# ONE TERM EDGEWORTH EXPANSION TO STUDENT'S $t$ STATISTIC

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Abstract. We evaluate the rate of approximation of the distribution function of Student's $t$ statistic based on $N$ iid observations by its one term Edgeworth expansion. The rate is $o(N^{-1/2})$ if the distribution of observations is non-lattice and has finite third absolute moment. If the fourth absolute moment is finite and Cramér's condition holds the rate is $O(N^{-1})$.

## 1. Introduction and Results

Let $X_1, \dots, X_N, \dots$ be independent identically distributed random variables. Write $\mathbf{E}\, X_1 = \mu$. Let

$$\mathbf{t} = \mathbf{t}(X_1, \dots, X_N) = (\overline{X} - \mu)/\hat{\sigma}$$

denote the Student statistic, where

$$\overline{X} = N^{-1}(X_1 + \dots + X_N), \qquad \hat{\sigma}^2 = N^{-1} \sum_{i=1}^{N} (X_i - \overline{X})^2.$$

Assume that $\sigma^2 := \mathbf{E}\, (X_1 - \mu)^2$ is finite and positive. Then the statistics $T_N = \sqrt{N}\mathbf{t}$ is asymptotically standard normal as $N \to \infty$.

The rate of the normal approximation and asymptotic expansions to the distribution function

$$F(x) = \mathbf{P}\, \{\sqrt{N}\mathbf{t} \le x\}$$

were studied by a number of authors, Chung (1946), Bhattacharya and Ghosh (1978), Chibisov (1980), Helmers and van Zwet (1982), Babu and Singh (1985), Helmers (1985), Slavova (1985), Hall (1987), Hall (1988), Praskova (1989), Friedrich (1989),

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Bentkus and Götze (1996), Bentkus, Bloznelis and Götze (1996), Bentkus, Götze and van Zwet (1997), Gine, Götze and Mason (1997), Putter and van Zwet (1998), etc.

It is interesting to note that some fundamental problems in this area were solved only recently. The necessary and sufficient conditions for the asymptotic normality of $T_N$ were found by Gine, Götze and Mason (1997). The Berry-Esseen bound $\sup_x |F(x) - \Phi(x)| < cN^{-1/2}\beta_3/\sigma^3$, was constructed by Bentkus and Götze (1996). Here $\Phi(x)$ denotes the standard normal distribution function and $\beta_3 := \mathbf{E}\,|X_1 - \mu|^3$.

The problem of establishing the asymptotic expansions under optimal moment and smoothness conditions so far remains open. Probably the most general and precise result concerned with a higher order asymptotics of Student's $t$ statistic is due to Hall (1987) who proved the validity of a $k$-term Edgeworth expansion of $F(x)$ with remainder $o(N^{-k/2})$, for every integer $k$, provided that $\mathbf{E}\,|X_1|^{k+2} < \infty$ and the distribution $F_0$ of $X_1$ is non-singular. The moment conditions in Hall (1987) are the minimal ones, but the smoothness condition on the distribution $F_0$ of the observations is too restrictive.

The aim of the present paper is to prove the validity of one term Edgeworth expansion under *optimal* conditions. We approximate $F(x)$ by the one-term Edgeworth expansion (also called the second order approximation)

$$G(x) = \Phi(x) + \frac{\kappa_3}{6\sqrt{N}}\,(2\,x^2 + 1)\,\Phi'(x), \qquad \kappa_3 = \mathbf{E}\,(X_1 - \mu)^3/\sigma^3$$

and construct bounds for the remainder

$$\Delta_N = \sup_x |F(x) - G(x)|.$$

The best rate that can be achieved by the second order approximation is $O(N^{-1})$. Write $\beta_s = \mathbf{E}\,|X_1 - \mu|^s$, for $s \geq 1$, and

$$\rho_x = 1 - \sup\bigl\{|\,\mathbf{E}\,\exp\{i\,t\,(X_1 - \mu)\}|:\ \sigma^2/(9\beta_3) \leq |t| \leq x/\sigma\bigr\}, \qquad \rho = \rho_{\sqrt{N}}$$

**Theorem 1.** *There exists an absolute constant $c > 0$ such that, for every $N = 2, 3, \dots,$*

$$\Delta_N \leq \frac{c}{\rho^2 N}\,\frac{\beta_4}{\sigma^4}\,, \tag{1.1}$$

*whenever $\rho > 0$.*

Note that the Cramér condition

$$\limsup_{|t|\to\infty} |\,\mathbf{E}\,\exp\{itX_1\}| < 1 \tag{C}$$

implies $\liminf_N \rho_{\sqrt{N}} > 0$. Therefore, if $\beta_4 < \infty$, under Cramér's (C) condition Theorem 1 implies $\Delta_N = O(N^{-1})$. This result was conjectured by Bentkus, Götze and van Zwet (1997). In their fundamental work Bentkus, Götze and van Zwet (1997) constructed a second order approximation to a general (nonlinear) symmetric statistic with the remainder $O(N^{-1})$. This general result was applied to a number of important statistics and established the validity of one term Edgeworth expansion with the remainder $O(N^{-1})$ under optimal conditions for each case considered with the sole exception of the Student statistic. For this particular statistic the bound

$$\Delta_N \leq \frac{c(\varepsilon)}{\rho_N^2 N} \Big( \frac{\beta_3 \beta_{4+\varepsilon}}{\sigma^{7+\varepsilon}} + \frac{\beta_4^3}{\sigma^6} \Big)$$

was obtained which implies $\Delta_N = O(N^{-1})$ under Cramér's (C) condition provided that $\mathbf{E}\,|X_1|^{4+\varepsilon} < \infty$, for some $\varepsilon > 0$, see Bentkus, Götze and van Zwet (1997).

The minimal smoothness condition which allows to prove the validity of one-term Edgeworth expansion, i.e., to prove the bound $\Delta_N = o(N^{-1/2})$, is the non-latticeness of the distribution $F_0$. Note that for a non-lattice distribution $F_0$ we have $\rho_x > 0$, for every $x > 0$.

**Theorem 2.** *Assume that for some decreasing functions $f_1$ and $f_2$ with $f_i(x) \to 0$ as $x \to +\infty$,*

$$\rho(x) \geq f_1(x), \qquad \mathbf{E}\,|X_1 - \mu|^3 \mathbb{I}\{(X_1 - \mu)^2 > x\} \leq f_2(x), \qquad x > x_0, \tag{1.2}$$

*for some $x_0 > 0$. Then there exists a sequence $\varepsilon_N$ (depending only on $f_1$ and $f_2$) with $\varepsilon_N \to 0$ as $N \to \infty$ such that*

$$\Delta_N \leq N^{-1/2} \varepsilon_N, \qquad \text{for} \quad N = 2, 3, \ldots.$$

Theorem 2 provides a bound for $\Delta_N$ which is uniform over the class of distributions satisfying (1.2) with given functions $f_1$ and $f_2$. An immediate consequence of Theorem 2 is the following result. If the distribution of $X_1$ is nonlattice and $\mathbf{E}\,|X_1|^3 < \infty$ then

$$\Delta_N = o(N^{-1/2}) \qquad \text{as} \quad N \to \infty. \tag{1.3}$$

Theorem 2 improves earlier results of Babu and Singh (1985), Helmers (1991) and Putter and van Zwet (1998) where the bound (1.3) was established assuming that $F_0$ is non-lattice and increasingly sharp moment conditions, the sharpest to date being $\mathbf{E}\,|X_1|^{3+\varepsilon} < \infty$, for some $\varepsilon > 0$, obtained in the latter paper.

Our approach differs from that used by Hall (1987) and that of Putter and van Zwet (1998). We use and elaborate some ideas and techniques, e.g. "data depending smoothing" from Bentkus and Götze (1996) and Bentkus, Götze and van Zwet (1997).

The rest of the paper is organized as follows. In Section 2 we present proofs of our results. Some more technical steps of the proofs are given in Section 3. Auxiliary results are collected in Section 4.

## 2. Proofs

The proofs are rather technical and involved. The only excuse for such complex proofs is that the results obtained are optimal.

We may and shall assume that $\mathbf{E}\, X_1 = 0$ and $\sigma^2 = 1$.

In what follows $c$, $c_1$, $c_2$,... denote generic absolute constants. We write $c(a, b, ...)$ to denote a constant that depends only on the quantities $a, b, ...$. We shall write $A \ll B$ to denote the fact that $A \le c\, B$. Furthermore, $\exp\{itx\}$ is abbreviated by $\mathrm{e}\{x\}$.

In what follows $\theta_1$, $\theta_2$, ... denote independent random variables uniformly distributed in $[0, 1]$ and independent of all other random variables considered below. For a vector valued smooth function $H$, we shall use the mean value formula, $H(x) - H(0) = \mathbf{E}\, H'(\theta_1\, x)\, x$ and write $\|H\| = \sup_x |H(x)|$.

Let $g : R \to R$ denote a function which is infinitely many times differentiable with bounded derivatives and such that

$$\frac{8}{9} \le g(x) \le \frac{8}{7}, \quad \text{for all} \quad x \in R, \quad \text{and} \quad g(x) = \frac{1}{\sqrt{x}}, \quad \text{for} \quad \frac{7}{8} \le x \le \frac{9}{8}.$$

Write $c_g = \|g\| + \|g'\| + \|g''\| + \|g'''\|$.

Let $a$ denote the largest nonnegative solution of the equation

$$a^2 = \mathbf{E}\, X_1^2\, \mathbb{I}\{X_1^2 \le a^2 N\}.$$

For $1 \le i, j \le N$ and $1 \le k \le 4$, write $Y_i = a^{-1} N^{-1/2} X_i \mathbb{I}\{X_i^2 \le a^2 N\}$ and denote

$$\begin{aligned} &Y = Y_1 + \cdots + Y_N, \qquad \eta = \eta_1 + \cdots + \eta_N, \qquad D = d_1 + \cdots + d_N, \\ &\eta_i = Y_i^2 - \mathbf{E}\, Y_i^2, \qquad d_i = Y_i\, \eta_i, \qquad Q_{i,j} = Y_i^2 d_j, \\ &b_k = \mathbf{E}\, |Y_1|^k, \quad \mathcal{M} = N b_4, \quad \gamma = N |\mathbf{E}\, Y_1|, \quad \gamma_0 = N^{-1/2}\, \mathbf{E}\, |X_1|^3 \mathbb{I}\{X_1^2 \ge N/2\}. \end{aligned}$$

Note that $|Y_i| \le 1$. By Hölder's inequality, $\beta_3^2 \le \sigma^2 \beta_4 = \beta_4$ and

$$b_3 \ge b_2^{3/2} = N^{-3/2}, \qquad \mathcal{M} \ge N\, b_2^2 = N^{-1}, \qquad (b_3 N)^2 \le N^2 b_2 b_4 = \mathcal{M}. \tag{2.1}$$

If $Q$ denotes the sum $q_1 + \cdots + q_k$ then write $Q^{(i,j)} = Q - q_i - q_j$. Similarly, $Q^{(i)} = Q - q_i$. Given a subset $A \subset \{1, \ldots, k\}$ write $Q_A = \sum_{j \in A} q_j$. Given $A = \{i_1, \ldots, i_m\} \subset \{1, \ldots, N\}$ we write $\mathbf{E}_A$ or $\mathbf{E}_{i_1, \ldots, i_m}$ to denote the conditional expectation given $\{X_j,\, j \notin A\}$.

*Proof of Theorem 1.* For clarity we start by outlining the main steps of the proof. Firstly, we use Lemma 4.3 below to replace the statistic $T_N$ by a statistic $S_1$, which is conditionally linear in the first m observations $X_1, \ldots, X_m$, given the remaining observations of the sample, $X_{m+1}, \dots, X_N$. With $F_1(x) = \mathbf{P}\{S_1 \le x\}$ denoting the distribution function of $S_1$, an application of "data depending smoothing" procedure then reduces the problem of bounding $|F_1(x) - G(x)|$ to that of bounding the difference $|\hat{F}_1(t) - \hat{G}(t)|$, where $\hat{F}_1$ and $\hat{G}$ denote the Fourier transforms of $F_1$ and $G$ respectively. The (conditional) linearity of $S_1$ produces a multiplicative component in $\hat{F}_1$ and in combination with the smoothness condition ($\rho > 0$) guarantees an exponential decay of $|\hat{F}_1(t)|$, for large $t$, $|t| \ge c(F)\sqrt{N}$. Finally we bound the difference $|\hat{F}_1(t) - \hat{G}(t)|$, for $|t| \le c(F)\sqrt{N}$.

We were not able to prove the bound $\Delta_N = O(N^{-1})$ under Cramér's (C) condition (the minimal smoothness condition that ensures such rate) using the conventional Esseen's (1945) smoothing lemma.

We may assume that for a sufficiently small $c_0 > 0$,

$$\beta_3/\sqrt{N} \le c_0, \qquad \beta_4/N \le c_0, \qquad \rho_N^{-1} \ln N < c_0 N. \tag{2.2}$$

Indeed, if the first inequality fails, the bound (1.1) follows from the simple inequalities $\Delta_N \ll 1 + \beta_3/\sqrt{N} \ll \beta_3^2/N \le \beta_4/N$. Hence, without loss of generality we may assume that $\beta_3/\sqrt{N} \le c_0$. Then $\Delta_N \ll 1 + \beta_3/\sqrt{N} \ll 1$ and the inequality $\beta_4/N \ge c_0$ implies $\Delta_N \ll \beta_4/N$. We obtain (1.1) again. Finally, if the last inequality of (2.2) fails we have $\rho_N^{-2} N^{-1} > c_0$ and this in combination with $\Delta \ll 1$ and $\beta_4 \ge 1$ implies (1.1). Thus, we may and shall assume that (2.2) holds. Note that now, by Lemma 4.2, $a > 3/4$. Therefore (2.2) implies

$$\gamma \le (4/3)^4 \beta_4/N, \quad \mathcal{M} \le (4/3)^4 \beta_4/N \le 4c_0, \quad N b_3 \le (4/3)^3 \beta_3 N^{-1/2} \le 3c_0. \tag{2.3}$$

Let $m$ be the smallest integer greater than $18\rho^{-1} \ln N$. By (2.2), $m < N$. Put $A = \{1, \dots, m\}$ and $B = \{m+1, \dots, N\}$ and split $Y = Y_A + Y_B$ and $\eta = \eta_A + \eta_B$. Write

$$S_1 = Y g(V_B) + Y \eta_A g'(V_B) + 2^{-1} Y_B \eta_A^2 g''(V_B), \qquad V_B = 1 + \eta_B - Y_B^2/N, \tag{2.4}$$

and denote $F_1(x) = \mathbf{P}\{S_1 \le x\}$. By Lemma 4.3, see below, the probability $\mathbf{P}\{|S_1 - \mathbf{t}| \ge \mathcal{M}\}$ is not greater than $c\rho^{-2}\beta_4/N$. Then Slucky's argument gives

$$\Delta_N \le \sup_x |F_1(x) - G(x)| + c\mathcal{M} \max_x |G'(x)| + c\rho^{-2}\beta_4/N.$$

But $\max_x |G'(x)| \le c$, by (2.2). Hence in order to prove (1.1) it remains to show that

$$\sup_x |F_1(x) - G(x)| \ll \rho^{-2}\beta_4/N. \tag{2.5}$$

We are going to apply the "data depending smoothing". This smoothing procedure was introduced by Bentkus, Götze and van Zwet (1997). To make the proof shorter we shall refer to Lemma 5.1 of Bentkus, Götze and van Zwet (1997). An inspection of the proof of this lemma shows that Pravitz's (1972) smoothing lemma applied to the conditional distribution function of $S_1$ given $Y_{m+1}, \dots, Y_N$, yields

$$\sup_x |F_1(x) - G(x)| \le c \big( \mathbf{E}\, \mathcal{J}_1 + |\, \mathbf{E}\, \mathcal{J}_2| + \mathcal{J}_3 + \mathbf{E}\, \mathcal{J}_4 + \mathcal{J}_5 \big),$$

$$\mathcal{J}_1 = \int_{H_1 \le |t| \le H} \frac{|\hat{F}_1^*(t)|}{|t|}\, dt, \quad \mathcal{J}_2 = \int_{|t| \le H_1} \frac{\mathrm{e}\{-x\} \hat{F}_1^*(t)}{H}\, dt, \quad \mathcal{J}_3 = \int_{|t| \ge H_1} \frac{|\hat{G}(t)|}{|t|}\, dt,$$
$$\mathcal{J}_4 = \int_{|t| \le H_1} \frac{|t|}{H^2}\, dt, \qquad \mathcal{J}_5 = \int_{|t| \le H_1} \frac{|\hat{F}_1(t) - \hat{G}(t)|}{|t|}\, dt.$$

Here

$$\hat{G}(t) = \exp\{-t^2/2\} - \frac{\kappa_3}{6\sqrt{N}} \big(2(it)^3 + 3it\big) \exp\{-t^2/2\} \qquad \text{and} \qquad \hat{F}_1(t) = \mathbf{E}\, \mathrm{e}\{S_1\}$$

denote the Fourier transforms of $G(x)$ and $F_1(x)$ respectively, and

$$\hat{F}_1^*(t) = \mathbf{E}_A\, \mathrm{e}\{S_1\} \quad \text{and} \quad H_1 = 1/(4N\, b_3), \quad H = \rho N / \big(16 \beta_4^{1/4} c_g (1 + \Theta_1 + \Theta_2)\big),$$

where $\Theta_1 = |Y_{m+1} + \dots + Y_k|$ and $\Theta_2 = |Y_{k+1} + \dots + Y_N|$ and $k \approx m + (N-m)/2$.

It remains to estimate $\mathbf{E}\, \mathcal{J}_k$, for $k = 1, 2, \dots, 5$.

*Estimation of* $\mathcal{J}_3$ and $\mathbf{E}\, \mathcal{J}_4$. It follows from the inequality $3/4 \le a \le 1$ that

$$H_1 \ge N^{1/2} / (\mathbf{E}\, |X_1|^3 \mathbb{I}\{X_1^2 \le a^2 N\}) \ge y, \qquad y := N^{1/2}/\beta_3.$$

Furthermore, by (2.2), $|\hat{G}(t)| \le \exp\{-t^2/2\}(1 + |t|^3)$ and, therefore, $\mathcal{J}_3 \le \exp\{-cy^2\}$. Note that $y$ is sufficiently large, by (2.2). The inequality $\exp\{-u\} < u^{-1}$ (which holds for sufficiently large $u$) applied to $u = cy^2$ implies

$$\mathcal{J}_3 \ll \beta_3^2/N \le \beta_4/N.$$

Consider $\mathcal{J}_4$. By (2.1),

$$H_1 \le 4^{-1}/(N b_2^{3/2}) = 4^{-1} N^{1/2}. \tag{2.6}$$

It follows from (2.6) and (4.1) that

$$\mathbf{E}\, \mathcal{J}_4 = H_1^2\, \mathbf{E}\, H^{-2} \ll \rho^{-2} N^{-1}.$$

*Estimation of* $\mathbf{E}\, \mathcal{J}_1$. We shall show that

$$|\, \mathbf{E}\, \mathcal{J}_1| \ll \rho^{-2} \mathcal{M}. \tag{2.7}$$

Split $Y_A\eta_A = D_A + U_A$, where

$$D_A = \sum_{i\in A} d_i, \qquad U_A = \sum_{i,j\in A,\, i\neq j} Y_i\eta_j.$$

Write also $\eta_A^2 = \tilde D_A + \tilde U_A$, where

$$\tilde D_A = \sum_{i\in A} \eta_i^2, \qquad \tilde U_A = \sum_{i,j\in A,\, i\neq j} \eta_i\eta_j.$$

Denote $L = L_1 + \cdots + L_m$, where $L_i = l_{i,1} + l_{i,2} + l_{i,3} + l_{i,4}$,

$$l_{i,1} = Y_i g(V_B), \quad l_{i,2} = \eta_i Y_B g'(V_B), \quad l_{i,3} = d_i g'(V_B), \quad l_{i,4} = 2^{-1}\eta_i^2 Y_B g''(V_B).$$

Furthermore, write $Z = L + U_A g'(V_B) + W$, where $W = Y_B g(V_B)$. Then $S_1 = Z + X$, where $X = 2^{-1}\tilde U_A Y_B g''(V_B)$. Expanding $\mathrm{e}\{Z + X\}$ in powers of $itX$ we obtain

$$|\,\mathbf{E}\,\mathcal{J}_1| \le \mathbf{E}\, I_1 + \mathbf{E}\, I_2 + R, \qquad I_1 = \int_{H_1\le|t|\le H} \frac{|\,\mathbf{E}_A\, \mathrm{e}\{Z\}|}{|t|}\, dt,$$
$$I_2 = \int_{H_1\le|t|\le H} |\,\mathbf{E}_A\, \mathrm{e}\{Z\} X| dt, \qquad R = 2^{-1}\,\mathbf{E}\, X^2 \int_{H_1\le|t|\le H} |t| dt.$$

A simple calculation shows $\mathbf{E}_A \tilde U_A^2 \ll m^2 b_4^2$. Invoking the inequality $H^2 \le \rho^2 N^2 \beta_4^{-1/2}$ we obtain

$$R \ll \rho^2 N^2 \beta_4^{-1/2} m^2 b_4^2 \ll \mathcal{M}^{3/2} \ll \mathcal{M}.$$

Let us show $\mathbf{E}\, I_2 \ll \rho^{-2}\mathcal{M}$. Expanding $\mathrm{e}\{Z\} = \mathrm{e}\{L + W + U_A g'(V_B)\}$ in powers of $itU_A g'(V_B)$ we get $\mathbf{E}\, I_2 \le \mathbf{E}\, I_3 + R$, where

$$I_3 = \int_{H_1\le|t|\le H} |\,\mathbf{E}_A\, \mathrm{e}\{L + W\} X| dt, \qquad R = \mathbf{E}\, |X U_A g'(V_B)| \int_{H_1\le|t|\le H} |t| dt.$$

We have $R \ll \rho^2 N^2 \beta_4^{-1/2}\, \mathbf{E}\, |X U_A| \ll \mathcal{M}$. Here we estimated

$$\mathbf{E}\, |X U_A| \ll \mathbf{E}\, |\tilde U_A U_A| \ll m^2 N^{-2} \mathcal{M}^{3/2}.$$

To prove the last inequality we combine the bounds

$$\mathbf{E}\, \tilde U_A^2 \ll m^2 b_4^2, \qquad \mathbf{E}\, U_A^2 \ll (m/N)^2 \mathcal{M}, \tag{2.8}$$

(see Lemma 4.1 for the second inequality) and the inequality $ab \le a^2\tau + b^2\tau^{-1}$ applied to $a = |\tilde{U}_A|$, $b = |U_A|$ and $\tau = \mathcal{M}^{-1/2}$.

Let us show $\mathbf{E}\, I_3 \ll N^{-1}$. By the symmetry,

$$\mathbf{E}_A\, \mathrm{e}\{L\}X = m(m-1)Y_B g''(V_B)\, \mathbf{E}_A\, \mathrm{e}\{L_2 + L_3\}\eta_2\eta_3 h^{m-2}, \quad h = \mathbf{E}_1\, \mathrm{e}\{L_1\}.$$

We shall prove below that

$$|h| \le 1 - \rho/2. \tag{2.9}$$

Since $W$ is independent of $X_i$, $i \in A$, we have $|\,\mathbf{E}_A\, \mathrm{e}\{L + W\}X| \le |\,\mathbf{E}_A\, \mathrm{e}\{L\}X|$ and combining (2.9) and the expression for $\mathbf{E}_A\, \mathrm{e}\{L\}X$ given above we obtain

$$|\,\mathbf{E}_A\, \mathrm{e}\{L + W\}X| \ll c_g m^2 |Y_B| N^{-2}(1 - \rho/2)^{m-2}.$$

Finally, the inequalities (4.1) and

$$(1 - \rho/2)^{m-2} \le \exp\{-\rho(m-2)/2\} \le e^{\rho} \exp\{-\rho m/2\} \le e\, N^{-3}$$

complete proof of the bound $\mathbf{E}\, I_3 \ll N^{-1}$.

Let us prove (2.9). Expanding the exponent in powers of $it(l_{1,2} + l_{1,3} + l_{1,4})$ we get

$$|h - \mathbf{E}_1\, \mathrm{e}\{l_{1,1}\}| \ll c_g H\big(\mathbf{E}_1(|\eta_1| + \eta_1^2)|Y_B| + \mathbf{E}\,|d_1|\big) \le \rho/4. \tag{2.10}$$

In the last step we estimated $\mathbf{E}_1|\eta_1| \le 2/N$ and

$$\mathbf{E}_1\eta_1^2 \le b_4 = N^{-1}\mathcal{M} \le N^{-1}, \qquad \mathbf{E}\,|d_1| \le (\,\mathbf{E}\, Y_1^2)^{1/2}(\,\mathbf{E}\,\eta_1^2)^{1/2} \le N^{-1},$$

see (2.3). In (2.11) below we show that $\mathbf{E}\,|\,\mathrm{e}\{l_{1,1}\} - \mathrm{e}\{z_1\}| \le \rho/5$, where $z_1 = a^{-1}N^{-1/2}X_1 g(V_B)$. Furthermore, the inequalities $3/4 \le a \le 1$ and $8/9 \le |g| \le 8/7$ imply $|\,\mathbf{E}_1\, \mathrm{e}\{z_1\}| \le 1 - \rho$, for $H_1 \le |t| \le H$ and invoking (2.10) we obtain (2.9). For $|t| \le H$, we have

$$\begin{aligned} \big|\,\mathbf{E}_1\, \mathrm{e}\{l_{1,1}\} - \mathbf{E}_1\, \mathrm{e}\{z_1\}\big| &\le a^{-1}N^{-1/2}|tg(V_B)|\, \mathbf{E}_1|X_1|\mathbb{I}\{X_1^2 > a^2 N\} \\ &\le (8/7)a^{-4}N^{-2}H\beta_4 \le (8/7)(4/3)^4 16^{-1}\rho\beta_4/N \\ &\le (c_0/3)\rho \le \rho/5. \end{aligned} \tag{2.11}$$

Recall that $c_0$ is (sufficiently) small absolute constant and $3/4 \le a \le 1$, by Lemma 4.2.

Let us show that $\mathbf{E}\, I_1 \ll \beta_4/N$. Split $A = A_1 \cup A_2 \cup A_3$, so that $A_p \cap A_q = \emptyset$, for $p \neq q$, and $|A_p| \approx m/3$, for every $p$. Write

$$U_{p,q} = \sum_{i \in A_p,\, j \in A_q,\, i \neq j} y_i \eta_j g'(V_B), \qquad U^* = \sum_{1 \le p,q \le 3,\, p \neq q} U_{p,q}.$$

Then $U_A g'(V_B) = U_{1,1} + U_{2,2} + U_{3,3} + U^*$ and $\mathrm{e}\{Z\} = \mathrm{e}\{W + L + U^*\} g_1 g_2 g_3$, where we denote $g_p = \mathrm{e}\{U_{p,p}\}$. By the mean value theorem, $g_p = 1 + \varkappa_p$, where $\varkappa_p = i\, t\, U_{p,p}\, \mathbf{E}_{\,\theta}\, \mathrm{e}\{\theta U_{p,p}\}$. Write

$$a_1 = 1, \quad a_2 = -g_1, \quad a_3 = -g_2, \quad a_4 = -g_3, \quad a_5 = g_1 g_2, \quad a_6 = g_1 g_3, \quad a_7 = g_2 g_3.$$

The identity $g_1 g_2 g_3 = (a_1 + \dots + a_7) + \varkappa_1 \varkappa_2 \varkappa_3$ implies

$$\mathbf{E}\, I_1 \le (\, \mathbf{E}\, I_{1,1} + \dots + \mathbf{E}\, I_{1,7}) + R, \qquad I_{1,i} = \int_{H_1 \le |t| \le H} \frac{|\, \mathbf{E}_{\,A}\, \mathrm{e}\{L + W + U^*\} a_i|}{|t|}\, dt,$$

$$R \le \mathbf{E} \int_{H_1 \le |t| \le H} \frac{|\varkappa_1 \varkappa_2 \varkappa_3|}{|t|} \le \big( N \rho \beta_4^{-1/4} \big)^3\, \mathbf{E}\, |U_{1,1} U_{2,2} U_{3,3}| \ll \beta_4/N.$$

In the last step we used the inequalities

$$\mathbf{E}_{\,A} \prod_{1 \le p \le 3} |U_{p,p}| = \prod_{1 \le p \le 3} \mathbf{E}_{\,A} |U_{p,p}| \le \prod_{1 \le p \le 3} (\, \mathbf{E}_{\,A} U_{p,p}^2)^{1/2}, \quad \mathbf{E}_{\,A} U_{p,p}^2 \le c_g m^2 b_4/N,$$

cf. (2.8). In order to complete the proof of (2.7) it remains to show $|\, \mathbf{E}\, I_{1,i}| \ll \mathcal{M}$, for $i = 1, 2, \dots, 7$. We shall prove that $|\, \mathbf{E}\, I_{1,7}| \ll \rho^{-2} \mathcal{M}$ since the proof for the rest $i = 1, \dots, 6$ is almost the same or simpler. Write

$$\eta^* := \eta_A - \eta_{A_1}, \qquad Y^* := Y_A - Y_{A_1}, \qquad U_{A_1} := Y_{A_1} \eta^* + \eta_{A_1} Y^*.$$

We have

$$|\, \mathbf{E}_{\,A}\, \mathrm{e}\{L + W + U^*\} a_7| \le \mathbf{E}_{\,A} |\, \mathbf{E}_{\,A_1} \psi|, \qquad \psi = \mathrm{e}\{L_{A_1} + U_{A_1} g'(V_B)\}. \tag{2.12}$$

Write $\mathbb{I}_\eta = \mathbb{I}\{100 c_g |\eta^*| < 1\}$ and $\mathbb{I}_Y = \mathbb{I}\{100 c_g |Y^*| < 1\}$. Then

$$\mathbf{E}_{\,A} |\, \mathbf{E}_{\,A_1} \psi| = \mathbf{E}_{\,A} |\, \mathbf{E}_{\,A_1} \psi \mathbb{I}_\eta \mathbb{I}_y| + R, \qquad R \le \mathbf{E}_{\,A} |1 - \mathbb{I}_\eta| + \mathbf{E}_{\,A} |1 - \mathbb{I}_Y|. \tag{2.13}$$

By Chebyshev's inequality, $R \ll \mathbf{E}\, (\eta^*)^2 + \mathbf{E}\, (Y^*)^4 \ll m^2 b_4$. Therefore,

$$\mathbf{E} \int_{H_1 \le |t| \le H} \frac{R}{|t|}\, dt \ll m^2 b_4 \ln N \ll \rho^{-2} \mathcal{M}. \tag{2.14}$$

It remains to estimate $\mathbf{E}_{A_1}\psi\mathbb{I}_\eta\mathbb{I}_Y$. Write Write $m_1 = |A_1|$. By the symmetry

$$\mathbf{E}_{A_1}\psi\mathbb{I}_\eta\mathbb{I}_Y = h_1^{m_1}, \qquad h_1 = \mathbf{E}_1\,\mathrm{e}\{L_1 + (Y_1\eta^* + \eta_1 Y^*)g'(V_B)\}\mathbb{I}_\eta\mathbb{I}_Y.$$

We shall show that

$$|h_1| \le 1 - \rho/2. \tag{2.15}$$

Then

$$\big|\mathbf{E}_{A_1}\psi\mathbb{I}_\eta\mathbb{I}_Y\big| \le \exp\{-m_1\rho/2\} \ll \exp\{-\rho m/6\} \le N^{-3}$$

and this inequality in combination with (2.12), (2.13) and (2.14) gives $\mathbf{E}\,I_{1,7} \ll \rho^{-2}\mathcal{M}$. It remains to prove (2.15). Proceeding as in (2.10) we obtain

$$|h_1 - h_2| \le \rho/4, \qquad \text{where} \quad h_2 = \mathbf{E}_1\,\mathrm{e}\{l_{1,1} + (Y_1\eta^* + \eta_1 Y^*)g'(V_B)\}\mathbb{I}_\eta\mathbb{I}_Y.$$

Furthermore, expanding the exponent in $h_2$ in powers of $it\eta_1 Y^*$ we obtain

$$|h_2 - h_3| \le c_g H\ \mathbf{E}_1|\eta_1 Y^*|\mathbb{I}_Y \le \rho/100, \quad h_3 = \mathbf{E}_1\,\mathrm{e}\{l_{1,1} + Y_1\eta^* g'(V_B)\}\mathbb{I}_\eta\mathbb{I}_Y.$$

We complete the proof of (2.15) by showing that $|h_3| \le 1 - (4/5)\rho$. We have

$$\big|\mathbf{E}_1\,\mathrm{e}\{z_2\}\mathbb{I}_\eta\big| \le 1 - \rho, \quad z_2 = a^{-1}N^{-1}X_1\big(g(V_B) + \eta^* g'(V_B)\big).$$

But $|h_3 - \mathbf{E}_1\,\mathrm{e}\{z_2\}| \le \rho/5$, cf. (2.11). Hence, $|h_3| \le 1 - (4/5)\rho$ and this completes the proof of (2.15). We arrive at (2.7).

*Estimation of* $\mathbf{E}\,\mathcal{J}_5$. Write

$$\varphi(t) = \exp\{-t^2/2\} - (N/6)\,\mathbf{E}\,Y_1^3\big(3it + 2(it)^3\big)\exp\{-t^2/2\}$$
$$f(t) = \mathbf{E}\,\mathrm{e}\{S\}, \qquad S = Yg(V), \qquad V = 1 + \eta - Y^2/N.$$

We have

$$\mathbf{E}\,\mathcal{J}_5 \le I_1 + I_2 + I_3, \qquad I_k = \int_{|t|\le H_1} \frac{|\delta_k(t)|}{|t|}\,dt,$$
$$\delta_1(t) = f(t) - \varphi(t), \qquad \delta_2(t) = \hat{F}_1(t) - f(t), \qquad \delta_3(t) = \varphi(t) - \hat{G}(t).$$

The inequality $I_1 \ll \beta_4/N$ is proved in Section 3.

The bound $I_3 \ll \beta_4/N$ is a consequence of the inequalities

$$\mathbf{E}\,\big(Y_1^3 - a^{-3}N^{-3/2}X_1^3\big) = a^{-3}N^{-3/2}\,\mathbf{E}\,|X_1|^3\mathbb{I}\{X_1^2 > a^2N\} \ll N^{-2}\beta_4,$$
$$a^{-3} - 1 = \frac{1 + a + a^2}{a^3}\,(1 - a) \ll \beta_4/N.$$

In the last step we used (4.2) and the inequality $3/4 \le a \le 1$, see Lemma 4.2.

Let us show that $I_2 \ll \rho^{-2}\beta_4/N$. Write

$$W_1 = -Y_A^2/N, \qquad W_2 = -2\,Y_A Y_B/N. \tag{2.16}$$

Then $V = V_B + \eta_A + W_1 + W_2$. Expanding $g$ in powers of $W_1 + W_2$ and then in powers of $\eta_A$ we obtain $S = S_1 + (Q_1 + \dots + Q_4)$, where

$$\begin{aligned} &Q_1 := 2^{-1} Y_A \eta_A^2\, \mathbf{E}_\theta g''(V_B + \theta\eta_A), \qquad Q_2 := 6^{-1} Y_B \eta_A^3\, \mathbf{E}_\theta g'''(V_B + \theta\eta_A), \\ &Q_3 := Q_{3.1} + Q_{3.2}, \quad Q_{3.i} := Y_A W_i \zeta, \quad Q_4 := Q_{4.1} + Q_{4.2}, \quad Q_{4.i} := Y_B W_i \zeta, \quad i = 1, 2, \end{aligned}$$

where we denote $\zeta = \mathbf{E}_\theta g'(V_B + \eta_A + \theta(W_1 + W_2))$. The inequality

$$|\,\mathrm{e}\{x\} - 1| \le 2|x|^\alpha, \qquad \text{for} \quad x \in R, \quad 0 \le \alpha \le 1, \tag{2.17}$$

implies

$$|f(t) - \mathbf{E}\,\mathrm{e}\{S_1 + Q_4\}| \ll R_1, \qquad R_1 = \mathbf{E}\left(|tQ_1|^{4/5} + \left|t(Q_2 + Q_{3.1})\right|^{2/3} + |tQ_{3.2}|\right).$$

Write

$$h = \mathbf{E}\,\mathrm{e}\{S_1\} itQ_{4.2}, \qquad h_1 = \mathbf{E}\,\mathrm{e}\{S_2\} itQ_{4.2}, \quad S_2 = Y g(V_B).$$

Expanding $\mathrm{e}\{Q_4\}$ in powers of $itQ_{4.1}$ and then in powers of $itQ_{4.2}$ we get

$$\mathbf{E}\,\mathrm{e}\{S_1 + Q_4\} = \hat{F}_1(t) + h + R_2, \qquad \text{where} \quad |R_2| \ll \mathbf{E}\,|tQ_{4.1}| + \mathbf{E}\,(tQ_{4.2})^2.$$

Furthermore, by (2.17),

$$|\,\mathrm{e}\{S_1\} - \mathrm{e}\{S_2\}| \ll |tY\eta_A| + |tY_B\eta_A^2|^{3/4}$$

and therefore

$$|h - h_1| \ll R_3, \quad R_3 = \mathbf{E}\,|Q_{4.2}|\big(t^2|Y\eta_A| + |t|^{7/4}|Y_B\eta_A^2|^{3/4}\big).$$

Combining these inequalities we obtain $|f(t) - \hat{F}_1(t) - h_1| \ll R_1 + R_2 + R_3$. A direct calculation shows

$$\begin{aligned} &R_1 \le |t|^{4/5} m^2 b_4 + |t|^{2/3}\big(mb_4 + m^2 N^{-5/3}\big) + |t|\, m\, N^{-2}, \\ &|R_2| \le m|t|N^{-2} + mt^2 N^{-3}, \qquad R_3 \le m^2 \mathcal{M}(t^2 N^{-3/2} + |t|^{7/4} N^{-2}). \end{aligned}$$

Write

$$h_2 = \mathbf{E}\,\mathrm{e}\{S_2\} itQ_5, \quad Q_5 = Y_B W_2 g'(V_B), \quad \text{and} \quad Q_6 = Q_{4.2} - Q_5.$$

By the mean vakue theorem $|Q_6| \ll |Y_B W_2|\,(|\eta_A| + |W_1 + W_2|)$. A simple calculation shows that

$$|h_1 - h_2| \le |t|\,\mathbf{E}\,|Q_6| \ll R_4, \qquad R_4 = |t|\,N^{-2}(1 + m(Nb_4)^{1/2}).$$

Furthermore, we shall show below that

$$|h_2| \ll R_5, \qquad R_5 = |t|N^{-1}(m/N)^{1/2}\exp\{-c(m/N)t^2\}, \quad \text{for} \quad |t| \le H_1. \tag{2.18}$$

Collecting the estimates for $R_i$, $i = 1, \dots, 5$, given above we get

$$I_2 \ll \int_{0 \le |t| \le H_1} \frac{R_1 + R_2 + R_3 + R_4 + R_5}{|t|}\,dt \ll \rho^{-2}\beta_4/N.$$

It remains to prove (2.18). Split

$$h_2 = h_{2,1} + h_{2,2}, \quad h_{2,j} = (-2it/N)\,\mathbf{E}\,Y_{A_j} Y_B^2\,\mathrm{e}\{S_2\} g'(V_B),$$

where $A_1 = \{1, \dots, k\}$ and $A_2 = A \setminus A_1$, and where $k \approx m/2$. It suffices to show that $|h_{2,j}| \ll R_5$. We have

$$|h_{2,1}| \ll N^{-1}|t|\;\mathbf{E}\,Y_B^2\zeta_1\zeta_2, \qquad \zeta_1 := \mathbf{E}_{A_1}|Y_{A_1}|, \quad \zeta_2 := \big|\,\mathbf{E}_{A_2}\,\mathrm{e}\{S_2\}\big|.$$

By Hölder's inequality, $\zeta_1^2 \le k/N < m/N$. Furthermore, by the symmetry, $\zeta_2 \le |u|^n$, where $u = \mathbf{E}_m\,\mathrm{e}\{Y_m g(V_B)\}$, and where $n := |A_2|$. The inequality

$$|\,\mathbf{E}\,\mathrm{e}\{\tau\}|^2 \le 1 - t^2\,\mathbf{E}\,(\tau - \mathbf{E}\,\tau)^2 + (4/3)|t|^3\,\mathbf{E}\,|\tau|^3,$$

see, e.g., Petrov (1995), applied to the random variable $\tau = Y_m g(V_B)$ conditionally, given $Y_{m+1}, \dots, Y_N$, implies

$$|u|^2 \le 1 - t^2 g^2(V_B)\Big(\frac{1}{N} - (\,\mathbf{E}\,Y_m)^2\Big) + \frac{4}{3}\,|t|^3 b_3 |g(V_B)|^3 \le 1 - \frac{t^2}{4N}\,, \tag{2.19}$$

for $|t| \le H_1$, provided that $c_0$ is sufficiently small. Here we estimated $|\,\mathbf{E}\,Y_m| \le a^{-4}N^{-2}\beta_4 \le 4c_0/N$, by Chebyshev's inequality and (2.2), and used the inequalities $8/9 \le g \le 8/7$. We obtain

$$\zeta_2 \le (1 - t^2/(4N))^{n/2} \le \exp\{-c(m/N)t^2\}$$

thus completing the proof of the bound $|h_{2,1}| \ll R_5$. Clearly, the same bound holds for $h_{2,2}$ as well. We arrive to (2.18).

*Estimation of* $\mathbf{E}\,\mathcal{J}_2$. We shall show that

$$|\,\mathbf{E}\,\mathcal{J}_2| \ll \rho^{-1}\beta_4/N.$$

Write $K = \{1, \dots, r\}$ and denote $\Theta = |Y_{r+1} + \cdots + Y_N|$. It suffices to show that for any $r \geq N/2$,

$$\mathbf{E}\,I_1 \leq c\rho^{-1}\beta_4/N, \qquad \text{where} \quad I_1 = \frac{\beta_4^{1/4}}{\rho N}\int_{20}^{H_1} |\,\mathbf{E}_K\, \mathrm{e}\{S_1\}|(1+\Theta)dt. \tag{2.20}$$

Let us prove (2.20). In the first step we replace $S_1$ by $S$. It follows from (4.5), see below, and (2.17) that

$$|\,\mathrm{e}\{S\} - \mathrm{e}\{S_1\}| \ll |t|(R_1 + \cdots + R_4) + |t|^{2/5}R_5^{2/5} + |t|^{1/3}R_6^{1/3} =: L_1,$$

where the random variables $R_i$ are defined in the proof of Lemma 4.3, see below. A simple calculation gives

$$\begin{aligned}\mathbf{E}\,L_1(1+\Theta) &\ll |t|N^{-1}\,\mathbf{E}\,\big(|Y_A|^3 + Y_A^2 + |Y_A|\big) + |t|^{2/5}\,\mathbf{E}\,(Y_A^2 + |\eta_A|) + |t|^{1/3}\,\mathbf{E}\,|\eta_A| \\ &\ll |t|N^{-1}\big(\beta_4^{1/2}(m/N)^{3/2} + (m/N)^{1/2}\big) + \big(|t|^{2/5} + |t|^{1/3}\big)(m/N).\end{aligned}$$

Furthermore, invoking the inequality $H_1 \leq 4^{-1}N^{1/2}$, see (2.6), we obtain

$$\frac{\beta_4^{1/4}}{\rho N}\int_{20}^{H_1} \mathbf{E}\,L_1(1+\Theta)dt \ll \frac{\beta_4^{3/4} + \beta_4^{1/4}}{\rho N} \leq \rho^{-1}\beta_4/N.$$

It remains to show that

$$\frac{\beta_4^{1/4}}{\rho N}\int_{20}^{H_1} |\,\mathbf{E}_K\,\mathrm{e}\{S\}|(1+\Theta)dt \ll \rho^{-1}\beta_4/N.$$

Let $m_0 = m_0(t)$ be the smallest integer greater than $20\,N\,t^{-2}\ln|t|$. Clearly, $m_0 \leq N/2$, for $|t| \geq 20$. We may and shall assume that $m_0 > 10$, since $N$ is sufficiently large, by (2.2). Write

$$A_0 = \{1, \dots, m_0\}, \quad B_0 = \{m_0 + 1, \dots, N\}, \quad \text{and} \quad V_{B_0} = 1 + \eta_{B_0} - Y_{B_0}^2/N,$$

Then $V = V_{B_0} + \eta_{A_0} - \tilde{W}$, where $\tilde{W} = Y_{A_0}^2/N + 2Y_{A_0}Y_{B_0}/N$. Expanding $g$ in powers of $\tilde{W}$ and then in powers of $\eta_{A_0}$ we get

$$S = S_3 + r_1 + r_2 + r_3 + r_4, \qquad S_3 = Y_{A_0}g(V_{B_0}) + \eta_{A_0}Y_{B_0}g'(V_{B_0})$$

where

$$|r_1| \ll |Y\tilde{W}|, \qquad |r_2| \ll |Y_{A_0}\eta_{A_0}|, \qquad |r_3| \ll |Y_{A_0}|\eta_{A_0}^2, \qquad |r_4| \ll |Y_{B_0}|\eta_{A_0}^2.$$

An application of (2.17) gives

$$|\,\mathrm{e}\{S\} - \mathrm{e}\{S_3\}| \ll |tY\tilde{W}| + |tY_{A_0}\eta_{A_0}|^{2/3} + |tY_{B_0}\eta_{A_0}^2|^{1/2} + |tY_{A_0}\eta_{A_0}^2|^{2/5}.$$

Finally, combining the inequalities

$$\begin{aligned}
&\mathbf{E}\,|Y\tilde{W}|(1+\Theta) \ll N^{-1}, \quad \mathbf{E}\,|Y_{B_0}\eta_{A_0}^2|^{1/2}(1+\Theta) \ll m_0/N \ll t^{-2}\ln|t|,\\
&\mathbf{E}\,|Y_{A_0}\eta_{A_0}|^{2/3}(1+\Theta) \ll \mathbf{E}\,(Y_{A_0}^2 + |\eta_{A_0}|) \ll m_0/N \ll t^{-2}\ln|t|,\\
&\mathbf{E}\,|Y_{A_0}\eta_{A_0}|^{2/5}(1+\Theta) \ll \mathbf{E}\,(Y_{A_0}^2 + |\eta_{A_0}|) \ll m_0/N \ll t^{-2}\ln|t|,
\end{aligned}$$

we obtain

$$\frac{\beta_4^{1/4}}{\rho N}\int_{20}^{H_1} \mathbf{E}\,|\,\mathrm{e}\{S\} - \mathrm{e}\{S_3\}|(1+\Theta)dt \ll \rho^{-1}\beta_4/N.$$

We complete the proof of (2.20) by showing

$$|\,\mathbf{E}_{A_0}\,\mathrm{e}\{S_3\}| \ll |t|^{-2} + c\,c_g N^{-3/4}\beta_4^{3/4}|Y_{B_0}|^{3/2}. \tag{2.21}$$

In order to prove (2.21) write $\mathbb{I}_B = \mathbb{I}\{100c_g N Y_{B_0}^2\beta_4 < 1\}$. We have

$$|\,\mathbf{E}_{A_0}\,\mathrm{e}\{S_3\}| \le \zeta + (1-\mathbb{I}_B), \qquad \zeta := |\,\mathbf{E}_{A_0}\,\mathrm{e}\{S_3\}\mathbb{I}_B|.$$

But

$$1-\mathbb{I}_B \ll c_g N^{-3/4}\beta_4^{3/4}|Y_{B_0}|^{3/2}.$$

Therefore, it suffices to estimate $\zeta$. By the symmetry

$$\zeta \le |v\mathbb{I}_B|^{m_0}, \qquad v := \mathbf{E}_1\,\mathrm{e}\{Y_1 g(V_{B_0}) + \eta_1 Y_{B_0} g'(V_{B_0})\}.$$

Expanding the exponent in powers of $it\eta_1 Y_{B_0} g'(V_{B_0})$ we obtain

$$v = u + R, \qquad u = \mathbf{E}_1\,\mathrm{e}\{Y_1 g(V_{B_0})\}, \quad \text{with} \quad |R| \le t^2\,\mathbf{E}_1\eta_1^2 Y_{B_0}^2 g'(V_{B_0}) \le c_g t^2 b_4 Y_{B_0}^2.$$

Note that $|R|\mathbb{I}_B \le t^2 N^{-1}/100$. Then

$$|v\mathbb{I}_B| \le |u| + |R|\mathbb{I}_B \le 1 - \frac{t^2}{8N} + \frac{t^2}{100N} \le 1 - \frac{t^2}{9N}\,. \tag{2.22}$$

Here we used the inequality $|u| \le 1 - t^2/(8N)$ which follows from the inequality (2.19) applied to $\tau = Y_1 g(V_{B_0})$, for $|t| \le H_1 \le 4^{-1}N^{1/2}$, use also (2.6). It follows from (2.22) that

$$\zeta \le \bigl(1 - t^2/(9N)\bigr)^{m_0} \le \exp\{-t^2 m_0/(9N)\} \le |t|^{-20/9} < t^{-2}.$$

We arrive to (2.21) thus completing the proof of (2.20).

Collecting the estimates of $\mathbf{E}\,\mathcal{J}_k$, for $k = 1, \dots, 5$ we arrive to (2.5) thus completing the proof of the theorem.

*Proof of Theorem 2.* Theorem 2 is a consequence of the following bound. For each $N = 2, 3, ...$ and each $1 < x < N^{1/6}$, we have $\Delta_N \le cW$, where

$$W := \Bigl(\frac{1}{x\sqrt{N}} + \frac{1}{\rho_x^2 N} + \frac{\beta_3^2}{\sigma^6 N} + \frac{\mathbf{E}\, X^4 \mathbb{I}\{X^2 < \sigma^2 N\}}{\sigma^4 \rho_x^2 N} + \frac{\mathbf{E}\, |X|^3 \mathbb{I}\{X^2 \ge \sigma^2 N\}}{\sigma^3 \sqrt{N}}\Bigr).$$

The scheme of the proof of this bound is similar to that of the proof of Theorem 1. Only now we use the conventional Esseen's (1945) smoothing lemma, what makes the proof considerably simpler, see Bloznelis and Putter (1998) for details.

## 3. Expansions

In this section we prove the inequality

$$I_1 := \int_{|t| \le H_1} |\,\mathbf{E}\, \mathrm{e}\{S\} - \varphi(t)|\,\frac{dt}{|t|} \ll \frac{\beta_4}{N}, \qquad H_1^{-1} = 4N\,\mathbf{E}\,|Y_1|^3,$$
$$S = Y g(V), \quad V = 1 + \eta - \frac{Y^2}{N}, \quad \varphi(t) = \exp\{-t^2/2\}\Bigl(1 - \frac{N\,\mathbf{E}\, Y_1^3 \;\; 3it + 2(it)^3}{6}\Bigr).$$

It is convenient to split the integral $I_1 = J_1 + J_2$, where

$$J_i = \int_{A_i} |\,\mathbf{E}\, \mathrm{e}\{S\} - \varphi(t)|\,\frac{dt}{|t|}, \qquad i = 1, 2,$$

and where $A_1 = \{|t| \le 1\}$ and $A_2 = \{1 \le |t| \le H_1\}$. The inequality $I_1 \ll \beta_4/N$ is a consequence of the two inequalities $J_i \ll \beta_4/N$, $i = 1, 2$. The bound $J_1 \ll \beta_4/N$ follows form Lemma 3.1 applied to the smooth function $H(u) = \exp\{itu\}$. The bound $J_2 \ll \beta_4/N$ can be obtained by combining the proof of Lemma 3.1 and some ideas of the proof of the Berry-Esseen bound for Student's $t$ statistic given in Bentkus and Götze (1996). Detailed calculations yielding the inequality $J_2 \ll \beta_4/N$ can be found in Bloznelis and Putter (1998).

Let $\xi, \xi_1, \xi_2, \dots$ be a sequence of independent standard normal r.v. We assume that this sequence is independent of $X_1, \dots, X_N$. By $\tilde{\xi}$ we denote the sum $\tilde{\xi}_1 + \dots + \tilde{\xi}_N$, $\tilde{\xi}_i = N^{-1/2}\xi_i$.

**Lemma 3.1.** *Let $H : \mathbf{R} \to \mathbb{C}$ be a bounded infinitely many times differentiable function with bounded derivatives. Assume that (2.2) holds. Then*

$$\big|\mathbf{E}\, H(S) - \mathbf{E}\, H(\xi) - \Gamma\big| \ll c_H \mathcal{R}, \qquad \Gamma = -\frac{N}{6}\, b_3\big(3\,\mathbf{E}\, H'(\xi) + 2\,\mathbf{E}\, H'''(\xi)\big),$$

*where $\mathcal{R} = \mathcal{M} + \gamma$ and $c_H = \|H'\| + \cdots + \|H^{vi}\|$.*

*Proof of Lemma 3.1.* We shall write $g \sim h$ if $|g - h| \ll c_H\, \mathcal{R}$. A simple calculation shows that $\Gamma \sim \Gamma_1 + \Gamma_2 + \Gamma_3$,

$$\Gamma_1 = \frac{N}{6}\,\mathbf{E}\, H'''(\xi) Y_1^3, \quad \Gamma_2 = -\frac{N}{2}\,\mathbf{E}\, H'(\xi) d_1, \quad \Gamma_3 = -\frac{N(N-1)}{2}\,\mathbf{E}\, H'''(\xi) Y_1^2 d_2.$$

Therefore, the lemma is a consequence of the following two facts,

$$\mathbf{E}\, H(Y) \sim \mathbf{E}\, H(\xi) + \Gamma_1, \qquad \mathbf{E}\, H(S) \sim \mathbf{E}\, H(Y) + \Gamma_2 + \Gamma_3. \tag{3.1}$$

For the proof of the first part of (3.1) we refer to Bentkus, Götze, Paulauskas and Račkauskas (1990), where the inequality $|\,\mathbf{E}\, H(Y_1 + \cdots + Y_N) - \mathbf{E}\, H(\xi)| \ll c_H(\mathcal{M} + (b_3 N)^2)$ was proved in the case of centered summands $Y_i$. It remains to prove the second part of (3.1).

Expanding in powers of $Y^2/N$ and using the bound $\mathbf{E}\,|Y|^3 \ll 1$, see Lemma 4.1, we get $\mathbf{E}\, H(S) \sim h_1$, where $h_1 = \mathbf{E}\, H(Y g(1+\eta))$. Write

$$g_1(\eta) = \mathbf{E}_{\theta} g'(1+\theta\eta), \qquad Y\eta = U + D, \qquad U = \sum_{i \neq j} Y_i \eta_j, \qquad D = \sum_i d_i.$$

By the mean value theorem, $g(1+\eta) = 1 + g_1(\eta)\,\eta$. Then $Y\, g(1+\eta) = Y + (U+D) g_1(\eta)$. Expanding $H$ in powers of $U\, g_1(\eta)$ we get $h_1 = h_2 + h_3 + R$,

$$h_2 = \mathbf{E}\, H(W), \qquad h_3 = \mathbf{E}\, H'(W) U g_1(\eta) \quad \text{where} \quad W = Y + D g_1(\eta)$$

and where $|R| \ll c_H\, \mathbf{E}\, U^2$. By Lemma 4.1, $\mathbf{E}\, U^2 \ll \mathcal{R}$. Therefore, $h_1 \sim h_2 + h_3$.

Let us show that $h_3 \sim \Gamma_3$. The inequality $|g_1(\eta) - g'(1)| \ll |\eta|$ implies

$$|h_3 - h_4| \ll c_H\, \mathbf{E}\, |U\eta|, \quad h_4 = \mathbf{E}\, H'(W)\, U\, g'(1) = -\frac{N(N-1)}{2}\,\mathbf{E}\, H'(W)\, Y_1 \eta_2.$$

Furthermore, the inequalities $|U\eta| \le U^2 + \eta^2$, $\mathbf{E}\, U^2 \ll \mathcal{R}$ and $\mathbf{E}\, \eta^2 \ll \mathcal{M}$, see Lemma 4.1, imply $h_3 - h_4 \sim 0$. Hence, $h_3 \sim h_4$. Next we show that

$$h_4 \sim -\frac{N(N-1)}{2}\, h_5, \qquad h_5 = \mathbf{E}\, H'''(Y^{(1,2)}) Y_1^2 d_2. \tag{3.2}$$

By the mean value theorem, $g_1(\eta) - g_1(\eta^{(1)}) = w\,\eta_1$ with some $|w| \ll 1$. Then

$$D\,g_1(\eta) = D^{(1)} g_1(\eta) + d_1 g_1(\eta) = W_1 + W_2,$$

where we denote $W_1 = D^{(1)} g_1(\eta^{(1)})$ and $W_2 = d_1 g_1(\eta) + D^{(1)} w \eta_1$. We have $W = Y + W_1 + W_2$. Expanding $H'$ in powers of $W_2$ we get

$$|\,\mathbf{E}\,H'(W) Y_1 \eta_2 - h_6| \ll c_H R_1, \quad h_6 = \mathbf{E}\,H'(Y + W_1) Y_1 \eta_2, \quad R_1 = \mathbf{E}\,|Y_1 \eta_2 W_2|.$$

Furthermore, expanding $H'$ in powers of $Y_1$ we obtain $|h_6 - h_7 - h_8| \ll c_H R_2$, where

$$h_7 = \mathbf{E}\,H''(Y^{(1)} + W_1) Y_1^2 \eta_2, \qquad h_8 = \mathbf{E}\,H'''(Y^{(1)} + W_1) Y_1^3 \eta_2,$$

and $R_2 = \mathbf{E}\,|\eta_2|(\,\mathbf{E}\,Y_1^4 + |\,\mathbf{E}\,Y_1|)$. A simple calculation shows that $N^2 |R_i| \ll \mathcal{R}$, $i = 1, 2$. We obtain

$$h_4 \sim -\frac{N(N-1)}{2}\,(h_7 + h_8).$$

We complete the proof of (3.2) by showing that

$$N^2 h_8 \sim 0 \qquad \text{and} \qquad N^2 (h_7 - h_5) \sim 0. \tag{3.3}$$

By the mean value theorem, $g_1(\eta^{(1)}) - g_1(\eta^{(1,2)}) = w\,\eta_2$, with some $w \ll 1$. Then we can write $W_1 = W_3 + W_4$, where

$$W_3 = D^{(1,2)} g_1(\eta^{(1,2)}) \qquad \text{and} \qquad W_4 = d_2 g_1(\eta^{(1)}) + D^{(1,2)}\, w\, \eta_2.$$

Expanding $H''$ and $H'''$ in powers of $W_4$ we get

$$\begin{aligned} |h_7 - h_9| &\ll c_H R_1, \quad h_9 = \mathbf{E}\,H''(Y^{(1)} + W_3) Y_1^2 \eta_2, \quad R_1 = \mathbf{E}\,Y_1^2 |\eta_2 W_4|, \\ |h_8 - h_{10}| &\ll c_H R_2, \quad h_{10} = \mathbf{E}\,H'''(Y^{(1)} + W_3) Y_1^3 \eta_2, \quad R_2 = \mathbf{E}\,Y_1^3 |\eta_2 W_4|. \end{aligned}$$

Furthermore, expanding in powers of $Y_2$ we get $|h_{10}| \ll c_H R_3$, with $R_3 = \mathbf{E}\,|Y_1^3 d_2|$. Finally, expanding $H''$ in powers of $Y_2$ and then in powers of $W_3$ we obtain

$$|h_9 - h_5| \ll c_H R_4, \qquad \text{with} \qquad R_4 = \big(\,\mathbf{E}\,Y_1^2 Y_2^2 |\eta_2| + \mathbf{E}\,Y_1^2 |d_2 W_3|\big).$$

A straightforward calculation shows $N^2 R_i \ll \mathcal{R}$, $i = 1, 2, 3, 4$. Hence, $N^2 h_8 \sim N^2 h_{10} \sim 0$ and $N^2 h_7 \sim N^2 h_9 \sim N^2 h_5$. We arrive to (3.3) thus completing the proof of (3.2).

Next we replace $Y^{(1,2)}$ by $\tilde{\xi}^{(1,2)}$ in $h_5$. The error of this replacement,

$$\big|\,\mathbf{E}\,H'''(Y_3 + \cdots + Y_N) - \mathbf{E}\,H'''(\tilde{\xi}^{(1,2)})\big| \ll c_H (b_3 N + N\gamma). \tag{3.4}$$

For the proof of this inequality we refer to Bentkus, Götze, Paulauskas and Račkauskas (1990), where a similar bound was proved in the case of centered summands $Y_i$. Combining (3.4) and the inequalities

$$|\,\mathbf{E}\,H'''(\tilde{\xi}^{(1,2)}) - \mathbf{E}\,H'''(\xi)| \ll c_H\,\mathbf{E}\,|\tilde{\xi}_1+\tilde{\xi}_2| \ll N^{-1/2}, \quad \mathbf{E}\,Y_1^2|d_2| \ll b_3/N \tag{3.5}$$

we obtain

$$N^2\big|h_5 - \mathbf{E}\,H'''(\xi)Y_1^2 d_2\big| \ll c_H(b_3N + N\gamma + N^{-1/2})b_3N \ll c_H\mathcal{R}.$$

Hence, $-N(N-1)h_5/2 \sim \Gamma_3$ and this completes the proof of $h_3 \sim \Gamma_3$.

It remains to show that $h_2 \sim \mathbf{E}\,H(Y) + \Gamma_2$. We start by showing

$$h_2 \sim \mathbf{E}\;H(Y) + h_{11}, \qquad \text{where} \quad h_{11} = -\frac{N}{2}\;\mathbf{E}\;H'(Y^{(1)})d_1. \tag{3.6}$$

Expanding $g_1$ and then $H$ in powers of $\eta$ we get

$$|h_2 - h_{12}| \ll c_H R, \qquad h_{12} = \mathbf{E}\;H(Y + D\,g'(1)), \qquad |R| = \mathbf{E}\;|\eta D|.$$

The inequalities $|\eta D| \le \eta^2 + D^2$, $\mathbf{E}\,\eta^2 \ll \mathcal{M}$ and $\mathbf{E}\,D^2 \ll \mathcal{R}$, see Lemma 4.1, imply $R \sim 0$. Therefore, $h_2 \sim h_{12}$. Furthermore, expanding $H$ in powers of $Dg'(1)$ we get

$$h_{12} = \mathbf{E}\;H(Y) + h_{13} + R, \qquad h_{13} = \mathbf{E}\;H'(Y)\,D\,g'(1), \tag{3.7}$$

where $|R| \ll c_H\,\mathbf{E}\;D^2 \sim 0$, see Lemma 4.1. Hence, $h_{12} \sim \mathbf{E}\,H(Y) + h_{13}$. By the symmetry, $h_{13} = -2^{-1}N\;\mathbf{E}\,H'(Y)d_1$. Now, expanding $H'$ in powers of $Y_1$ we obtain $h_{13} \sim h_{11}$. This together with (3.7) yields (3.6). Finally, we replace $Y^{(1)}$ in $h_{11}$ by $\tilde{\xi}^{(1)}$. The error this replacement, $|h_{11} - \Gamma_2| \ll c_H\mathcal{R}$, cf (3.4) and (3.5). We obtain $h_{11} \sim G_2$ thus completing the proof of the lemma.

## 4. Auxiliary Results

**Lemma 4.1.** *Assume that (2.2) holds. Then*

$$\mathbf{E}\,|Y_B|^s \le c(s), \tag{4.1}$$

*for each $s > 0$ and each subset $B \subset \{1,\dots,N\}$. Furthermore, for every $m \le n$,*

$$\mathbf{E}\,D^2 \ll mb_4 + m^2b_3^2 \ll \mathcal{M}, \qquad D = \sum_{1\le j\le m} d_j,$$

$$\mathbf{E}\,U^2 \ll (m/N)^2\mathcal{M}, \qquad U = \sum_{1\le i,j\le m,\, i\ne j} Y_i\eta_j.$$

*Write $A = \{1, \dots, m\}$, where $m < n$. We have $\mathbf{E}\, \eta_A^2 \ll m b_4$,*

*Proof of Lemma 4.1.* The last inequality is trivial. The inequality $\mathbf{E}\, |Y_B|^s \le c(s)$ is proved in Bentkus and Götze (1996).

To estimate $\mathbf{E}\, D^2$ write

$$\mathbf{E}\, D^2 \le \mathbf{E}\, T_1 + \mathbf{E}\, |T_2|, \qquad T_1 = \sum_{1 \le i \le m} d_i^2, \quad T_2 = \sum_{1 \le i,j \le m,\, i \ne j} d_i d_j.$$

Using the inequality $|Y_i| \le 1$ we get $\mathbf{E}\, T_1 \ll m b_4$. Furthermore, $\mathbf{E}\, |T_2| \ll m^2 b_3^4$. Finally, by (2.1), $(N b_3)^2 \le \mathcal{M}$.

Let us estimate $\mathbf{E}\, U^2$. Write

$$U = U_1 + U_2, \qquad U_1 = \sum_{1 \le i,j \le m,\, i \ne j} \mathbf{E}\, Y_i \eta_j, \quad U_2 = \sum_{1 \le i,j \le m,\, i \ne j} (Y_i - \mathbf{E}\, Y_i) \eta_j.$$

A simple calculation shows

$$\begin{aligned} \mathbf{E}\, U_1^2 &= \big((m-1)\, \mathbf{E}\, Y_1\big)^2\, \mathbf{E}\, (\eta_1 + \dots + \eta_m)^2 \ll (m/N)^3 \gamma^2 \mathcal{M}, \\ \mathbf{E}\, U_2^2 &= m(m-1)\, \mathbf{E}\, (Y_1 - \mathbf{E}\, Y_1)^2\, \mathbf{E}\, \eta_2^2 \ll (m/N)^2 \mathcal{M}. \end{aligned}$$

Clearly, the bound $\mathbf{E}\, U^2 \ll (m/N)^2 \mathcal{M}$ is an easy consequnce of these inequalities.

The last inequality, $\mathbf{E}\, \eta_A^2 \ll m\, b_4$, is trivial.

**Lemma 4.2.** *Assume that (2.2) holds. Then*

$$1 - 2\beta_4/N \le a \le 1. \tag{4.2}$$

*In particular we have $3/4 \le a \le 1$.*

*Proof of Lemma 4.2.* Clearly, $a^2 \le \sigma^2 \le 1$. Furthermore, write $\varkappa(u) = \mathbf{E}\, X_1^2 \mathbb{I}\{X_1^2 \le u\, N\}$, for $u \ge 0$. We have

$$\tau := 1 - \varkappa(1/2) = \mathbf{E}\, X_1^2 \mathbb{I}\{X_1^2 > N/2\} \le 2\beta_4/N. \tag{4.3}$$

Note that $2\beta_4/N < 1/4$, by (2.2). We have $\tau < 1/2$. The function $\varkappa(u)$ is nondecreasing. Therefore, (4.3) implies $\varkappa(1-\tau) \ge \varkappa(1/2) = 1 - \tau$. But, $\varkappa(1) \le \sigma^2 = 1$. Then there exists a solution of the equation $u = \varkappa(u)$ in the interval $1 - \tau \le u \le 1$. This implies $a^2 \ge 1 - \tau$ and we obtain $a \ge a^2 \ge 1 - \tau$. This inequality in combination with (4.3) yields (4.2) thus completing the proof of the lemma.

**Lemma 4.3.** *Assume that (2.2) holds. Let $S_1$ be given by (2.4). Then*

$$\mathbf{P}\left\{|S_1 - \mathbf{t}| > \mathcal{M}\right\} \ll \rho^{-2} N^{-1} \beta_4. \tag{4.4}$$

*Proof of Lemma 4.3.* The inequalities

$$\mathbf{P}\left\{\sqrt{N}\,\mathbf{t} \neq \frac{Y}{\sqrt{1+\eta - Y^2/N}}\right\} \leq N\ \mathbf{P}\left\{X_1^2 > a^2 N\right\} \leq (4/3)^3 \gamma_0 \ll \beta_4/N,$$
$$\mathbf{P}\left\{\frac{Y}{\sqrt{1+\eta - Y^2/N}} \neq S\right\} \leq\ \mathbf{P}\left\{|\eta| > 1/4\right\} +\ \mathbf{P}\left\{Y^2/N > 1/4\right\}$$

and

$$\mathbf{P}\left\{|\eta| > 1/4\right\} \leq 16\,\eta^2 \leq 32 N\ \mathbf{E}\ Y_1^4, \qquad \mathbf{P}\left\{Y^2/N > 1/4\right\} \leq 4\,N^{-1}\,\mathbf{E}\ Y^2 = 4/N$$

imply $\mathbf{P}\left\{\mathbf{t} \neq S\right\} \ll \mathcal{R}$.

In order to prove (4.4) it suffices to show $\mathbf{P}\left\{|S - S_1| > \mathcal{M}\right\} \leq c\rho^{-2} N^{-1}\beta_4$. We have $V = V_B + \eta_A - W_1 - W_2$, where $W_1$ and $W_2$ are given by (2.16). Expanding $g$ in powers of $W_1 + W_2$ and then in powers of $\eta_A$ we get

$$\begin{aligned} |S - S_1| \leq & c_g\big(R_1 + \cdots + R_6\big), \qquad \text{where} \quad R_1 = |Y_A W_1|, \quad R_2 = |Y_A W_2|, \\ & R_3 = |Y_B W_1|, \quad R_4 = |Y_B W_2|, \quad R_5 = |Y_A|\eta_A^2, \quad R_6 = |Y_B \eta_A^3|. \end{aligned} \tag{4.5}$$

We complete the proof by showing $P_i := \mathbf{P}\left\{|R_i| > \mathcal{M}/6\right\} \ll \rho^{-2}\mathcal{M}$, where we abbreviate $\rho_N$ by $\rho$. In what follows we use the inequalities $N\mathcal{M} \geq 1$ and $\mathbf{E}\,|Y_D|^s \leq c(s)$, see (2.1) and Lemma 4.1. By Chebyshev's inequality,

$$\begin{aligned} &P_1 \ll (\mathcal{M}\,N)^{-4/3}\,\mathbf{E}\,Y_A^4 \ll\ \mathbf{E}\,Y_A^4 \ll m^3 b_4 \ll \rho^{-2}\mathcal{M}, \\ &P_i \ll (\mathcal{M}\,N)^{-2}\,\mathbf{E}\,Y_B^2\,\mathbf{E}\,Y_A^4 \ll\ \mathbf{E}\,Y_A^4 \ll \rho^{-2}\mathcal{M}, \quad i = 1,\, 2, \\ &P_4 \ll (\mathcal{M}\,N)^{-4}\,\mathbf{E}\,Y_B^8 Y_A^4 \ll\ \mathbf{E}\,Y_A^4 \ll \rho^{-2}\mathcal{M}, \\ &P_5 \ll \mathcal{M}^{-4/5}\,\mathbf{E}\,|Y_A\eta_A|^{4/5} \ll \mathcal{M}^{-4/5}\big(\mathbf{E}\,Y_A^4 + \mathbf{E}\,\eta_A^2\big) \ll \mathcal{M}^{-4/5} m^3 b_4 \ll \rho^{-2}\mathcal{M}, \end{aligned}$$

Here we applied the inequality $a\,b^4 \leq a^5 + b^5$ to $a = |Y_A|$ and $b = |\eta_A|^{1/2}$. Finally, the inequalities

$$P_6 \ll \mathcal{M}^{-2/3}\,\mathbf{E}\,|Y_B\eta_A^3| \ll \mathcal{M}^{-2/3}\,\mathbf{E}\,\eta_A^2 \ll \mathcal{M}^{-2/3} m b_4 \ll \rho^{-1}\mathcal{M}$$

complete the proof of (4.4). Lemma is proved.

## References


Babu Jogdesh G. and Kesar Singh, *Edgeworth expansions for sampling without replacement from finite populations*, J. Multivar. Analysis. **17** (1985), 261–278.

Bentkus, V. Bloznelis, M. and Götze, F., *The Berry–Esseen bound for Student's Statistic in the non-i.i.d. case*, J. Theoret. Probab. **9** (1996), 765-796.

Bentkus, V. and Götze, F., *The Berry–Esseen bound for Student's Statistic*, Ann. Probab. **24** (1996), 491–503.

Bentkus. V., Götze, F. and van Zwet, W. R., *An Edgeworth expansion for symmetric Statistics,*, Ann. Statist. **25** (1997), 851–896.

Bentkus, V., Götze F., Paulauskas V. and Račkauskas A., *The accuracy of Gaussian approximation in Banach spaces*, SFB 343 Preprint 90 -100, Universität Bielefeld (In Russian: Itogi Nauki i Tehniki, ser. Sovr. Probl. Matem., Moskva, VINITI **81** (1991), 39–139; to appear in English in Encyclopedia of Mathematical Sciences, Springer), 1990.

Bhattacharya, R.N. and Ghosh, J.K., *On the validity of the formal Edgeworth expansion*, Ann. Statist. **6** (1978), 435–451.

Bloznelis, M. and Putter, H., *Second order and bootstrap approximation to Student's t statistic*, Preprint 98 - 053 SFB 343, Bielefeld University 1998, pp 1-40, Publication available at the internet: http://www.mathematik.uni-bielefeld.de/sfb343/Welcome.html.

Chibisov, D.M., *Asymptotic expansion for the distribution of a statistic admitting a stochastic expansion I*, Theor. Probab. Appl. **25** (1980), 732–744.

Chung K.L., *The approximate distribution of Student's statistic*, Ann. Math. Stat. **17** (1946), 447-465.

Esseen C.–G., *Fourier analysis of distribution functions. A mathematical study of the Laplace–Gaussian law*, Acta Math. **77** (1945), 1–125.

Friedrich K.O., *A Berry–Esseen bound for functions of independent random variables*, Ann. Statist. **17** (1989), 170–183.

Gine, E., Götze, F. and Mason, D.M., *When is the Studen t-statistic asymptotically standard normal?*, Ann. Probab. **25** (1997), 1514–1531.

Hall P., *Edgeworth expansion for Student's t statistics under minimal moment conditions*, Ann. Statist. **15** (1987), 920–931.

Hall P., *On the effect of random norming on the rate of convergence in the Central Limit Theorem*, Ann. Probab. **16** (1988), 1265–1280.

Helmers R., *On the Edgeworth expansion and the bootstrap approximation for a Studentized U-statistic*, Ann. Statist. **19** (1991), 470–484.

Helmers R., *The Berry–Esseen bound for studentized U-statistics*, Canad. J. Stat. **13** (1985), 79–82.

Helmers R., *On the Edgeworth expansion and the bootstrap approximation for a Studentized U-statistic*, Ann. Statist. **19** (1991), 470–484.

Helmers R. and van Zwet W., *The Berry–Esseen bound for U-statistics*, Stat. Dec. Theory Rel. Top. (S.S. Gupta and J.O. Berger, eds.), vol. 1, Academic Press, New York, 1982, pp. 497–512.

Petrov, V.V., *Limit Theorems in Probability Theory. Sequences of Independent Random Variables* (1995), Oxford University Press, New York.

Praskova, Z., *Sampling from a finite set of random variables: the Berry–Esseen bound for the Studentized mean*, Proceedings of the Fourth Prague Symposium on Asymptotic Statistics, Charles Univ., Prague, 1989, pp. 67–82.

Prawitz, H., *Limits for a distribution, if the characteristic function is given in a finite domain*, Scand. AktuarTidskr. (1972), 138–154.

Putter, H. and van Zwet, W.R., *Empirical Edgeworth expansions for symmetric statistics*, To appear in Ann. Statist. (1998).

Singh Kesar, *On the asymptotic accuracy of Efron's bootstrap*, Ann. Statist. **9** (1981), 1187-1195.
Slavova V.V., *On the Berry–Esseen bound for Student's statistic*, LNM **1155** (1985), 335–390.
van Zwet, W.R., *A Berry-Esseen bound for symmetric statistics*, Z. Wahrsch. verw. Gebiete **66** (1984), 425–440.