathbb{I}_{|X_1|>\sigma w} + \frac{c}{w\,\sigma^3}\,\mathbf{E}\,|X_1|^3\,\mathbb{I}_{|X_1|\le\sigma w}. \tag{1.2}$$

Theorems 1.1 and 1.2 can be considered as a particular extension to the case of simple random sampling of Berry–Esseen bounds for Student's statistic based on i.i.d. observations, proved recently by Bentkus and Götze (1996). Indeed, the case where $n \to \infty$ and N is fixed corresponds to the i.i.d. situation and in this way we obtain Theorems 1.1 and 1.2 of Bentkus and Götze (1996) as corollaries of Theorems 1.1 and 1.2. It could be mentioned that our techniques are related to those of Bentkus and Götze (1996), Bloznelis and Götze (1997) and Höglund (1978).

Next we apply Theorem 1.2 to prove the CLT for Studentized mean. Consider a sequence of populations $\{x\}_n = \{x_{n\,1}, \dots, x_{n\,n}\}$ such that $\sum_i x_{n\,i} = 0$, for every $n = 2, 3, \dots$. Let $\mathbb{X}_{n\,N} = \{X_{n\,1}, \dots, X_{n\,N}\}$ denote a sample of size $N = N_n$ drawn without replacement from $\{x\}_n$. Write $\sigma_n^2 = \mathbf{E}\,X_{n\,1}^2$ and assume that $\sigma_n^2 > 0$, for every $n = 2, 3, \dots$. Write $p_n = N_n/n$ and $q_n = 1 - p_n$. Erdős and Rényi (1959) proved that if

$$\forall \varepsilon > 0, \qquad \lim_{n\to\infty} \sigma_n^{-2}\mathbf{E}\,X_{n\,1}^2\mathbb{I}_{|X_{n\,1}|\ge\varepsilon\sigma_n w_n} = 0, \qquad w_n^2 = n\,p_n\,q_n, \tag{1.3}$$

then the sequence $S_n = S(\{x\}_n) = (X_{n\,1} + \dots + X_{n\,N_n})/(\sigma_n w_n)$ converges in distribution to the standard normal distribution as $n \to \infty$. Note that (1.3) implies $N_n \to \infty$ as $n \to \infty$. Hajek (1960) showed that Erdős–Rényi condition (1.3) is also necessary for the asymptotic normality of S_n. One consequence of Theorem 1.2 is that this condition is sufficient also for the asymptotic normality of the Studentized mean.

Corollary 1.3. *Assume that* (1.3) *holds. Then* $\sqrt{N_n/q_n}\ \mathbf{t}\big(\mathbb{X}_{n\,N_n}\big)$ *converges in distribution to the standard normal distribution.*

May be more interesting is the fact that it may happen that $\sqrt{N_n/q_n}\,\mathbf{t}(\mathbb{X}_{n\,N_n})$ is asymptotically standard normal when S_n doesn't. Such a situation is exhibited in the following example.

Example. Let $\{x\}_n$ be a sequence of populations as above. Assume that this sequence satisfies (1.3) and that $\sigma_n = 1$. Construct a new sequence of populations $\{\tilde{x}\}_{n+2}$ by putting $\{\tilde{x}\}_{n+2} = \{x\}_n \cup \{-n,\, n\}$. Choose the sequence N_n so that $N_n p_n \to 0$ and let $\tilde{\mathbb{X}}_{n\,N_n}$ denote a simple random sample of size N_n drawn from the population $\{\tilde{x}\}_n$. It is easy to see that in this case (1.3) fails and $S(\{\tilde{x}\}_n)$ converges to a degenerate distribution. Furthermore, since

$$\mathbf{P}\{\{-n,\, n\} \subset \tilde{\mathbb{X}}_{n+2\,N_{n+2}}\} \le 2N_{n+2}p_{n+2} \to 0$$

the limiting behavior (as $n \to \infty$) of distributions of $\mathbf{t}(\mathbb{X}_{n\,N_n})$ and $\mathbf{t}(\tilde{\mathbb{X}}_{n+2\,N_{n+2}})$ is the same, i.e., both are asymptotically standard normal.

Remark. All the results stated above remain valid if instead of the standardized Student statistic $\sqrt{N}\mathbf{t}$ one considers the selfnormalized sums

$$\frac{X_1 + \cdots + X_N}{\sqrt{X_1^2 + \cdots + X_N^2}}\,.$$

In particular Theorems 1.1 and 1.2 hold with δ_N replaced by δ'_N, where

$$\delta'_N := \sup_x \Big| \mathbf{P}\Big\{ \frac{X_1 + \cdots + X_N}{\sqrt{X_1^2 + \cdots + X_N^2}} < \sqrt{q}\, x \Big\} - \Phi(x) \Big|.$$

In contrast to the case of independent and identically distributed observations, where the normal approximation of the Studentized mean and related statistics was studied by a number of authors, see, for instance, Chung (1946), Efron (1969), Logan, Mallows, Rice and Shepp (1973), Chibisov (1980), Helmers and van Zwet (1982), van Zwet (1984), Slavova (1985), Bhattacharya and Ghosh (1978), Hall (1988), Griffin and Mason (1991), Sharakhmetov (1995), Bentkus and Götze (1996), Bentkus, Bloznelis and Götze (1996), Gine, Götze and Mason (1997), Bentkus, Götze and van Zwet (1997), Putter and van Zwet (1998) etc. there are only few results concerned with the rate of the the normal approximation of finite population Student's statistic. Praškova (1989) constructed a Berry–Esseen bound for the Studentized mean based on the observations drawn without replacement from a finite set of *random variables*, assuming that each of them is of zero mean. Rao and Zhao (1994) proved the Berry–Esseen bound,

$$\delta_N \le \frac{c}{\sqrt{q}}\ \frac{\mathbf{E}\,|X_1|^4}{\sqrt{N}\ \sigma^4}\,,$$

which establishes the rate $O(N^{-1/2})$ but involves the fourth moment. Babu and Singh (1985) studied a higher order asymptotics of the distribution function of $\sqrt{N}\,\mathbf{t}$. Berry–Esseen bounds for some other nonlinear finite population statistics were obtained by Zhao and Chen (1990), Kokic and Weber (1990) and, as a particular case of the rate of convergence of general multivariate sampling statistics, by Bolthausen and Götze (1993).

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2. Proofs

The section is organized as follows. In the beginning we formulate a general result, see Theorem 2.1 below. Then we give proofs of Theorems 1.1 and 1.2 and Corollary 1.3 which are simple consequences of Theorem 2.1. The proof of Theorem 2.1, is postponed to the end of the section.

Define the number $a \geq 0$ by the truncated second moment equation,

$$a^2 = \sup\bigl\{b: \ \mathbf{E}\, X_1^2 \mathbb{I}_{X_1^2 \leq b\, w^2} \geq b\bigr\}.$$

It is easy to check that $a \leq \sigma$ and a is the largest solution of the equation

$$a^2 = \mathbf{E}\, X_1^2 \mathbb{I}_{|X_1| \leq aw}.$$

In the case where a is positive we write

$$\gamma = a^{-2}\sigma^2 - 1, \quad \alpha = w^2|\mathbf{E}Y_1|, \quad \mu = w^2\mathbf{E}|Y_1|^3, \quad Y_1 = a^{-1}w^{-1}X_1\mathbb{I}_{|X_1|\leq aw}.$$

and note that $|Y_1| \leq 1$, $\mathbf{E}Y_1^2 = w^{-2}$ and $N^{-1/2} \leq w^{-1} \leq \mu$, by Lyapunov's inequality $(\mathbf{E}Y_1^2)^3 \leq (\mathbf{E}|Y_1|^3)^2$.

Theorem 2.1. *There exists an absolute constant $c > 0$ such that*

$$\delta_N \leq c\, w^2 \mathbf{P}\{|X_1| > a\, w\} + c(\mathcal{R} + \gamma\, \mathbb{I}_{p>q}), \qquad \mathcal{R} = \alpha + \mu, \tag{2.1}$$

whenever $a > 0$.

Proof of Theorem 1.1. Theorem 1.1 is an immediate consequence of Theorem 1.2.

Proof of Theorem 1.2. We may and shall assume without loss of generality that $\sigma = 1$. This implies $a \leq 1$.

In the case where $a^2 \geq 1/4$ we derive (1.2) from (2.1). Introduce the events $\Delta_1 = \{|X_1| > aw\}$, $\Delta_2 = \{aw < |X_1| \leq w\}$ and $\Delta_3 = \{|X_1| > w\}$. Combining the identity $\mathbb{I}_{\Delta_1} = \mathbb{I}_{\Delta_2} + \mathbb{I}_{\Delta_3}$ (here $\mathbb{I}_\Delta$ denotes the indicator function of the event Δ) and Chebyshev's inequality we get

$$\begin{aligned}
&\mathbf{P}\{|X_1| > a\,w\} = \mathbf{E}\mathbb{I}_{\Delta_2} + \mathbf{E}\mathbb{I}_{\Delta_3} \le \frac{1}{a^3w^3}\,\mathbf{E}|X_1|^3\mathbb{I}_{\Delta_2} + \frac{1}{w^2}\,\mathbf{E}X_1^2\mathbb{I}_{\Delta_3},\\
&a^2\gamma = \sigma^2 - a^2 = \mathbf{E}X_1^2\mathbb{I}_{\Delta_1} = \mathbf{E}X_1^2\mathbb{I}_{\Delta_2} + \mathbf{E}X_1^2\mathbb{I}_{\Delta_3} \le \frac{1}{aw}\,\mathbf{E}|X_1|^3\mathbb{I}_{\Delta_2} + \mathbf{E}X_1^2\mathbb{I}_{\Delta_3},\\
&aw|\mathbf{E}Y_1| = |\mathbf{E}X_1\mathbb{I}_{\Delta_1}| \le \mathbf{E}|X_1|\mathbb{I}_{\Delta_2} + \mathbf{E}|X_1|\mathbb{I}_{\Delta_3} \le \frac{1}{a^2w^2}\,\mathbf{E}|X_1|^3\mathbb{I}_{\Delta_2} + \frac{1}{w}\,\mathbf{E}X_1^2\mathbb{I}_{\Delta_3}.
\end{aligned}$$

In the last step we used $\mathbf{E}X_1 = 0$. Using these inequalities we obtain bounds for $\mathbf{P}\{|X_1| > aw\}$, α, γ and μ. Substitution of these bounds in the right hand side of (2.1) yields (1.2).

In the case where $a^2 < 1/4$ we have $\mathbf{E}\,X_1^2\mathbb{I}_{|X_1|\le w/2} < 1/4$ and therefore, $\mathbf{E}\,X_1^2\mathbb{I}_{|X_1|>w/2} \ge 3/4$. Furthermore,

$$3/4 \le \mathbf{E}\,X_1^2\mathbb{I}_{|X_1|>w/2} \le 2w^{-1}\mathbf{E}\,|X_1|^3\mathbb{I}_{w/2<|X_1|\le w} + \mathbf{E}\,X_1^2\mathbb{I}_{|X_1|>w}.$$

Since $\delta_N \le 1$, we obtain

$$\delta_N \le 1 \le \frac{8}{3}\,w^{-1}\mathbf{E}\,|X_1|^3\mathbb{I}_{w/2<|X_1|\le w} + \frac{4}{3}\,\mathbf{E}\,X_1^2\mathbb{I}_{|X_1|>w},$$

thus completing the proof of Theorem 1.2.

Proof of Corollary 1.3. We may and shall assume without loss of generality that $\sigma_n = 1$, for $n = 2, 3, \dots$.

Introduce the events $\Delta_{n\,1} = \{|X_{n\,1}| > w_n\}$ and $\Delta_{n\,2} = \{|X_{n\,1}| \le w_n\}$. In view of Theorem 1.2 it suffices to show that for every $\varepsilon > 0$,

$$\limsup_n\big(\mathbf{E}X_{n\,1}^2\mathbb{I}_{\Delta_{n\,1}} + w_n^{-1}\mathbf{E}|X_{n\,1}|^3\mathbb{I}_{\Delta_{n\,2}}\big) \le \varepsilon. \tag{2.2}$$

Let us show (2.2). Given $\varepsilon > 0$ introduce the events $\Delta_{n\,3} = \{|X_{n\,1}| > \varepsilon w_n\}$ and $\Delta_{n\,4} = \{|X_{n\,1}| \le \varepsilon w_n\}$. We have

$$\mathbf{E}X_{n\,1}^2\mathbb{I}_{\Delta_{n\,1}} + w_n^{-1}\mathbf{E}|X_{n\,1}|^3\mathbb{I}_{\Delta_{n\,2}} \le \mathbf{E}X_{n\,1}^2\mathbb{I}_{\Delta_{n\,3}} + \varepsilon\mathbf{E}X_{n\,1}^2\mathbb{I}_{\Delta_{n\,4}} \le \mathbf{E}X_{n\,1}^2\mathbb{I}_{\Delta_{n\,3}} + \varepsilon.$$

Now (2.2) follows from (1.3).

It remains to prove Theorem 2.1. We shall assume that $a > 0$ in what follows. Before the proof we introduce some notation. In what follows $c, c_1, \dots$ denote generic absolute constants. By $c(\alpha_1, \alpha_2, \dots)$ we denote constants which may depend only on the parameters $\alpha_1, \alpha_2, \dots$. We write $A \ll B$ if $A \le c\,B$. The expression $\exp\{ix\}$ is abbreviated by $\mathrm{e}\{x\}$.

For $k = 1, 2, \dots,$ write $\Omega_k = \{1, \dots, k\}$. Given a sum $S = s_1 + \dots + s_k$, denote $S^{(i)} = S - s_i$. Given $A \subset \Omega_k$ write $S_A = \sum_{j \in A} s_j$.

Let $\theta_1, \theta_2, \dots$ denote independent random variables uniformly distributed in $[0, 1]$ and independent of all other random variables considered. For a complex valued smooth function h we use the Taylor expansion

$$h(x) = h(0) + h'(0)x + \dots + h^{(n)}(0)\,\frac{x^n}{n!} + \mathbf{E}_{\theta_1} h^{(n+1)}(\theta_1 x)(1-\theta_1)^n\,\frac{x^{n+1}}{n!}\,.$$

Here $\mathbf{E}_{\theta_1}$ denotes the conditional expectation given all the random variables but θ_1. In particular, we have the mean value formula, $h(x) - h(0) = \mathbf{E}_{\theta_1} h'(\theta_1 x)x$.

Let g be a three times differentiable real function with bounded derivatives such that

$$g(x) = x^{-1/2}, \qquad \text{for} \quad |x-1| \le c_1, \quad \text{and} \quad |g(x) - 1| \le c_1, \quad \text{for} \quad x \in \mathbb{R}.$$

The (small) constant $0 < c_1 < 1$ will be specified later.

Let $\mathbb{X}^* = (X_1, \dots, X_n)$ denote a random permutation uniformly distributed over permutations of the sequence $\{x_1, \dots, x_n\}$. In particular, $X_1, \dots, X_N$ represents a simple random sample of size N drawn without replacement from $\{x\}$. Let $\overline{\nu} = (\nu_1, \dots, \nu_n)$ denote a sequence of independent Bernoulli random variables independent of $\mathbb{X}^*$ and having probabilities

$$\mathbf{P}\{\nu_i = 1\} = p, \qquad \mathbf{P}\{\nu_i = 0\} = q, \qquad 1 \le i \le n.$$

Given $A = \{i_1, \dots, i_k\} \subset \Omega_n$ let $\mathbf{E}_{\{i_1,\dots,i_k\}} = \mathbf{E}_A$ (respectively $\mathbf{E}^{(i_1,\dots,i_k)}$) denote the conditional expectation given all the random variables, but $\nu_{i_1}, \dots, \nu_{i_k}$ (respectively $X_{i_1}, \dots, X_{i_k}$).

Write

$$\begin{aligned}
Y_i &= \frac{1}{aw}\,X_i \mathbb{I}_{|X_i| \le aw}, \qquad Z_i = Y_i^2 - \mathbf{E}Y_i^2, \qquad 1 \le i \le n, \\
Y &= \sum_{i=1}^N Y_i, \qquad Z = \sum_{i=1}^N Z_i, \qquad Y' = \sum_{i=N+1}^n Y_i, \qquad Z' = \sum_{i=N+1}^n Z_i, \\
S &= (Y - \mathbf{E}Y)g(1 + qZ), \qquad S' = -(Y' - \mathbf{E}Y')g(1 - qZ'),
\end{aligned} \tag{2.3}$$

and note that

$$\mathbf{E}Z_i^2 \ll \mathbf{E}|Z_i|^{3/2} \ll \mathbf{E}|Y_i|^3 = w^{-2}\mu, \quad \mathbf{E}|Y_i - \mathbf{E}Y_i|^3 \le 8\mathbf{E}|Y_i|^3 = 8w^{-2}\mu. \tag{2.4}$$

Below we shall use the following simple inequality. Given $\{i_1, \dots, i_k\} \subset \Omega_n$ and $j \in \Omega_n \setminus \{i_1, \dots, i_k\}$ let X_j^* be a measurable function of X_j. We have

$$\mathbf{E}^{(i_1,\dots,i_k)}|X_j^*|^\alpha \le \frac{n}{n-k}\,\mathbf{E}|X_j^*|^\alpha, \qquad \text{for} \quad \alpha > 0. \tag{2.5}$$

We shall apply this inequality to random variables $Y_j, Z_j, Y_j - \mathbf{E}Y_j$, etc.

Given a random variable W, write $\Delta_W = \sup_x |\mathbf{P}\{W \le x\} - \Phi(x)|$. Let W' be a random variable defined on the same probability space as W. Then

$$\Delta_W \le \Delta_{W'} + \varepsilon \max_x |\Phi'(x)| + \mathbf{P}\{|W - W'| > \varepsilon\}, \qquad \forall \varepsilon > 0, \tag{2.6}$$

$$|\Delta_W - \Delta_{W'}| \le \mathbf{P}\{|W \ne W'|\}. \tag{2.7}$$

The proof of Theorem 2.1 consists of two steps. In the first step (see Lemma 2.1) we replace $X_1, \dots, X_N$ by truncated random variables $Y_1, \dots, Y_N$ and replace the statistic $\sqrt{N/q}\mathbf{t}$ by S (respectively by S') in the case where $p \le q$ (respectively $p > q$), see (2.3). Furthermore, the Berry–Esseen smoothing lemma reduces the problem of estimation $|P\{S \le x\} - \Phi(x)|$ to that of the estimation the difference $|\mathbf{E}\exp\{itS\} - \exp\{-t^2/2\}|$. In the second step we estimate this difference by means of expansions. For $p > q$, we estimate $|P\{S' \le x\} - \Phi(x)|$ in much the same way.

Lemma 2.1. *Assume that $a > 0$ and $N \ge 2$. Then*

$$\delta_N \le \Delta_S \mathbb{I}_{p \le q} + \Delta_{S'} \mathbb{I}_{p > q} + c\mathcal{R}_1, \quad \mathcal{R}_1 = w^2 \mathbf{P}\{|X_1| > a\,w\} + \alpha + \mu + \gamma\,\mathbb{I}_{p>q}. \tag{2.8}$$

Proof of Lemma 2.1. We may and shall assume that $\alpha < 1$ and $\mu < 1$. Otherwise (2.8) follows from the inequality $\delta_N \le 1$.

Let us prove (2.8) in the case where $p \le q$, i.e. $1/2 \le q$. Introduce the statistic $\tilde{S} = Yg(1 + qZ - qY^2/N)$ based on the sample $\mathbb{Y} = (Y_1, \dots, Y_N)$. Since $\sqrt{N/q}\mathbf{t}(\mathbb{X}) = \sqrt{N/q}\mathbf{t}(\mathbb{Y})$ on the event $A_1 = \{\mathbb{X} = aw\mathbb{Y}\}$ and $\sqrt{N/q}\mathbf{t}(\mathbb{Y}) = \tilde{S}$ on $A_2 = \{q|Z - Y^2/N| \le c_1\}$, we have

$$\mathbf{P}\{\sqrt{N/q}\mathbf{t}(\mathbb{X}) \ne \tilde{S}\} \le 1 - \mathbf{P}\{A_1 \cap A_2\} \le 1 - \mathbf{P}\{A_1\} + 1 - \mathbf{P}\{A_2\} \ll \mathcal{R}_1. \tag{2.9}$$

Indeed, $1 - \mathbf{P}\{A_1\} \le N\,\mathbf{P}\{|X_1| > a\,w\} \le 2w^2 \mathbf{P}\{|X_1| > a\,w\}$ and

$$1 - \mathbf{P}\{A_2\} \le \mathbf{P}\{|Z| > \tfrac{c_1}{2}\} + \mathbf{P}\{\tfrac{Y^2}{N} > \tfrac{c_1}{2}\} \le c\mathbf{E}|Z|^{3/2} + \tfrac{c}{N}\,\mathbf{E}Y^2 \ll \mu.$$

In the last step we used the inequalities

$$\mathbf{E}Y^2 \le c, \qquad \mathbf{E}|Z|^{3/2} \le c\,\mu \tag{2.10}$$

and $N^{-1/2} \le w^{-1} \le \mu$. To prove (2.10) we combine Hoeffding's (1963) Theorem 4 and the Marcinkiewicz-Zygmund inequality. It follows from (2.9) and (2.7) that

$$|\delta_N - \Delta_{\tilde{S}}| \ll \mathcal{R}_1. \tag{2.11}$$

Decompose $\tilde S = S + R_1 + R_2$, where $R_1 = g(1+qZ)\mathbf{E}Y$ and $R_2 = \tilde S - Yg(1+qZ)$ satisfy

$$|R_1| \le N|\mathbf{E}Y_1|(1+c_1) \le 4\alpha \qquad \text{and} \qquad |R_2| \le c|Y|^3N^{-1},$$

by the mean value theorem. Fix $\varepsilon = 5\alpha + N^{-1/2}$ and note that

$$\mathbf{P}\{|S-\tilde S| \ge \varepsilon\} \le \mathbf{P}\{|R_2| \ge N^{-1/2}\} \le N^{-1/2}\mathbf{E}|Y|^3 \ll N^{-1/2} \le \mu. \tag{2.12}$$

Here we used the inequality $\mathbf{E}|Y|^3 \le c$, which is proved in much the same way as (2.10). Finally, (2.6) applied to $\tilde S$ and S in combination with (2.12) and the simple bound $\max_x |\Phi'(x)| \le c$ implies $\Delta_{\tilde S} \le \Delta_S + c\alpha + c\mu$. This inequality together with (2.11) yields (2.8), for $p \le q$.

Let us prove (2.8) in the case, where $p > q$. We may and shall assume that $2\gamma < c_1/2$. Otherwise, (2.8) follows from the inequalities $\delta_N \le 1 \ll \gamma$.

It follows from the identities $\sum_{i=1}^n X_i = 0$ and $\sum_{i=1}^n X_i^2 = n\sigma^2$ that

$$\overline{X} = \frac{-X'}{N}, \quad \hat\sigma^2 = \frac{\sigma^2}{p} - \frac{1}{N}\sum_{i=N+1}^n X_i^2 - \frac{(X')^2}{N^2}, \quad \text{where} \quad X' = \sum_{i=N+1}^n X_i.$$

Therefore, on the event $A_3 = \{(X_{N+1},\dots,X_n) = aw(Y_{N+1},\dots,Y_n)\}$ we have

$$\sqrt{N/q}\,\mathbf{t}(\mathbb{X}) = -Y'(1-qZ'+R_3)^{1/2}, \qquad \text{where} \qquad R_3 = \gamma/p - qN^{-1}(Y')^2.$$

Furthermore, on the event $A_4 = \{q|Z' + (Y')^2/N| \le c_1/2\}$ we have $-Y'(1-qZ'-R_3)^{1/2} = \tilde S'$, where $\tilde S' = -Y'g(1-qZ'+R_3)$. Hence, $\sqrt{N/q}\,\mathbf{t}(\mathbb{X}) = \tilde S'$ on the event $A_3 \cap A_4$. It is easy to show, cf. (2.9), that $1 - \mathbf{P}\{A_3 \cap A_4\} \ll \mathcal{R}_1$. Therefore, by (2.7), $|\delta_N - \Delta_{\tilde S'}| \ll \mathcal{R}_1$. The remaining part of the proof is much the same as that of the case where $p \le q$.

Proof of Theorem 2.1. By Lemma 2.1, it suffices to show $\Delta_S \mathbb{I}_{p\le q} \ll \mathcal{R}$ and $\Delta_{S'}\mathbb{I}_{p>q} \ll \mathcal{R}$. We give the proof of the first inequality only. The proof of the second inequality is much the same.

We shall assume that $p \le 1/2 \le q$ in what follows and show that $\Delta_S \ll \mathcal{R}$. We may and shall assume that for a small constant c_2,

$$\alpha < c_2, \qquad \mu < c_2. \tag{2.13}$$

Indeed, if at least one of these inequalities fails we obtain $\Delta_S \le 1 \ll \mathcal{R}$.

Denote

$$\varphi(t) = \mathbf{E}\,\mathrm{e}\{tS\}, \qquad \psi(t) = \mathbf{E}\,\mathrm{e}\{t(Y - \mathbf{E}Y)\}, \qquad \phi_r(t) = \exp\{-t^2r^2/2\}, \quad r > 0.$$

Given two complex valued functions f and h write

$$I_{[d,e]}(f,h) = \int_{|t|\in(d;e]} |t|^{-1}|f(t)-h(t)|dt, \qquad e > d \geq 0.$$

The Berry-Esseen smoothing inequality, see Feller (1971), 538 p., yields

$$\Delta_S \ll I_{[0;H]}(\varphi,\phi_1) + H^{-1}, \qquad H = c_3 b^2 \mu_0^{-1}. \tag{2.14}$$

Here we denote

$$b^2 = w^2\mathbf{E}(Y_1 - \mathbf{E}Y_1)^2 = 1 - \alpha^2 w^{-2}, \qquad \mu_0 = w^2\mathbf{E}|Y_1 - \mathbf{E}Y_1|^3.$$

The (small) constant c_3 will be specified later. Since $\mu_0 \ll \mu$ and, by (2.13), $b^{-2} \leq c$, we have $H^{-1} \ll \mathcal{R}$. It remains to show $I_{[0;H]}(\varphi,\phi_1) \ll \mathcal{R}$. Write

$$I_{[0;H]}(\varphi,\phi_1) \leq I_{[0;H]}(\varphi,\psi) + I_{[0;H]}(\psi,\phi_b) + I_{[0;H]}(\phi_b,\phi_1).$$

Clearly, $I_{[0;H]}(\phi_b,\phi_1) \ll (1-b^2) \ll \mathcal{R}$, by (2.13). It follows from Höglund's (1978), formula (8), that $I_{[0;H]}(\psi,\phi_b) \ll b^{-3}\mu_0$, provided that c_3 is sufficiently small. By (2.13), $b^{-3}\mu_0 \ll \mu_0 \ll \mu$. Therefore, it remains to bound $I_{[0;H]}(\varphi,\psi)$. We split $I_{[0;H]}(\varphi,\psi) = I_{[0;c_4]}(\varphi,\psi) + I_{[c_4;H]}(\varphi,\psi)$ and estimate the summands separately.

Let us show

$$I_{[c_4;H]}(\varphi,\psi) \ll \mathcal{R}. \tag{2.15}$$

To this aim we represent the characteristic functions φ and ψ in Erdős-Rényi (1959) form, see (2.16) below. Write

$$T = \sum_{i=1}^n T_i, \qquad Q = \sum_{i=1}^n Q_i, \qquad S = \sum_{i=1}^n S_i,$$
$$T_i = (Y_i - \mathbf{E}Y_i)(\nu_i - p), \qquad Q_i = qZ_i(\nu_i - p), \qquad S_i = w^{-1}(\nu_i - p).$$

We have

$$\varphi = \lambda\int_{-\pi w}^{\pi w} \mathbf{E}\,\mathrm{e}\{tTg(1+Q) + sS\}ds, \qquad \psi = \lambda\int_{-\pi w}^{\pi w} \mathbf{E}\,\mathrm{e}\{tT + sS\}ds, \tag{2.16}$$

with $\lambda^{-1} = 2\pi w\mathbf{P}\{S=0\}$. Höglund (1978) showed that $2^{-1/2}\pi \leq \lambda^{-1} \leq (2\pi)^{1/2}$.

Given a number $L > 0$ and a complex valued bivariate function f write $f \prec L$ if

$$\int_{\mathcal{Z}} |t|^{-1}|f(s,t)|dsdt \ll L, \quad \text{where } \mathcal{Z} = \{(s,t) : c_4 \leq |t| \leq H,\ |s| \leq \pi w\}.$$

Given two complex valued functions f, h write $f \sim h$ if $f - h \prec \mathcal{R}$.
Introduce the integer valued function

$$m = m(s,t) \approx 2^{-1} c_4 n u^{-1} \ln u, \qquad u = t^2 + s^2, \qquad (s,t) \in \mathcal{Z}. \tag{2.17}$$

A simple calculation shows that $10 \le m(s,t) \le n/2$, for $(s,t) \in \mathcal{Z}$, provided that c_4 is sufficiently large. Write $z := mpqw^{-2} = m/n \ll u^{-1} \ln u$. We shall often use the following fact. For $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \ge 0$ satisfying $\alpha_3 + \alpha_4 > \alpha_1 + \alpha_2 + 1/2$,

$$(t^2)^{\alpha_1} (s^2)^{\alpha_2} z^{\alpha_3} u^{-\alpha_4} \prec c(\alpha_1, \alpha_2, \alpha_3, \alpha_4).$$

Denote

$$A = \Omega_m, \qquad B = \Omega_n \setminus \Omega_m, \qquad g_0 = g(1 + Q_B), \qquad g_1 = g'(1 + Q_B).$$

Split

$$T = T_A + T_B, \quad Q = Q_A + Q_B, \quad T_A Q_A = D_A + U_A, \quad T_B Q_B = D_B + U_B, \tag{2.18}$$

where, we denote

$$D_G = \sum_{j \in G} T_j Q_j, \qquad U_G = \sum_{i,j \in G,\, i \ne j} T_i Q_j, \qquad G \subset \Omega_n. \tag{2.19}$$

Introduce the random variables

$$\begin{aligned}
&v_j = v_j^* - 2^{-1} t T_j Q_j, \qquad v_j^* = t T_j g_0 + s S_j, \qquad v_j^\star = t T_j + s S_j, \qquad 1 \le j \le n, \\
&V = \sum_{j=1}^n v_j, \qquad V^* = \sum_{j=1}^n v_j^*, \qquad V^\star = \sum_{j=1}^n v_j^\star, \\
&H_G = |\mathbf{E}_G \, \mathrm{e}\{V_G\}|, \quad H_G^* = |\mathbf{E}_G \, \mathrm{e}\{V_G^*\}|, \quad H_G^\star = |\mathbf{E}_G \, \mathrm{e}\{V_G^\star\}|, \quad G \subset \Omega_n.
\end{aligned}$$

Several useful inequalities to be used below are collected in the next two lemmas.

Lemma 2.2. *Assume that (2.13) holds. We have*

$$H\mu \ll 1, \qquad H^2 \mathbf{E}(Y_1 - \mathbf{E}Y_1)^2 \le c_3^2, \tag{2.20}$$

$$\mathbf{E}U_A^2 \ll z^2 \mu, \qquad \mathbf{E}|U_A Q_A| \ll z^{3/2} \mu, \tag{2.21}$$

$$\mathbf{E}|T_B Q_A^2|^{3/4} \ll z\mu, \qquad \mathbf{E}|T_B Q_A|^{3/2} \ll z\mu, \tag{2.22}$$

$$\mathbf{E}|\sum_{j \in A} T_j Q_j^2| \ll z\mu \qquad \mathbf{E}|\sum_{j \in A} T_j Q_j Q_A^{(j)}| \ll z^{3/2} \mu^{3/2}. \tag{2.23}$$

For any $G \subset \Omega_n$ and $i_1, i_2, i_3 \in \Omega_n \setminus G$, we have

$$\mathbf{E}^{(i_1, i_2, i_3)} |T_G|^r \ll c, \qquad 0 < r \le 6. \tag{2.24}$$

Lemma 2.3. *Let $G \subset \Omega_n$ and $|G| \geq m/4$. There exists a small constant $c_* > 0$ such that the inequality $c_1, c_2, c_3, c_4^{-1} < c_*$ implies*

$$\mathbf{E}^{(i,j)} H_G^2 < u^{-10}, \quad \mathbf{E}^{(i,j)} (H_G^*)^2 < u^{-10}, \quad \mathbf{E}^{(i,j)} (H_G^\star)^2 < u^{-10}, \tag{2.25}$$

$$\mathbf{E}^{(i,j)} H_G < u^{-5}, \quad \mathbf{E}^{(i,j)} H_G^* < u^{-5}, \quad \mathbf{E}^{(i,j)} H_G^\star < u^{-5}, \tag{2.26}$$

for any $i, j \in \Omega \setminus G$.

These lemmas are proved in Section 3. We shall assume that c_1, c_2, c_3 and c_4^{-1} are choosen small enough so that (2.25) and (2.26) hold.

In view of the inequality $\lambda \leq 2^{1/2}\pi^{-1}$, (2.15) can be written as follows,

$$f \sim f^*, \qquad \text{where} \qquad f = \mathbf{E}\,\mathrm{e}\{tTg(1+Q) + sS\}, \quad f^* = \mathbf{E}\,\mathrm{e}\{tT + sS\}. \tag{2.27}$$

Let us prove (2.27). The proof consists of the following steps,

$$f \sim f_1, \qquad f_1 = \mathbf{E}\,\mathrm{e}\{W_1\}, \quad W_1 = V^* + tTQ_A g_1, \tag{2.28}$$

$$f_1 \sim f_2, \qquad f_2 = \mathbf{E}\,\mathrm{e}\{W_2 + tT_B Q_A g_1\}, \quad W_2 = V_A + V_B^*, \tag{2.29}$$

$$f_2 \sim f_3, \qquad f_3 = \mathbf{E}\,\mathrm{e}\{V_A + V_B^*\}, \tag{2.30}$$

$$f_3 \sim f_4, \qquad f_4 = \mathbf{E}\,\mathrm{e}\{V^*\}, \tag{2.31}$$

$$f_4 \sim f_5, \qquad f_5 = \mathbf{E}\,\mathrm{e}\{V_A^\star + V_B^*\}, \tag{2.32}$$

$$f_5 \sim f^*. \tag{2.33}$$

Proof of (2.28). Expanding in powers of Q_A we get $g(1+Q) = g_0 + Q_A g_1 + Q_A^2 r$, where r is a bounded function of Q_A, Q_B. Substituting this expansion we obtain $tTg(1+Q) + sS = W_1 + tTQ_A^2 r$ and therefore,

$$|f - f_1| \leq \mathbf{E}|\,\mathrm{e}\{tTQ_A^2 r\} - 1|. \tag{2.34}$$

By (2.18), $TQ_A^2 = R_1 + R_2 + R_3$, where $R_1 = T_B Q_A^2$, $R_2 = U_A Q_A$ and $R_3 = D_A Q_A$. Split

$$R_3 = R_{3.1} + R_{3.2}, \qquad R_{3.1} = \sum_{j \in A} T_j Q_j Q_A^{(j)}, \quad R_{3.2} = \sum_{j \in A} T_j Q_j^2.$$

Now, applying the inequality

$$|\,\mathrm{e}\{x\} - 1| \leq 2|x|^\tau, \qquad 0 \leq \tau \leq 1, \qquad x \in \mathbb{R}, \tag{2.35}$$

several times, with $\tau = 1$ and $\tau = 3/4$, we get from (2.34)

$$\begin{aligned} |f - f_1| &\ll |t|(\mathbf{E}|R_2| + \mathbf{E}|R_{3.1}|) + |t|^{3/4}(\mathbf{E}|R_1|^{3/4} + \mathbf{E}|R_{3.2}|^{3/4}) \\ &\ll |t|(z^{3/2}\mu + z^{3/2}\mu^{3/2}) + |t|^{3/4} z\mu, \end{aligned}$$

by Lemma 2.2. We obtain $|f - f_1| \prec \mathcal{R}$, thus proving (2.28).

Proof of (2.29). Write $TQ_A = T_B Q_A + D_A + U_A$, see (2.18), and expand $g_1 = g'(1+Q_B) = -2^{-1} + Q_B r$ to get $D_A g_1 = -2^{-1} D_A + D_A Q_B r$, where r is a bounded function of Q_B. Now we have

$$W_1 = W_2 + tT_B Q_A g_1 + w_1 + w_2, \qquad w_1 = tU_A g_1, \qquad w_2 = tD_A Q_B r.$$

Firstly we shall show $f_1 \sim f_6$, where $f_6 = \mathbf{E}\,\mathrm{e}\{W_2 + tT_B Q_A g_1 + w_1\}$. By (2.35), $|f_1 - f_6| \ll \mathbf{E}|w_2|$. Let us show $\mathbf{E}|w_2| \prec \mathcal{R}$. By the symmetry,

$$\mathbf{E}|w_2| \le m|t|\mathbf{E}|T_1 Q_1 Q_B| = m|t|\mathbf{E}|T_1 Q_1|\mathbf{E}^{(1)}|Q_B|. \tag{2.36}$$

Since $\nu_j - p$, $1 \le j \le n$, are independent centered random variables, we have

$$\mathbf{E}^{(1)} Q_B^2 = \sum_{j \in B} \mathbf{E}^{(1)} Q_j^2 = |B| pq \mathbf{E}^{(1)} Z_n^2,$$

by the symmetry. Furthermore, combining (2.5) and (2.4) we obtain $\mathbf{E}^{(1)} Q_B^2 \ll \mu$ and therefore, $\mathbf{E}^{(1)}|Q_B| \ll \mu^{1/2}$. Substituting this bound in (2.36) and estimating $\mathbf{E}|T_1 Q_1| \ll pq\mathbf{E}|Y_1|^3$ we obtain $\mathbf{E}|w_2| \ll |t| z \mu^{3/2} \ll |t|^{1/2} z\mu \prec \mathcal{R}$. In the last step we used the inequality $|t|\mu \ll 1$, which holds for $|t| \le H$, see (2.20).

Let us show $f_6 \sim f_2$. Expanding the exponent in powers of iw_1 we get

$$f_6 = f_2 + f_7 + R, \quad f_7 = \mathbf{E}\,\mathrm{e}\{W_2 + tT_B Q_A g_1\} iw_1, \quad \text{with} \quad |R| \ll t^2 \mathbf{E} U_A^2.$$

By (2.21), $|R| \ll t^2 z^2 \mu \prec \mathcal{R}$. Therefore, $f_1 \sim f_2 + f_7$. Next we show

$$f_7 \sim f_8, \qquad f_8 = \mathbf{E}\,\mathrm{e}\{W_2\} iw_1. \tag{2.37}$$

An application of (2.35) with $\tau = 3/4$ gives

$$|f_7 - f_8| \ll |t|^{7/4} \mathbf{E}|T_B Q_A|^{3/4} |U_A| \le |t|^{7/4} (\mathbf{E}|T_B Q_A|^{3/2})^{1/2} (\mathbf{E} U_A^2)^{1/2},$$

by Cauchy–Schwarz. Invoking inequalities of Lemma 2.2 we obtain $|f_7 - f_8| \ll |t|^{7/4} z^{3/2} \mu \prec \mathcal{R}$ and thus, (2.37) follows.

We complete the proof of (2.29) by showing $f_8 \prec \mathcal{R}$. By the symmetry,

$$f_8 = it(m^2 - m) f_9, \qquad f_9 = \mathbf{E}\,\mathrm{e}\{W_2\} T_1 Q_2 g_1, \tag{2.38}$$

Recall that $W_2 = V_A + V_B^*$ and write

$$f_9 = \mathbf{E}\,\mathrm{e}\{V_{A''} + V_B^*\}\,\mathrm{e}\{v_1 + v_2\} T_1 Q_2 g_1, \qquad A'' = A \setminus \{1, 2\}.$$

Expanding

$$\mathrm{e}\{v_1+v_2\} = (1+v_1r_1)\,\mathrm{e}\{v_2\} = \mathrm{e}\{v_2\} + v_1r_1(1+v_2r_2), \quad r_j = i\mathbf{E}_{\theta_j}\,\mathrm{e}\{\theta_j v_j\},$$

and using the fact that the conditional expectation of T_1 (respectively Q_2) given all the random variables, but ν_1 (respectively ν_2) is zero, we obtain

$$f_9 = -\mathbf{E}\,\mathrm{e}\{V_{A''} + V_B^*\}Rg_1, \qquad R = T_1Q_2v_1v_2r_1r_2.$$

Since $|g_1| \le c$ and R is a function of ν_1, ν_2, X_1, X_2 and $H_{A''}$ is a function of $X_3, \dots, X_m$ we can write

$$|f_9| \le \mathbf{E}|R|H_{A''} \ll \mathbf{E}|R|\mathbf{E}^{(1,2)}H_{A''},$$

Combining the inequality $\mathbf{E}^{(1,2)}H_{A''} < u^{-5}$, see Lemma 2.3, and the simple bound $\mathbf{E}|R| \ll p^2q^2w^{-4}u\mu$, we obtain $|f_9| \ll n^{-2}u^{-4}\mu$. Substituting this inequality in (2.38) we get $f_8 \prec \mathcal{R}$ thus completing the proof of (2.29).

Proof of (2.30). Split $A = A_1 \cup A_2 \cup A_3$ so that $A_i \cap A_j = \emptyset$, for $i \ne j$, and $|A_j| \approx m/3$ and $j \in A_j$, for $j = 1, 2, 3$. Write

$$tT_BQ_Ag_1 = w_1 + w_2 + w_3, \qquad w_j = tT_BQ_{A_j}g_1, \quad j = 1, 2, 3,$$

and denote $W_3 = W_2 + w_2 + w_3$. Firstly we show

$$f_2 \sim f_{10} + f_{11}, \qquad f_{10} = \mathbf{E}\,\mathrm{e}\{W_3\}, \qquad f_{11} = \mathbf{E}\,\mathrm{e}\{W_3\}iw_1. \tag{2.39}$$

Expanding the exponent in $f_2 = \mathbf{E}\,\mathrm{e}\{W_3 + w_1\}$ in powers of iw_1, we obtain

$$f_2 = f_{10} + f_{11} + f_{12}, \qquad f_{12} = \mathbf{E}\,\mathrm{e}\{W_3\}w_1^2r_1,$$

where r_1 is a bounded function of w_1.

Let us show $f_{12} \prec \mathcal{R}$. Expanding

$$\mathrm{e}\{w_2+w_3\} = (1+w_2r_2)\,\mathrm{e}\{w_3\} = \mathrm{e}\{w_3\} + w_2r_2(1+w_3r_3),$$

where r_j is a bounded function of w_j, for $j = 2, 3$, we obtain

$$f_{12} = f_{12.1} + f_{12.2} + f_{12.3}, \qquad f_{12.1} = \mathbf{E}\,\mathrm{e}\{W_2 + w_3\}w_1^2r_1,$$
$$f_{12.2} = \mathbf{E}\,\mathrm{e}\{W_2\}w_1^2w_2r_1r_2, \qquad f_{12.3} = \mathbf{E}\,\mathrm{e}\{W_2\}w_1^2w_2w_3r_1r_2r_3.$$

We shall show that $f_{12.j} \prec \mathcal{R}$, for $j = 1, 2, 3$. Clearly,

$$|f_{12.1}| \ll \mathbf{E}H_{A_2}w_1^2, \qquad |f_{12.2}| \ll \mathbf{E}H_{A_3}w_1^2|w_2|, \qquad |f_{12.3}| \ll \mathbf{E}w_1^2|w_2w_3|.$$

Using the symmetry and the fact that conditionally, given $\mathbb{X}^*$, the random variables Q_j, $j \in \Omega_n$ are uncorrelated we construct bounds for $f_{12.j}$, $j = 1, 2, 3$. We have

$$|f_{12.3}| \le t^4 \mathbf{E} T_B^4 Q_{A_1}^2 |Q_{A_2} Q_{A_3}| = t^4 |A_1| \mathbf{E} T_B^4 Q_1^2 |Q_{A_2} Q_{A_3}| \le t^4 m^3 \mathbf{E} T_B^4 Q_1^2 |Q_2 Q_3|,$$

Combining the bound $\mathbf{E}^{(1,2,3)} T_B^4 \le c$, see (2.24), and the inequalities

$$\mathbf{E} Q_1^2 |Q_2 Q_3| \ll p^3 q^3 \mathbf{E} Z_1^2 |Z_2 Z_3| \ll p^3 q^3 (\mathbf{E} |Z_1|^{3/2})(\mathbf{E}|Z_2|)(\mathbf{E}|Z_3|) \ll p^3 q^3 w^{-6} \mu$$

(here we use (2.4) and (2.5)) we obtain $f_{12.3} \ll t^4 z^3 \mu \prec \mathcal{R}$.
Similarly,

$$|f_{12.2}| \ll |t|^3 m^2 \mathbf{E} H_{A_3} |T_B|^3 Q_1^2 |Q_2| \ll |t|^3 m^2 p^2 q^2 \mathbf{E} Z_1^2 |Z_2| \mathbf{E}^{(1,2)} H_{A_3} |T_B|^3. \tag{2.40}$$

By Hölder's inequality, (2.25) and (2.24),

$$\mathbf{E}^{(1,2)} H_{A_3} |T_B|^3 \le (\mathbf{E}^{(1,2)} H_{A_3}^2)^{1/2} (\mathbf{E}^{(1,2)} T_B^6)^{1/2} \ll u^{-5}. \tag{2.41}$$

Substituting (2.41) in (2.40) and then using the inequalities

$$\mathbf{E} Z_1^2 |Z_2| \ll \mathbf{E} Z_1^2 \mathbf{E} |Z_2| \ll w^{-4} \mu,$$

(here we apply (2.5) and (2.4)) we obtain $f_{12.2} \ll |t|^3 u^{-5} \mu \prec \mathcal{R}$. Finally,

$$|f_{12.1}| \ll t^2 |A_1| \mathbf{E} H_{A_3} T_B^2 Q_1^2 \ll t^2 mpq \mathbf{E} Z_1^2 \mathbf{E}^{(1)} H_{A_3} T_B^2.$$

Combining the inequalities $\mathbf{E}^{(1)} H_{A_3} T_B^2 \ll u^{-5}$, cf. (2.41), and $\mathbf{E} Z_1^2 \ll w^{-2} \mu$, see (2.4), we obtain $|f_{12.1}| \ll t^2 u^{-5} z \mu \prec \mathcal{R}$, thus completing the proof of (2.39).
Let us show

$$f_{11} \prec \mathcal{R}, \quad \text{where} \quad f_{11} = \mathbf{E}\,\mathrm{e}\{W_3\} i w_1, \quad W_3 = V_A + V_B^* + w_2 + w_3. \tag{2.42}$$

By the symmetry, $f_{11} = it|A_1| \mathbf{E}\,\mathrm{e}\{W_3\} T_B g_1 Q_1$. Expanding the exponent in powers iv_1 and using the fact that the conditional expectation of Q_1 given all the random variables but ν_1 is zero, we get

$$f_{11} = i^2 t |A_1| \mathbf{E}\,\mathrm{e}\{V_{A'} + V_B^* + w_2 + w_3\} T_B g_1 Q_1 v_1 r_1, \qquad A' = A \setminus \{1\},$$

where r_1 is a bounded function of v_1. Clearly,

$$|f_{11}| \ll |t| m \mathbf{E} |Q_1 v_1 T_B| H_{A_1'} \le |t| m \mathbf{E} |Q_1 v_1| \mathbf{E}^{(1)} T_B | H_{A_1'}, \qquad A_1' = A_1 \setminus \{1\}.$$

Combining the inequality $\mathbf{E}^{(1)} H_{A_1'} |T_B| \ll u^{-5}$, cf. (2.41), and the simple bound $\mathbf{E}|Q_1 v_1| \ll pq(|t|+|s|)w^{-3}$ we obtain $|f_{11}| \ll (|t|+|s|)u^{-5}w^{-1} \prec w^{-1} \le \mu$, thus proving (2.42).

Let us show $f_{10} \sim f_3$. Write $w_4 := w_2 + w_3$. We have $W_3 = V_A + V_B^* + w_4$. Expanding the exponent in f_{10} in powers of iw_4 we obtain

$$f_{10} = f_3 + f_{13} + f_{14}, \qquad f_{13} = \mathbf{E}\,\mathrm{e}\{V_A + V_B^*\} iw_4, \qquad f_{14} = \mathbf{E}\,\mathrm{e}\{V_A + V_B^*\} w_4^2 r,$$

where r is a bounded function of w_4. The proof of $f_{13} \prec \mathcal{R}$ (respectively $f_{14} \prec \mathcal{R}$) is much the same as that of $f_{11} \prec \mathcal{R}$ (respectively $f_{12.1} \prec \mathcal{R}$) above. Therefore, $f_{10} \sim f_3$. Now, invoking (2.39) and (2.42), we obtain (2.30).

Proof of (2.31). Split $A = A_1 \cup A_2$ so that

$$A_1 \cap A_2 = \emptyset \quad \text{and} \quad |A_j| \approx m/2 \quad \text{and} \quad j \in A_j, \quad \text{for} \quad j = 1,2. \tag{2.43}$$

Write $D_A = D_{A_1} + D_{A_2}$, see (2.19), and denote $w_j = tD_{A_j} g_1$, for $j = 1,2/$ We have $f_3 = \mathbf{E}\,\mathrm{e}\{V^* + w_1 + w_2\}$. Expanding the exponent in powers of iw_1 and iw_2 we get

$$f_3 = f_4 + f_{15} + f_{16}, \qquad f_{15} = \mathbf{E}\,\mathrm{e}\{V^*\} w_1 r_1, \quad f_{16} = \mathbf{E}\,\mathrm{e}\{V^* + w_1\} w_2 r_2,$$

where r_j is a bounded function of w_j, $j = 1,2$. By the symmetry,

$$|f_{15}| \ll |t| \mathbf{E}|D_{A_1}| H_{A_2}^* \le |t||A_1| \mathbf{E}|T_1 Q_1| H_{A_2}^*.$$

Similarly, $|f_{15}| \le |t||A_2| \mathbf{E}|T_2 Q_2| H_{A_1}$. Combining the inequalities $\mathbf{E}^{(1)} H_{A_2}^* \ll u^{-5}$ and $\mathbf{E}^{(2)} H_{A_1} \ll u^{-5}$, see Lemma 2.3, and the simple bound $\mathbf{E}|T_i Q_i| \ll pqw^{-2}\mu$ we obtain $f_{15} \prec \mathcal{R}$ and $f_{16} \prec \mathcal{R}$ thus proving (2.31).

Proof of (2.32). Split $V^* = V_A^* + V_B^*$ and $V_A^* = V_{A_1}^* + V_{A_2}^*$, where $A_1 \cup A_2 = A$ satisfy (2.43). In order to prove (2.32) we shall show

$$f_4 \sim f_{17}, \qquad f_{17} = \mathbf{E}\,\mathrm{e}\{W_4\}, \qquad W_4 = V_{A_1}^\star + V_{A_2 \cup B}^*, \tag{2.44}$$

and $f_{17} \sim f_5$.

Let us prove (2.44). Expanding $g_0 = g(1 + Q_B) = 1 - Q_B/2 + Q_B^2 r$ we get

$$V_{A_1}^* = V_{A_1}^\star + w_1 + w_2, \qquad \text{with} \qquad w_1 = -tT_{A_1} Q_B/2, \quad w_2 = tT_{A_1} Q_B^2 r,$$

where r is a bounded function of Q_B. Furthermore, expanding the exponent in $f_4 = \mathbf{E}\,\mathrm{e}\{W_4 + w_1 + w_2\}$ and in powers of iw_2 and iw_1 to obtain

$$\begin{aligned} &f_4 = f_{17} + f_{18} + f_{19} + f_{20}, \quad f_{18} = \mathbf{E}\,\mathrm{e}\{W_4\} iw_1, \\ &f_{19} = \mathbf{E}\,\mathrm{e}\{W_4\} iw_1^2 r_1, \quad f_{20} = \mathbf{E}\,\mathrm{e}\{W_4 + w_1\} iw_2 r_2, \end{aligned}$$

where r_j is a bounded function of w_j, $j = 1, 2$.

To show $f_{19} \prec \mathcal{R}$ we use symmetry, and the fact that conditionally, given all the random variables but ν_i, $i \in B$, the random variables Q_i, $i \in B$ are uncorrelated,

$$|f_{19}| \ll t^2 \mathbf{E} Q_B^2 T_{A_1}^2 H_{A_2}^* = t^2 |B| pq \mathbf{E} Z_n^2 T_{A_1}^2 H_{A_2}^*.$$

Combining the bounds $\mathbf{E} Z_n^2 \ll w^{-2}\mu$ and $\mathbf{E}^{(n)} T_{A_1}^2 H_{A_2}^* \ll u^{-5}$, cf. (2.41), we obtain $f_{19} \prec \mathcal{R}$. The proof of $f_{20} \prec \mathcal{R}$ is much the same.

Let us show $f_{18} \prec \mathcal{R}$. By the symmetry,

$$f_{18} = -2^{-1} it |A_1||B| \mathbf{E}\, \mathrm{e}\{W_4\} T_1 Q_n.$$

Write $V_{A_1}^\star = V_{A_1'}^\star + v_1^\star$, where $A_1' = A_1 \setminus \{1\}$ and $V_{A_2 \cup B}^* = V_{A_2 \cup B'}^* + v_n^*$, where $B' = B \setminus \{n\}$. Expanding $g_0 = g(1 + Q_{B'} + Q_n) = g(1 + Q_{B'}) + Q_n r_n$ we get $V_{A_2 \cup B'}^* = W_5 + w_3$, where

$$W_5 = t T_{A_2 \cup B'} g(1 + Q_{B'}) + s S_{A_2 \cup B'} \qquad \text{and} \qquad w_3 = v_n^* + t T_{A_2 \cup B'} Q_n r_n.$$

Here r_n is a bounded function of Q_n. We have $W_4 = V_{A_1'}^\star + W_5 + v_1^\star + w_3$ and therefore,

$$f_{18} = -2^{-1} it |A_1||B| \mathbf{E}\, \mathrm{e}\{V_{A_1'}^\star + W_5 + v_1^\star + w_3\} T_1 Q_n.$$

Expanding the exponent in powers of $iv_1^\star$ and then in powers of iw_3 and using the fact that the conditional expectation of T_1 (respectively Q_n) given all the random variables, but ν_1 (respectively ν_n) is zero, we get

$$f_{18} = 2^{-1} it |A_1||B| \mathbf{E}\, \mathrm{e}\{V_{A_1'}^\star + W_5\} T_1 v_1^\star Q_n w_3 r_3,$$

where r_3 is a bounded function of $v_1^\star$ and w_3. Clearly,

$$|f_{18}| \ll |t||A_1||B| \mathbf{E} |T_1 v_1^\star|\, |Q_n w_3| H_{A_1'}^\star.$$

Combining the bound $\mathbf{E}^{(1,n)} H_{A_1'}^\star < u^{-5}$, see (2.26), and the simple inequality

$$\mathbf{E} |T_1 v_1^\star|\, |Q_n w_3| \ll p^2 q^2 (|t| + |s|) w^{-4} \mu$$

we obtain $f_{18} \prec \mathcal{R}$ thus completing the proof of (2.44). The proof of $f_{17} \sim f_5$ is much the same. We arrive at (2.32).

Proof of (2.33). Expanding

$$g_0 = g(1 + Q_B) = 1 + Q_B g_2(Q_B), \qquad g_2(Q_B) = \mathbf{E}_{\theta_1} g'(1 + \theta_1 Q_B),$$

we obtain $V_B^* = V_B^\star + tT_BQ_Bg_2(Q_B)$. Split $T_BQ_B = U_B + D_B$ and write

$$V_B^* = V_B^\star + w_1 + w_2, \qquad w_1 = tU_Bg_2(Q_B), \qquad w_2 = tD_Bg_2(Q_B).$$

We have $f_5 = \mathbf{E}\,\mathrm{e}\{V^\star + w_1 + w_2\}$. Expanding in powers of iw_1 and iw_2 we get

$$f_5 = f^* + f_{21} + f_{22} + f_{23}, \qquad f_{21} = \mathbf{E}\,\mathrm{e}\{V^\star\}iw_1,$$
$$f_{22} = \mathbf{E}\,\mathrm{e}\{V^\star\}w_1^2r_1, \qquad f_{23} = \mathbf{E}\,\mathrm{e}\{V^\star + w_1\}w_2r_2,$$

where r_j is a bounded function of w_j, $j = 1, 2$.

Let us show $f_{22} \prec \mathcal{R}$ and $f_{23} \prec \mathcal{R}$. Using the fact that given $\mathbb{X}^*$, the random variables $T_{i_1}Q_{j_1}$ and $T_{i_2}Q_{j_2}$, for $i_1 \neq j_1$, $i_2 \neq j_2$, are conditionally uncorrelated unless the sets $\{i_1, j_1\}$ and $\{i_2, j_2\}$ coincide, we get

$$\mathbf{E}_BU_B^2 = \sum_{i,j\in B,\, i\neq j} \mathbf{E}_B\tilde{Z}_{i,j}, \qquad \tilde{Z}_{i,j} = T_i^2Q_j^2 + T_iQ_jT_jQ_i. \tag{2.45}$$

Therefore, by the symmetry,

$$|f_{22}| \ll t^2\mathbf{E}U_B^2H_A^\star = t^2(|B|^2 - |B|)\mathbf{E}\tilde{Z}_{n,n-1}H_A^\star.$$

Furthermore,

$$|f_{23}| \ll |t|\mathbf{E}|D_B|H_A^\star \leq |t||B|\mathbf{E}|T_nQ_n|H_A^\star.$$

Combining the bound $\mathbf{E}^{(1,2)}H_A^\star < u^{-5}$, see (2.26), and the inequalities $\mathbf{E}|T_nQ_n| \ll pqw^{-2}\mu$ and $\mathbf{E}|\tilde{Z}| \ll p^2q^2w^{-4}\mu$ we obtain $f_{22} \ll |t|u^{-5}\mu \prec \mathcal{R}$ and $f_{23} \ll t^2u^{-5}\mu \prec \mathcal{R}$.

We complete the proof of (2.33) by showing $f_{21} \prec \mathcal{R}$. By the symmetry,

$$f_{21} = (|B|^2 - |B|)itf_{24}, \qquad f_{24} = \mathbf{E}\,\mathrm{e}\{V^\star\}T_nQ_{n-1}g_2(Q_B). \tag{2.46}$$

Write $Q_B = Q_{B'} + Q_n$, $B' = B \setminus \{n\}$. Expanding g_2 in powers of Q_n we get

$$f_{24} = f_{25} + R_1, \quad f_{25} = \mathbf{E}\,\mathrm{e}\{V^\star\}T_nQ_{n-1}g_2(Q_{B'}), \quad |R_1| \ll \mathbf{E}|T_nQ_nQ_{n-1}|H_A^\star.$$

Combining (2.26) and the simple bound $\mathbf{E}|T_nQ_nQ_{n-1}| \ll p^2q^2w^{-4}\mu$, we obtain $|R_1| \ll n^{-2}u^{-5}\mu$.

Expanding the exponent in powers of $v_n^\star$ and using the fact that the conditional expectation of T_n given all the random variables, but ν_n is zero, we obtain

$$f_{25} = f_{26}, \qquad f_{26} = \mathbf{E}\,\mathrm{e}\{V_{\Omega_{n-1}}^\star\}T_nQ_{n-1}g_2(Q_{B'})v_n^\star r_n^\star,$$

where $r_n^\star$ is a bounded function of $v_n^\star$.

Write $B'' = B' \setminus \{n-1\}$. Expanding g_2 in powers of Q_{n-1} we obtain $f_{26} = f_{27} + R_2$, where f_{27} is defined in the same way as f_{26}, but with $g_2(Q_{B'})$ replaced by $g_2(Q_{B''})$ and

$$|R_2| \ll \mathbf{E}|T_n v_n^\star| Q_{n-1}^2 H_A^\star \ll u^{-5}(|t|+|s|) n^{-2}\mu.$$

In the last inequality we apply (2.26) and the simple bound $\mathbf{E}|T_n v_n^\star| Q_{n-1}^2 \ll (|t|+|s|) p^2 q^2 w^{-4}\mu$.

Finally, expanding the exponent in f_{27} in powers of $v_{n-1}^\star$ and using the fact that the conditional expectation of Q_{n-1} given all the random variables, but ν_{n-1} is zero, we obtain

$$|f_{27}| \ll \mathbf{E}|T_n v_n^\star Q_{n-1} v_{n-1}^\star| H_A^\star \ll (|t|+|s|)^2 u^{-5} n^{-2}\mu, \tag{2.47}$$

by (2.26) and the simple bound $\mathbf{E}|T_n v_n^\star Q_{n-1} v_{n-1}^\star| \ll (|t|+|s|)^2 p^2 q^2 w^{-4}\mu$.

It follows from (2.47) and the bounds for R_1, R_2 that $|f_{24}| \ll u^{-4} n^{-2}\mu$. Now, by (2.46), $f_{21} \prec \mathcal{R}$. We obtain (2.33) and, thus, complete the proof of (2.27).

We arrive at (2.15). The proof of the inequality $I_{[0;c_4]} \ll \mathcal{R}$ is similar to the proof of (2.15), but simpler. We have $I_{[0;H]} \ll \mathcal{R}$ and this completes the proof of the theorem.

3. Auxiliary inequalities

Denote, for brevity, $Y_j^\star = Y_j - \mathbf{E}Y_j$, $1 \le j \le n$.

Proof of Lemma 2.2. Let us prove (2.20). It follows from the inequalities $\mathbf{E}|Y_1|^3 \le 4\mathbf{E}|Y_1^\star|^3 + 4|\mathbf{E}Y_1|^3$ and $\mathbf{E}|Y_1^\star|^3 \ge (\mathbf{E}|Y_1^\star|^2)^{3/2} = w^{-3}b^3$ that $\mu \le 4\mu_0 + 4w^{-4}\alpha^3$ and $\mu_0 \ge w^{-1}b^3$. Therefore, $\mu_0^{-1}\mu \le 4 + 4w^{-3}b^3\alpha^3$ and $\mu_0^{-2}\mathbf{E}|Y_1^\star|^2 \le b^{-4}$. Finally, by (2.13),

$$H\mu = c_3 b^2 \mu_0^{-1}\mu \le c \qquad \text{and} \qquad H^2 \mathbf{E}|Y_1^\star|^2 = c_3^2 b^4 \mu_0^{-2}\mathbf{E}|Y_1^\star|^2 \le c_3^2.$$

Let us prove (2.21). We have, see (2.45),

$$\mathbf{E}U_A^2 = (|A|^2 - |A|)\mathbf{E}(T_1^2 Q_2^2 + T_1 Q_2 T_2 Q_1). \tag{3.1}$$

Combining the bounds

$$\mathbf{E}(Y_i^\star)^2 \ll w^{-2}, \quad \mathbf{E}Z_i^2 \ll \mathbf{E}|Z_i|^{3/2} \ll w^{-2}\mu, \quad \mathbf{E}|Y_i^\star Z_i| \ll w^{-2}\mu, \tag{3.2}$$

and (2.5) we obtain

$$\mathbf{E}T_1^2 Q_2^2 = p^2 q^4 \mathbf{E}(Y_1^\star)^2 Z_2^2 \ll z\mu, \quad \mathbf{E}|T_1 Q_2 T_2 Q_1| = p^2 q^4 \mathbf{E}|Y_1^\star Z_1 Y_2^\star Z_2| \ll z^2\mu^2.$$

These inequalities in combination with (3.1) and (2.13) give $\mathbf{E}U_A^2 \ll z^2\mu$.

The second inequality in (2.21) follows from $\mathbf{E}U_A^2 \ll z^2\mu$ and $\mathbf{E}Q_A^2 \ll z\mu$, by Cauchy–Schwarz. To prove $\mathbf{E}Q_A^2 \ll z\mu$ we use the identity $\mathbf{E}_A Q_A^2 = \sum_{i\in A}\mathbf{E}_A Q_i^2$, the symmetry and (3.2):

$$\mathbf{E}Q_A^2 = \mathbf{E}(\mathbf{E}_A Q_A^2) = |A|\mathbf{E}Q_1^2 = mpq^3\mathbf{E}Z_1^2 \ll mpqw^{-2}\mu = z\mu. \tag{3.3}$$

Let us prove (2.22). An application of Marcinkiewicz–Zygmund inequality conditionally given all the random variables, but ν_i, $i\in A$, gives $\mathbf{E}_A|Q_A|^{3/2} \ll \sum_{i\in A}\mathbf{E}_A|Q_i|^{3/2}$. Therefore, by the symmetry,

$$\mathbf{E}|T_B|^{3/4}|Q_A|^{3/2} \ll |A|\mathbf{E}|Q_1|^{3/2}|T_B|^{3/4} \ll mpq\mathbf{E}|Z_1|^{3/2}\mathbf{E}^{(1)}|T_B|^{3/4}.$$

Finally, combining (2.24) and (3.2) we obtain the first inequality of (2.22). The proof of the second one is much the same.

Let us prove (2.23). By the symmetry and (3.2),

$$\mathbf{E}|\sum_{j\in A}T_jQ_j^2| \le m\mathbf{E}|T_1Q_1^2| \ll mpq^3\mathbf{E}|Y_1^\star|Z_1^2 \ll mpq\mathbf{E}|Y_1^\star||Z_1| \ll z\mu,$$
$$\mathbf{E}|\sum_{j\in A}T_jQ_jQ_A^{(j)}| \le m\mathbf{E}|T_1Q_1||Q_A^{(1)}| = mpq^2\mathbf{E}|Y_1^\star Z_1|\mathbf{E}^{(1)}|Q_A^{(1)}| \ll z^{3/2}\mu^{3/2}.$$

In the last step we used the bound $\mathbf{E}^{(1)}|Q_A^{(1)}| \ll z^{1/2}\mu^{1/2}$, which follows from $\mathbf{E}^{(1)}(Q_A^{(1)})^2 \ll z\mu$, cf. (3.3), by Cauchy–Schwarz.

It remains to prove (2.24). The proof for $r=6$ is straightforward. Using (2.24), with $r=6$, and Lyapunov's inequality we obtain (2.24) for $0<r<6$.

Proof of Lemma 2.3. The inequalities (2.26) follows from (2.25), by Cauchy-Schwarz. Let us prove (2.25). We shall prove the first inequality only. The proof of the remaining two inequalities is similar, but simpler. Write

$$H_G^2 \le \prod_{k\in G}\xi_k, \qquad \xi_k = |\mathbf{E}_{\{k\}}\,\mathrm{e}\{v_k\}|^2. \tag{3.4}$$

We shall majorize ξ_k by a random variable, say ζ_k, which is a function of X_k, and apply Hoeffding's (1963) Theorem 4 to the expectation of the product of ζ_k, $k\in G$.

Since $\nu_k^2=\nu_k$, we can write $(\nu_k-p)^2 = \nu_k - 2\nu_kp + p^2$. Therefore,

$$T_kQ_k = (\nu_k-p)^2Y_k^\star qZ_k = (\nu_k-p)(1-2p)Y_k^\star qZ_k + r, \qquad r = (p-p^2)Y_k^\star qZ_k,$$

and we write

$$v_k = (\nu_k-p)b_k - 2^{-1}tr, \qquad b_k = ta_kY_k^\star + sw^{-1} \qquad a_k = g_0 - 2^{-1}(1-2p)qZ_k.$$

Since r does not depend on ν_k, we have

$$\xi_k \le |\beta(b_k)|^2, \qquad \text{where} \qquad \beta(x) = \mathbf{E}\, \mathrm{e}\{x(\nu_1 - p)\}, \quad x \in \mathbb{R}.$$

Höglund (1978) showed that, for any $z_0 \in [0, \pi)$ and z satisfying $|z| \le \pi + z_0$,

$$|\beta(z)|^2 \le 1 - pq(z)^2\Theta(z_0), \qquad \Theta(z_0) = \Big(\frac{2}{\pi} \ \frac{\pi - z_0}{\pi + z_0} \Big)^2.$$

We apply this inequality to those b_k satisfying $|a_k Y_k^\star| \le H^{-1}$. We have $|b_k| \le \pi+1$ and therefore, $\xi_k \le 1 - pqb_k^2\Theta(1)$. Combining this inequality with the obvious bound $\xi_k \le 1$, $k = 1, 2, \dots, n$, we obtain

$$\xi_k \le 1 - pq\, b_k^2\Theta(1)\mathbb{I}_k, \qquad \mathbb{I}_k = \mathbb{I}_{|Ha_kY_k^\star| \le 1}, \qquad 1 \le k \le n. \tag{3.5}$$

Write $b_k^\star = tY_k^\star + sw^{-1}$. The simple inequality $(x+y)^2 \ge x^2/2 - y^2$ gives

$$b_k^2 \ge (b_k^\star)^2/2 - (b_k - b_k^\star)^2 \ge (b_k^\star)^2/2 - d_k^2, \quad d_k = |tY_k^\star|(c_1 + |Z_k|). \tag{3.6}$$

Here we estimated $|b_k - b_k^\star| \le d_k$, using $|g_0 - 1| \le c_1$. Furthermore, since $|Z_k| \le 2$ and $|g_0| \le 1 + c_1 \le 2$, we have $|a_k| \le 3$, and therefore $\mathbb{I}_k \ge \mathbb{I}_k^\star := \mathbb{I}_{|3HY_k^\star| \le 1}$. This inequality in combination with (3.6) and (3.5) gives

$$\xi_k \le \zeta_k, \qquad \zeta_k = 1 - 2^{-1}pq\big((b_k^\star)^2 - 2d_k^2\big)\Theta(1)\mathbb{I}_k^\star, \qquad 1 \le j \le n. \tag{3.7}$$

Assume without loss of generality that $1 \in G$. By Hoeffding's (1963) Theorem 4,

$$\mathbf{E}^{(i,j)} \prod_{k \in G} \zeta_k \le \prod_{k \in G} \mathbf{E}^{(i,j)}\zeta_k = \big(\mathbf{E}^{(i,j)}\zeta_1\big)^{|G|}. \tag{3.8}$$

In the last step we used the symmetry. Next we show that, for some $c_5 > 0$,

$$\mathbf{E}^{(i,j)}\zeta_1 < 1 - c_5 n^{-1}u, \qquad u = t^2 + s^2. \tag{3.9}$$

Note that by (3.9) and (2.17), the right-hand side of (3.8) is less than

$$(1 - c_5n^{-1}u)^{m/4} \le \exp\{-\frac{c_5}{4}\ \frac{m}{n}\, u\} \le \exp\{-\frac{1}{8}\, c_5c_4 \ln u\} < u^{-10},$$

provided that the constant c_4 in the definition of m is sufficiently large. This bound in combination with (3.7) and (3.4) implies $\mathbf{E}^{(i,j)}H_G^2 < u^{-10}$.

In order to prove (3.9) we show that

$$I_1 := \mathbf{E}^{(i,j)}(b_1^\star)^2\mathbb{I}_1^\star \ge 2^{-1}uw^{-2} \qquad \text{and} \qquad \mathbf{E}^{(i,j)}d_1^2 \le t^2 8^{-1}w^{-2}. \tag{3.10}$$

The second inequality follows from the crude bound $\mathbf{E}d_1^2 \le 32t^2w^{-2}(c_1^2+\mu)$ and (2.13), provided that c_1 and c_2 are sufficiently small. To prove the first inequality write

$$I_1 = \frac{n}{n-2}\,I_2 - \frac{1}{n-2}\,I_3, \qquad I_2 = \mathbf{E}(b_1^\star)^2\mathbb{I}_1^\star, \qquad I_3 = (b_i^\star)^2\mathbb{I}_i^\star + (b_j^\star)^2\mathbb{I}_j^\star,$$
$$I_2 = I_4 - I_5, \quad I_4 = \mathbf{E}(b_1^\star)^2 = uw^{-2} - t^2w^{-4}\alpha^2, \quad I_5 = \mathbf{E}(b_1^\star)^2\mathbb{I}_{|3HY_1^\star|>1}.$$

Now it is easy so see that the first inequality of (3.10) follows from

$$I_3 \le 20^{-1}u(pq)^{-1}, \qquad I_5 \le 20^{-1}uw^{-2} \tag{3.11}$$

and the inequality $t^2w^{-4}\alpha^2 \le t^2w^{-4}c_2^2$, provided that c_2 is sufficiently small.

Let us prove the bound for I_3. It follows from the inequalities

$$(b_k^\star)^2 \le 2t^2(Y_k^\star)^2 + 2s^2w^{-2}, \tag{3.12}$$
$$(Y_i^\star)^2 + (Y_j^\star)^2 \le 2^{1/3}\big(|Y_i^\star|^3 + |Y_j^\star|^3\big)^{2/3} \le 2^{1/3}(n\mathbf{E}|Y_1^\star|^3)^{2/3} \le 8(\,\frac{\mu}{pq}\,)^{2/3}.$$

that $I_3 \le 16u(\mu^{2/3}(pq)^{-2/3} + w^{-2})$. This bound in combination with (2.13) yields the first inequality of (3.11) provided that c_2 is sufficiently small.

To prove the bound (3.11) for I_5 we combine (3.12) and Chebyshev's inequality,

$$I_5 \le 2\,\frac{t^2}{w^2}\,I_6 + 2\,\frac{s^2}{w^2}\,I_7, \qquad I_6 = w^2\mathbf{E}(Y_1^\star)^2|3HY_1^\star|, \qquad I_7 = \mathbf{E}|3HY_1^\star|^2.$$

By the definition of H, see (2.14), $I_6 = 3c_3b^2 \le 3c_3$. By (2.20), $I_7 \le 9c_3^2$. Choosing c_3 small enough we obtain the second inequality of (3.11) thus, completing the proof of the lemma.

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