## Chapter 4. FIXED INCOME SECURITIES

**Objectives:**

- To set the price of securities at the specified moment of time.
- To simulate mathematical and real content situations, where the values of securities need to be calculated.
- To evaluate the income of securities. Examined study results:
- Will know the types of securities.
- Will calculate the prices of securities.
- Will assess the income of securities.
- Will simulate mathematical and real content situations, where the methods for calculation the security price and income will be analysed.

**Student achievement assessment criteria:**

- Correct use of concepts.
- Proper use of formulas.
- Precise intermediate and final answers.
- Correct answers to questions.

**Repeat the concepts:** Simple interest. Simple interest and discount rates. Discount factor in the case of simple interest. The calculation of the future and present value in the case of simple interest. Compound interest. Nominal and effective interest rates. Discount rate and discount factor in the case of compound interest. Interest conversion period. The future and present value calculations in the case of compound interest. The future and present value calculation using the precision method and the continuous conversion of interest.

### 4.1 Promissory notes

**Remark** Promissory note often is called *bill.*

The participants of economic activity will inevitably face a number of different financial transactions. Perhaps the most important attribute of financial relations is the activity crediting. In other words, borrowing is inevitable for the expansion of activities, as well as for the management of adequate working capital. There exist different borrowing and lending instruments. The publication will discuss various credit instruments in details. This section will explore the specific debt securities, which are called promissory notes. We consider some of the promissory notes.

**Definition.** A promissory note is called a written undertaking concluded in the manner provided by the appropriate legislation in which the person is unconditionally obliged to pay a specified sum of money with interest or without interest at the specified time for the specified entity.

Below we give a number of concepts that are used for the creation of promissory notes.

The amount of money indicated in the promissory note is called a *face value.*

The *lender, or drawee,* of a promissory note is a person who borrows money.

The *payer (investor)* of a promissory note is a person who lends money.

*Interest rate of the promissory note* is the annual simple interest rate applicable to nominal value of the promissory note.

The *release date* is the moment of time when the promissory note was drawn and appeared in the circulation (was signed).

The promissory note *payment date* is a moment in time when a promissory note is paid *redeemed.*

*Interest period* is the period between the drawing of a promissory note (issue) and its maturity.

The *final value or maturity* is a face value plus interest.

The *discount period* is the time interval between the purchase (sale) of a promissory note and the moment of its maturity.

The promissory note without additional conditions are called *simple promissory note* or we shortly call *promissory note.*

The *promissory notes of exchange* are financial documents that require the individual or business that is addressed in the document to pay a specified amount of money on a date that is cited within the text of the document. Considered to be a negotiable instrument, the date for the demand to pay generally ranges from the current date to a date within the next six calendar months. A promissory note of exchange will also require the authorized signature of the debtor in order to be considered legal and binding.

The promissory note of exchange sometimes is called a *accommodation- promissory note.*

Sometimes, provisions are included concerning the payee's rights in the event of a default, which may include foreclosure of the maker's assets. *Demand promissory notes* are notes that do not carry a specific maturity date, but are due on demand of the lender. Usually the lender will only give the borrower a few days notice before the payment is due. For loans between individuals, writing and signing a promissory note are often instrumental for tax and record keeping.

DEMAND NOTE

Date: ____________________

On demand, the undersigned, for value received, jointly and severally promises to pay to the order of ______________________ [*insert name of lender*] the sum of ______________________________ dollars ($________.00) together with interest thereon from the date hereof until paid at the rate of _______% per annum.

In the event this note is not paid when due, the undersigned shall pay all attorney's fees and reasonable costs of collection.

NOTICE TO BORROWER: THIS IS A DEMAND NOTE AND SO MAY BE COLLECTED BY THE LENDER AT ANY TIME. A NEW NOTE MUTUALLY AGREED UPON AND SUBSEQUENTLY ISSUED MAY CARRY A HIGHER OR LOWER RATE OF INTEREST.

______________________________
[Maker's signature]

______________________________
[Maker's typed or printed name]

NOTICE TO CO-SIGNER: YOUR SIGNATURE ON THIS NOTE MEANS THAT YOU ARE EQUALLY LIABLE FOR REPAYMENT OF THIS LOAN. IF THE BORROWER DOES NOT PAY, THE LENDER HAS A LEGAL RIGHT TO COLLECT FROM YOU.

______________________________
[Co-signer's signature]

______________________________
[Co-signer's typed or printed name]

Witnessed:

______________________________
[Witness's signature]

______________________________
[Witness's typed or printed name]



fig 3.1 demand note

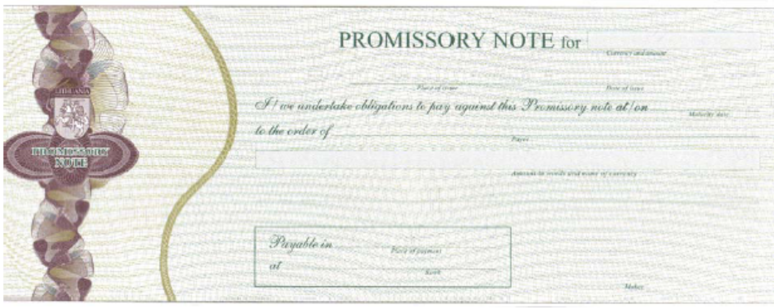
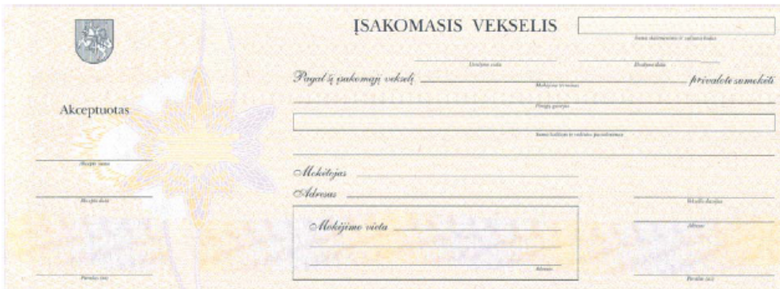




fig 3.2 promissory note



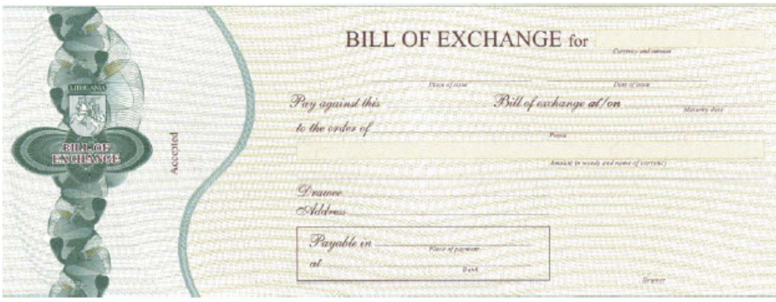




fig 3.2 promissory note of exchange

Making promissory note, promissory note of exchange or demand note are used the same notions. So in what follows we discuss about the notions which are related to promissory note because this kind of promissory note are used widely.

Basic promissory note form:

1. The top of the promissory note should indicate whether this is a promissory note or a promissory note of exchange. Further text must match the language of the of the promissory note;
2. The unconditional order to pay the amount should be given;
3. The title or the name and surname of the owing entity (the payer);
4. Maturity;
5. Place of payment;
6. The title or the name and surname of the owing entity to whom or on the order of whom it shall be paid;
7. Signing place and date;

8. Signature of the entity who issues the promissory note .

In the case the promissory note the third item is not usually provided. A promissory notes shall be void if at least one of the above-mentioned items is missing.

In different countries, the maturity of a promissory note has certain "tolerance" limits, which creates preconditions for the payment of a promissory note not necessarily at the point of time, which is indicated on the promissory note. In other words, the payment date may be delayed for several days (often for two or three days).

For example, if it is told that the promissory note lasts for 60 days, during the calculation of its value the number of interest days is applied as 62, in the case the two days of tolerance are applied.

**Note.** We generally are not going to apply the tolerance term and we are going to calculate the interest rates of a promissory note for the indicated term.

The promissory note s are two kinds: 1) promissory note s with interest; 2) promissory note s non interest bearing.

If the promissory note is given with interest, when the interest period includes the days after the drawing of a promissory note until the maturity date, with the additional condition that the first of the next days after the drawing of a promissory note, the last day is the maturity of a promissory note. If the promissory note is presented in weeks, months, years, when the date of maturity of promissory note and the date of the drawing of a promissory note are the same. The interest rate depends on the interest interval (ordinary yearly).

Suppose that promissory note is with interest, $A$ is nominal, $T$ is duration of the promissory note , $S$ maturity value and $r-$ simple interest rate. Then maturity value can be find by formula

$$S = (1 + rT)P.$$

If the promissory note is non interest bearing then $S = A$.

**Example.** The six-month promissory note was signed on March. 30, 2005, with the interest rate of 12.5%, and the final value (maturity) of 2000. Determine the face value.

Let us examine this situation in more detail. On the one hand, as six months is a half of the year, in the formulas they can be marked as t = 0.5, but in this case the promissory note should also mention that this is the half-year promissory note . On the other hand, the exact period of six months covers 184 days. If the promissory note would be include two days of tolerance, the number of 186 days should be used for the calculation of its value. The promissory note is redeemed on September 30, 2005.

We have that

$S = 2000;\ \ r = 0.125; t = \frac{184}{365}$. Then

$$P = \frac{20000}{1 + 0.125 \cdot \frac{184}{365}} = \frac{2000}{1.063013} \approx 1881.44.$$

## 4.2 Discounting of the promissory note

The person (company) managing promissory note s may sell these securities to other buyer. The purchase (sale) process is usually called the promissory note discounting. Let us discuss this method in more details.

By purchasing promissory notes the buyer invests money in the hope of obtaining the income from the deal in the future. This means that when purchasing a promissory note the entity has to pay less than the redemption value of a promissory note. We will analyse the method of this value determination. The following concepts are important in determining the discounted value of a promissory note.

1) The discount interest rate is called an interest rate used for the promissory note discounting. This interest rate at time moment $t$ in the following we denote by $i(t)$.

2) The promissory note discount rate is called an annual discount rate used in the promissory note discounting.

3) The discount term is the moment of time at which the discounted value of a promissory note is calculated.

4) The discount period is the interval of time between the discount term and maturity, i.e. the point in time when a promissory note must be paid.

5) The promissory note income is called an amount paid by the buyer to the manager of a promissory note. This amount shall be marked with the letter $B$.

6) The promissory note discount $D$ will be called a gap between the redemption price and the income of a promissory note:

$$D = S - B.$$

**Note.** If the promissory note has the interval of tolerance, during the calculation of the future value the interval of tolerance is added to the indicated interest period of the promissory note, and during the analogous discounting we have to take into account that in the calculation of discounting period at the end we need to extend it by the interval of tolerance.

For example the one-month promissory note signed on 1 April with the three-day period of tolerance was discounted on April 16. We have that the period of the promissory note is $30 + 3 = 33$ days, and the discount period is $14 + 3 = 19$ days. As we have mentioned already we are not going to use the concept of tolerance in calculations.

Let us form a formula for the promissory note income calculation. First of all, suppose that the promissory note face value is $A$, and the interest rate is $r$. Let $T$ be the promissory note interest rate period. Let us assume that the promissory note is sold (purchased) at the time when the time $t$ remains until the redemption of the promissory note, and at that moment the value of money is it. Then the purchase price of the promissory note will be determined as follows:

$$B = \frac{A(1 + rT)}{1 + ti(t)} = \frac{S}{1 + ti(t)}.$$

The promissory note discount is often referred to as the borrowing costs. The discount can be written in the following way

$$D = A(1 + rT) - \frac{A(1 + rT)}{1 + ti(t)} = A(1 + rT)\frac{i(t)t}{1 + ti(t)} = Sdt.$$

As we can see the borrowing costs (discount) satisfy the inequality:

$$0 \leq D \leq A(1 + rT).$$

**Note** If the market interest rate is $i = 0$, we see that the borrowing costs are also equal to zero. Thus, the purchase price of the promissory note coincides with its final price. Borrowing costs are increasing and approaching the future value of the promissory note when the market interest rate growth can be observed. Thus, the higher the market interest rate is, the lower income of the promissory note can be obtained, and the borrowing costs of the promissory note approach the redemption (final) value. In this case, the purchase price of the promissory note B approaches zero.

The promissory note of 150 days and 10Let us determine the final value of the promissory note .

$A = 120000; \quad r = 0.1; \quad T = \frac{150}{365}$. Then

$$S = 120000(1 + 0.1 \cdot \frac{150}{365}) = 115263.16.$$

Discounting we have $S = 115263,16; \quad i = 0,13; \quad t = \frac{58}{365}$.

Then

$$B = \frac{115263.16}{1 + 0.13 \cdot \frac{55}{365}} = 113048.65,$$

Here B is the value of the promissory note during the discount term. The promissory note discount is $115263.16 - 113048.65 = 2214.51$.

**Definition**. The promissory note is called an non interest bearing promissory note , if the final value coincides with its nominal value.

In other words, if the promissory note is the non interest bearing one, then at the time of maturity its face value is paid. If the promissory note is purchased (sold) not at the final term, then while determining its acquisition price the face value is discounted, and the discount factor includes the market-prevailing interest rate. It is sometimes said that a promissory note is discounted according to the current cash value. It is easy to understand that the face value of the non interest bearing promissory note includes the interest accumulated at the beginning of the period which at different points in time is discounted with different interest rates.

Thus, in this case, while determining the purchase price of the promissory note $B(t)$ at any point in time when the interest rate (cash value) at this moment is it (the annual interest rate), and the time interval the length of which is $t$ remains until the maturity, we apply the following formula:

$$B(t) = \frac{S}{1 + ti(t)}.$$

Find the value of the non interest bearing promissory note in the moment of time, when the promissory note was drawn up, if its face value is 95000, this is the three-month promissory note signed on September 30, and at the moment of drawing the cash value was 13.5%.

We have that

$S = 95000; \quad r = 0.135; \quad t = \frac{92}{365}$. Then

$$B = \frac{95000}{1 + 0.135 \cdot \frac{92}{365}} = 9187.35.$$

Find the income of the 120-day non interest bearing promissory note , with a face value of 40000, which was drawn up on October 2, and purchased on October 21, when the cash value of that time is 13%.

We have that the promissory note is to be redeemed on January 2, when 72 days are left until its maturity, and the cash value at the time of acquisition is 13%.

**Note.** If in the case, the promissory note would have two days of tolerance, when the number of 74 days would be used instead of 72 days in formulas. We have that

$S = 40000; \quad r = 0.13; \quad t = \frac{72}{365}$. Then

$$B = \frac{40000}{1 + 0.13 \cdot \frac{72}{365}} \approx 39000,$$

where $B$ is the current capital value, during the time of a focal date.

**Note.** The present value of the non interest bearing promissory note is the price of the promissory note at any moment of time, which is prior to a focal date in the case of the indicated cash value. In the case of the non interest bearing promissory note, while determining its present value, you need to know the final value of the promissory note . When calculating the final and current values of the promissory note two interest rates should be used:

1) when defining the final value of the promissory note the interest rate specified in the promissory note is used;

2) when calculating the present value of capital at the focal date point, i.e. when the promissory note is purchased or sold, the rate of cash value is used.

The two-month non interest bearing promissory note was signed on June 30 for the amount of 7000. Find the income of the promissory note and the discount size, if the promissory note is discounted on July 31, with 16payment date is August 30, the term of tolerance is September 2, the discount date is July 31, the discount period (July 31 to September 2) is $1 + 31 + 1 = 33$ (excluding September 2).

We have that

$$S = 7000; \quad r = 0.16; \quad t = \frac{33}{365}.$$

Then the promissory note income is

$$P = \frac{7000}{1 + 0.16 \cdot \frac{33}{365}} = 6900.$$

The discount rate is $7000 - 6900 = 100$.

Let us consider the problem of discounting with the help of other terms, i.e. the discount rate. Specifically, the discount rate s is used for the discounting procedure instead of the interest rate $i$. Let us recall that the discount rate is the percentage from the final value. Thus, when the borrowing interval of is $t$, the borrowing costs (discount) is $D = Sdt$, or

$$d = \frac{S - P}{S}.$$

Then the promissory note income can be expressed as follows: $B = S - D = (1 - dt)S$.

### 4.3 Demand promissory notes

We will consider another method of financial liabilities the demand loans, which are also called the demand promissory notes. Note that during the examination of this method of debt repayment the cases with the varied interest rate during the promissory note term have to be discussed. Let us analyse the essence of this method.

**Definition.** The *demand promissory note* is called a debenture, based on which the investor may demand the drawee of the promissory note to pay any part of this promissory note at any time. Sometimes this operation is called the protest of a promissory note. And vice versa, the drawee of a promissory note can also redeem a promissory note or to pay part of it at any time.

During the analysis of this sort of promissory notes the interest rates are not fixed, and even more at the change of the interest rate, the change in calculation of the promissory note interest can also be observed. The interest are calculated according to the contemporary cash value from the book value of the promissory note (the remaining debt), and usually, if it is not mentioned additionally, is paid every month.

The method will be called the *regular one*, when the interest is paid on a monthly basis and during the calculation of the interest rate the number of payments within the period of one month and the number of changed interest are taken into account. At the last day of the promissory note period the uncovered value of a promissory note with the accumulated interest is paid.

The method will be called the *declining balance* method if the promissory note interest is covered by payments. Interest may be covered partially, completely or with the surplus, at the same covering a part of the promissory note face value, from which the interest is calculated, as well.

Indeed, both methods are the same, only in the case of the regular method an interest is paid monthly, while in the case of the declining balance method, interest are covered by payments. After their formalization these methods are fundamentally the same.

**Note.** The discussed method can also be extended for the applications, by considering a book value of the loan not only a face part of the loan, but a part of the uncovered interest added to it, i.e. at any point in time the interest is calculated from the uncovered interest as well.

Assume that the term of the promissory note is $T$ days, its face value is $A$. Let only the periods of $t_{11}, \dots t_{1k}$, during which the change in the interest rate could be observed, have been noticed until the first interest payment $P_1$, and the interest rates for the period of time are $r_{11}, \dots, r_{1k}$, respectively. For the amount of these periods the following interest accumulated:

$$I_1 = (r_{11}t_{11} + \cdot + r_{1k}t_{1k})A,$$

here $T_1 := t_{11} + \cdots + t_{1k} \leq T$.

If $I_1 > P_1$, then the absolute value of a difference $S_1 = P_1 - I_1$ is the value of the uncovered interest rate, which is not added to the fixed capital; however while paying the promissory note this amount will be deducted from a face value (in fact, it will be added, since this value is negative) and the interest later will be calculated from $A$. Otherwise, i.e. where $I_1 < P_1$, the value $S_1 = P_1 - I_1$ is deducted from the fixed capital and the interest later will be calculated from the value of $A_1 = A - S_1$ (book value of the capital decreases). Denote

$$A_1 = \begin{cases} A - S_1, \text{ if } S_1 \geq 0, \\ A, \text{ if } S_1 < 0. \end{cases}$$

Let us consider one more step:

Let the periods of $t_{21}, t_{22}, \dots t_{2k}$ have been noticed until the end of the second interest payment $P_2$, and during those periods the interest was $r_{21}, r_{22}, \dots r_{2k}$, respectively. For the amount of these periods the following interest accumulated:

$$I_2 = (r_{21}m_{21} + \cdot + r_{2k}m_{2k})A_1.$$

Set $T_2 := t_{21} + \cdots + t_{22} + \cdots + t_{2k} \le T_1$.

If at the second payment moment, when payment $P_2$ is made we have that $I_2 - S_1 > P_2$, the absolute value of the difference $S_2 = P_2 - I_2 + S_1$ is value of the uncovered interest rate after two payment, which is not added to the fixed capital, but when paying the promissory note, this amount is deducted from its face value (in fact, it will be added to it, since this value is negative) and the interest will later be calculated from the fixed capital value found after the last interest payment. If $I_2 - S_1 < P_2$, then the value $S_2 = P_2 - I_2 + S^1$ is deducted from the fixed capital and the interest are later calculated from the value of $A_2 = A - S_2$. The process is repeated depending on the number of times coincide with the interest period of the promissory note.

**Note.** Pay attention to the fact that the method of the demand promissory note payment can be varied and the promissory note payment principles are discussed individually. The above-described method is based on the assumption that if at any moment during the interest payment one pays more than the accumulated interest, then during the following steps a new capital value will be considered as the main one and the interest will be later calculated from a new capital value.

**Example** On April 16 the credit company lent the amount of 2000000 with the interest rate of 12% for the support of the furniture manufacturing companys working capital. The loan was formed by signing a demand promissory note with the variable interest rates. On August 16 the interest rate rose to 14%, and on November 16 to 16%. The company plans to repay the amount of 100000 on June 25, the amount of another 50000 on October 5, and another 120000 on November 30 for the credit company. Determine the amount that the company will have to return on January 15.

We have that A = 2000000. From April 16 to June 25, when the first payment has been made, the interest rate will not change; the number of days is 70. Thus, during this period the interest is $I_1 = A \cdot 0,12 \cdot \frac{70}{365} \approx 46027$. After the depreciation payment of $P_1 = 100,000$, the interest and part of the debt were covered, i.e. $S_1^0 = P_1 - I_1 = 100000 - 46027 = 53973$. Then the remaining debt is $A_1 = A - 53973 = 1946027$.

The next payment $P_2 = 50000$ was made on October 5. However, the change in the interest rate was noticed within this interval, i.e. on August 16 it increased up to 16%. 52 days separate the last payment and this date. Thus, the interest of $\frac{52 \cdot 0.12}{365} A_1 = 33269$, has accumulated during the period, and during the rest of the interval of time until October 5, the period includes 50 days, as a result, the generated interest is $\frac{50 \cdot 0.14}{365} A_1 = 37321$. Consequently, the following amount of interest has accumulated during this period $I_2 = 70590$. Since at this moment the payment of $P_2 = 50000$ was made, the total amount of the uncovered interest is $S_2^0 = 50000 - 70590 = -20590$.

Note that from October 5 to November 16, when the change in the interest rates occurred, 42 days have passed, as a result, during this period the interest of $\frac{42 \cdot 0,14}{365} A_1 = 31350$ accumulated and from November 16 when the rate change occurred to November 30, the time interval is 14 days, and the accumulated interest rate is $\frac{14 \cdot 0,16}{365} A_1 = 11943$. At this moment the third payment $P_3 = 120000$ was made. At that moment, the total amount of the uncovered interest was $I_3 = 20590 + 31350 + 11943 = 63883$. Then $S_3^1 = 120000 - 63883 = 56117$. This payment covers not only the accrued interest, but also a part of the amount as well. The remaining debt is

$$A_3 = A_1 - 56117 = 1889910.$$

Then at the end of the term, on January 15, (46 days from the final payment) the amount of the accumulated income would be

$$I_4 = \frac{46 \cdot 0,16}{365} A_3 = 38109.$$

Thus, the amount which will have to be repaid on the last day will be as follows

$$A_5 = 1889910 + 36109 = 9126019.$$

### 4.4 Forfeiting method

**Definition.** *Forfeiting* is called a method of credit transactions when while purchasing a product the buyer issues a set of promissory notes for the amount, which is equal to the value of a purchased product plus the interest paid for the credit. Terms of a promissory note (the terms of interest and a part of the loan payment) are uniformly distributed in time, usually for the periods of six months.

Loan repayment (redemption of the portfolio of promissory notes) is the task of the debt depreciation. Let us analyze this problem in the case of simple interest. Assume that the loan portfolio is $P$ and it is paid over the even time periods n, where each payment $S_t$ consists of the payment of the part promissory note face value $\frac{P}{n}$ and the interest $I_t = P(1 - \frac{t-1}{n})i, \quad t = 1, \ldots n,$ $i-$ while the actual interest rate (of the period, after which the payment is made) is $i$. Then

$$S_t = \frac{P}{n} + I_t = \frac{P}{n}(1 + i(n - t + 1)).$$

Note the sequence of interest is an arithmetic progression with a denominator of $v = 1$. Then the entire amount of interest is

$$I = \sum_{t=1}^{n} I_t = \frac{Pi}{n} \cdot \sum_{t=1}^{n} (n + 1 - t) = (n + 1)\frac{iP}{2}.$$

We obtain that the value of the promissory note portfolio (loan with interest) is

$$S = \sum_{t=1}^{n} P_t + I_t = P(1 + (n + 1)\frac{i}{2}).$$

Usually the manager of a promissory note portfolio can sell their existing portfolio, if necessary, but at any point in time earlier than the maturity of a portfolio. In order to determine the selling prices of a promissory note portfolio, specific assumptions have to be made. The mentioned assumptions will be called the loan repayment strategies. We will consider several loan repayment strategies.

**Strategy 1**

We make an assumption that any payment to the promissory note portfolio has a certain weight, which is characterized by the repayment period of this payment. In other words, the later the payment is made, the greater its weight is. I.e. the $k-$th payments (end of the $k-$th payment) is characterized by the value of $R_k = \frac{P}{n} \cdot (1 + ki)$. Thus, the larger $k$ is, the larger $R_k$ is, as well. By the way, the amount of all the $R_k$ is equal to the final value of a promissory note portfolio. When following this strategy, let us examine the techniques for the determination of the promissory note portfolio acquisition price. It is interesting that while determining the price of a promissory note in this case the attention has to be paid not only to the market interest rate, but also to the way the future payments are treated. Suppose that $s \geq 0$ is a non-negative integer number, which will be used for the indication for the $s-$th payment period. (The payments shall be made at the end of the term). When setting the price of a promissory note at any time $s$, the letter $A_s$ will mark the value of a promissory note portfolio at the beginning of $s+1-$th period. In the case of the analyzed strategy, by fixing the promissory note price during the interest payment period, we face two problems the buyer and sellers attitude towards the sold portfolio. By setting a price of the sold promissory note portfolio we discount the payments $R_k$. We will note that discounting is performed with the contemporary market rate.

Assume that we set the portfolio price at the point in time $s, 1 \leq s \leq n$. Where n is the total number of payment intervals, $s$ marks the beginning of the $s-$th payment interval, $i$ is the actual interest rate of the portfolio and $d$ is the actual discount, which is determined by the effective market rate of that time $j$. We would like to remind that in the case of the simple interest

$$d = \frac{j}{1+tj}, \quad j = \frac{d}{1-td},$$

$t$ is the number of the discounted payment periods. We remember, that notion *actual* means that we consider interest rate for the interest payment period. The portfolio value at the time of its conclusion, when the market interest rate (money market value) is $j$, is as follows:

$$A_1 = \sum_{t=1}^{n} R_t(1 - td_t),$$

$d_t$ is the discount rate of the payment period at the $t-$th payment. The value $A_0 = P$ means initial price of the portfolio.

The presented equality is written in the form, which indicates both the promissory note and the market interest rates.

Let us make an assumption that the promissory note portfolio is sold (acquired) by making the payment $s$ with an interest (we would like to remind that the total number of payment periods is $n$.) Thus, at the beginning of the $s+1-$th period $n-s$ payments are still left, which will be received together with interest by the promissory note portfolio manager. Then at the end of the $s-$ th payment period the promissory note portfolio acquisition price can be written in the following way:

$$A_s = \frac{P}{n}\Big((n-s) + \sum_{k=s+1}^{n} (1+i(k-s))(1 - \frac{(k-s)j}{1+j(k-s)}), \quad s = 0, \ldots n.$$

We assume that $A_n = 0$.

Suppose that time moment $t$ be arbitrary positive real number. Then

$$A_t = A_{[t]+1} + \frac{P}{n}\Big(i(t-[t])(n-[t]) + (1+i([t]+1-t))(\nu(j;[t]+1-t)),\Big)$$

here $[t]$ is whole part of the real number and

$$\nu(i;s) = \frac{1}{1+is}.$$

**Strategy 2**

We will discuss the promissory note portfolio pricing methodology in the case when promissory notes are repaid by the linear method. In this case at the end of $s-$th $(0 \leq s \leq n)$ payment period the price of a promissory note portfolio is calculated by discounting all the future payments. We get the following formula for the pricing:

$$A_s = \frac{P}{n}\sum_{t=s}^{n}(1+i(n-s))(1-\frac{(n-s)j}{1+j(n-s)}).$$

Having transformed this expression, we obtain that

$$A_s = \frac{P}{n}\Big(n-s+\frac{(i-j)}{n}\sum_{t=s}^{n}\frac{(n-t)}{1+j(n-t)}\Big).$$

It is easy to understand that the price of a promissory note of portfolio depends on the market interest rates j. Let us call the portfolio selling price the face value at any point in time, which coincides with the payment date if during the sale the market interest rate and the promissory note interest rate coincide, i.e. $i = j$. If the market and the portfolio cash values coincide, we can calculate the value As in the following way as well:

$$A_s = S - \sum_{t=1}^{s-1} S_t.$$

Let us consider the case when a promissory note portfolio is sold (purchased) at any point in time moment $t$, which does not necessarily coincide with the payment period. Suppose that $s-1 < t \leq s$.

Then

$$A_t = \frac{P}{n}\Big(n-s+\frac{(i-j)}{n}\sum_{t=s}^{n}\frac{(n-t)}{1+j(n-t)}\Big)+\frac{P}{n}(1+i(s-t))\nu(j;s-t).$$

Pricing portfolio of the promissory note in time moment $t \in [s, s+1]$ can be done using the following idea. At the first we find value of the portfolio at time moment $s$ using above indicated formulas, so we get value $A_s$. The next step- using value of the money $j$ (an actual interest rate) at time moment $t$ we evaluate amount of the interest and add it to value $A_s-$ thus we get flat price of the portfolio

$$FP(t) = A_s(1+j(t-s)).$$

Finally, subtracting from this value interest $AI$ evaluated using portfolio interest rate

$$AI = A_s i(t-s)$$

we obtain the formula for the price of the portfolio at time moment $t:$

$$QP(t) = FP(t) - AI.$$

## 4.5 promissory note discounting in the case of compound interest

The promissory notes drawn up for the period longer than a year are based on compound interest. These promissory notes can be purchased or sold before their maturity. As well as in the case of simple interest, promissory notes can include or exclude interest.

Assume that $S$ is the final value of a promissory note which can be equal to the face value (an interest-excluding promissory note (IE)), or to the accrued interest of a promissory note (an interest-including promissory note). Let $P$ be the face value of a promissory note.

We will structure the formula for the definition of the promissory note pricing at any point in time until its redemption. Suppose that the promissory note is of the $T-$term, the interest rate is P, the interest is converted k times per year. This promissory note was discounted when $t$ period of time remained until its maturity at the interest rate of $r$, which was converted m times a year. Then the purchase price of the promissory note is as follows:

$$B = \frac{P(1+\frac{p}{k})^{kT}}{(1+\frac{r}{m})^{mt}}.$$

If at the time of purchase the promissory note and the market interest rate coincide, then

$$B = P(1+\frac{p}{k})^{kT-mt}.$$

This discounted value is called of the income of a promissory note, or rather, the income of a promissory note at the discounted point in time. As well as earlier, we will call the promissory note discount at the promissory note discounting moment the difference:

$$D = S - P = (1-\nu^n)S.$$

Note that if a promissory note is the interest-excluding one, its face value is discounted, based on the same formula, i.e.

$$B = \frac{P}{(1+\frac{r}{m})^{mt}}.$$

Let us determine the discounted value of the interest-excluding promissory note, the face value of which is 150000 when two years and a quarter remain until its maturity, if the nominal interest rate is 15%, which is compounded monthly.

We have 150000, $i = \frac{15}{12} = 0,0125, \quad n = 27.$

Then

$$P = \frac{150000}{(1+0,0125)^{27}} = 107257.$$

While discounting interest-including promissory notes, the discounted value is calculated from the final value. Thus, while discounting promissory notes of such a type, one has to determine the final value of a promissory note first.

### Tasks for the practice

1. The eight-month promissory note , with the interest of 18.5%, the face value of 5050, was signed on April 16, and was discounted on September 18 with the interest rate of 17.5%. Find the discount size and the promissory note income.

2. Find the value of the promissory note at the moment of signing if the promissory note was signed on March 20 for four months, the cash value at the day of signing was 14.5%, the face value was 20000, and this is the non interest bearing promissory note .

3. The person borrowed 60000 from a bank on April 5 with the interest of 15%. By securing the repayment they signed a promissory note with the declining balance and variable interest to the bank. The person made the first payment of 5000 on of May 10, the second of 8000 on September 15 and the third of 1000 on October 15. On August 1 the interest rate went up to 17%, and on October 1 it rose up to 20%. Determine the size of the final payment which is to be made on December 30. How much of the interest will the person have pay?

4. A.B. borrowed the amount of 305000 from the SEB Bank on February 5, the interest is paid every month, i.e. on the 5th day of each month. By securing the repayment they signed a demand promissory note using the conventional methods. Initially the interest rate was 17%. The first payment of 60000 was returned to A.B. on July 25, while the second of 160000 was made on September 21. On April 16 the interest rate for rose to 20% and on October 1 fell to 16%. Determine the size of the final payment which is to be made on December 13. What is the nominal amount paid by A.B. to the bank?

5. The portfolio of 10 promissory notes, with the face value of 5000, and the interest rate of 14%, is paid using the forfeiting method. Determine the final value of this portfolio, if its maturity is 5 years. Set the value of the portfolio when 2.5 years remain until its maturity, if the interest rate currently is 16% (use both of the strategies).

6. The five-year promissory note with a face value of 1650000, and the interest rate of 15%, which is converted on a monthly basis, is discounted three years and four months after the signing date, with the interest of 8%, which is compounded every six months. Find the promissory note income and the discount size.

7. The seven-year promissory note with the face value of 100000, and the interest rate of 11%, which is compounded monthly, is discounted two years and six months before the maturity with the interest, which are compounded quarterly, and during this period provides the income of 40000. Find the income rate used for the promissory note discounting.

8. Ten years loan of 800000 is formed as promissory note (PM) portfolio. Interest rate of the PM is 10% , loan is repaid by semi-annually payments. Find:

a) future value of the portfolio;

b) value of the portfolio after three years when portfolio was formed if interest rate b1) 10%; $b2$) 16%;

c) value of the portfolio after six years two months and 20 days when portfolio was formed, if value of the money is 7.5% . We assume that year consists from 360 days.

### 4.6 Bonds

**Definition.** A bond is a security which is concluded while borrowing money from an investor for the specified interval of time.

At the end of the set time interval, which is commonly referred to as the bond period, the loaned money with an interest is returned to the direct investor. Bonds are the securities of a fixed income because the investor knows the amount which could be received during the bond management period.

The principles for the determination of the final value of bonds and promissory notes are different in respect of the method, although they can be characterized by a number of similarities. The similarity is that the final value of a bond, as well as that of a promissory note, is formed of two values  a face value and an interest. Bonds are different from promissory notes in that they are generally the long-term securities, while promissory notes are the short-term securities; in addition the methodology of the pricing for the acquisition of those securities and the methods of interest payment is different. Bonds, as well as promissory notes, can be freely bought and sold.

Typically an entity issues bonds in order to borrow money, undertaking to redeem them at a specified time by paying their face value (sometimes the face value with additional conditions) and interest.

Here are the mainly used types of bonds:

1) *Bonds with fixed coupons* (the fixed interest rate). In this case, the investor receives the fixed income when the time interval, called a coupon payment period, is over.

2) *Discounted bonds*. In this case bonds are sold at a lower price than their face value and interest of the bond maturity consists of the difference between the acquisition and the redemption prices. These bonds are sometimes referred to as the bond with the accrued interest.

3) *Bonds with variable interest rate*. In this case the interest is not fixed and is determined at the time of the bond redemption on the basis of the methodology set for this purpose.

4) *A bond is called the convertible* one when at the time of redemption an investor has the possibility to receive the face value or to purchase the shares of the company.

5) Bond nominal value plus interest of which is redeemed by using the method of an ordinary annuity, will be called the *periodic bonds*.

6) A bond is called the *serial* one if this security is redeemed by parts at a variety of points in time during the bond period.

We are going to discuss the general concepts that are used for the bond operations.

1) A face value is the value indicated in the security. A value of a bond (nominal) is obtained by multiplying the written number (nominal) by 10. We will use the concept of a face value, i.e., we will assume that the value of a bond matches its face value.

**Note.** In the following we assume that nominal and face value of the bond equals.

2) A bond rate or coupon rate is the rate of the interest calculated from a face value.

3) Redemption value is the price paid by the bond-issuing entity for the manager of this document.

4) The date of maturity or redemption date is the moment in time when the redemption value of a bond is paid and the interest payments are over (the last coupon).

5) Bond purchase price is a price which is paid for the bonds purchased at that time.

6) A bond coupon is a simple interest paid to the bond manager at the end of the interest period. A coupon is always calculated from the face value of a bond.

**Note.** If the investor has to pay taxes on the received income, these taxes are calculated from the received income, i.e. from a value of the received coupons.

Most of the bonds are redeemed by paying their face value. However, there are certain bonds, which, in order to attract more customers, are sold or purchased not necessarily only by their face value. In this case, it is said that the bonds will be redeemed with a premium. This means that the redeemed value of a bond will be higher than its face value. For example, for the bond, the face value of which is 1000 and which is redeemed with a premium of 110, i.e. the redemption value is equal to 110

Sometimes bonds can be bought with a value smaller than their face value. In this case it is said that a bond is purchased at a discount. The bond with the discount of 90 means that the face value of the bond is sold at the discount of ten percent.

An investor (bond buyer) expects to receive periodic incomes at the period of bond management, while at the time of its maturity they expect the amount indicated in a face value of a bond or the final value which is given at a discount or a premium. Thus, during the redemption the redemption value of a bond may not coincide with its face value.

### 4.7 Bonds with coupons

In order to facilitate the interest payments the majority of bonds have a fixed term interest (coupons) that can be processed in banks at the time of an interest payment period. For example, the 20-year bond, the face value of which is 1000 and the interest rate is 10%, is paid every six months, has 40 correlated coupons, each of which is worth 50. If these coupons are not processed during the bond period the total value of an interest will be equal to the future value of an annuity of $50 \cdot 40_{\rceil 0.05}$. In determining the bond value (sale price) at the moments in time, which are noticed prior to the maturity of the bond, the two interest rates, i.e. the bond interest rate p and the market prevailing rate of compound interest $r$, sometimes referred to as the market cash value which is usually indicated in percentage, are involved. When setting the current price these two rates must be used. The symbol $i = \frac{r}{m}$ marks the interest rate of the payment period, or in other words, the actual interest rate. In determining the value of a bond a compound interest rate of that time is used; thus, coupons are considered as reinvested at the time of their payment with the market interest rate, which was recorded at the time of the bond acquisition.

**Brief note:**

$R$ a coupon rate (face value);

$n$ the number of coupon payment periods until its maturity;

$i$ the effective interest rate per coupon period. This interest rate can be called a bond discount interest rate, since it is used to determine the bond value before its maturity;

$p$ bond interest rate. This interest rate is used to determine the size of a bond coupon;

$k$ the number of coupons paid per year;

$m$ the number of interest conversion periods per year.

### The case of an simple annuity

We will consider a situation when bonds are acquired at the moment of coupon (interest) payments, and in addition, throughout the entire validity period of a bond, the interest rates of a conversion period coincide with the number of coupon payments during the interval of a year.

Suppose that the bond has the value (face value) of $A$ and the interest rate $p$, the interest (coupons) are paid $k$ times per year in the period of $T$ years. We will find the acquisition price of the bond, when n periods remain until the redemption, if the market interest rate is $r$, the interest is compounded $m$ times per year.

**Definition.** A bond price at the time of purchase (sale) will be called a bond flat price and will be marked by the letters $FP$.

Let $A$ be a face value of the bond, $k$ be the number of coupons per year, and $p$ be the bond interest rate. Then a value of the fixed interest or otherwise, a coupon, is determined in the following

$$R = \frac{A \cdot p}{k}.$$

The total coupon number between the periods of acquisition and maturity is $n = T \cdot k$, where $T$ is the time in years until maturity.

Suppose that the bond, the value of which is $A$, with the coupon interest rate of $p$, the coupons are paid $k$ times per year, is redeemed at the value of $K$ (which can be both a premium and a discount). Then the redemption value is $S = \frac{A \cdot K}{100}$. We will find the bond purchase price at the time of the coupon payment, if the market interest rate is $r$, the interest is converted m times per year. We have $S = \frac{A \cdot K}{100}$. The coupon value is

$$R = \frac{A \cdot p}{k}.$$

Then $n = T \cdot k$, $T$ is the time expressed in years until the maturity term and $i = \frac{r}{m}$. The bond purchase price at the time of coupon payment is formed by discounting the bond redemption value and by discounting all the coupons with the discount interest rate found at the time of acquisition. A flat price is formed as follows:

$$FP := P + A_n := S(1+i)^{-n} + Ra_{n\rceil i}.$$

In the case when a bond is redeemed by its face value (au pair), its redemption value is equal to a face value, $S = A$.

**Note.** A coupon value is always calculated from a face value.

The bond-issuing entity usually secures a value (face value) of bonds by their owned property. The bonds, which are not covered by the assets, are known as a debenture. It has been mentioned that bonds can be freely sold or purchased. If an investor purchases a bond, they acquire two debentures at the same time:

1) a redemption value paid at maturity;

2) coupons be paid periodically.

On the other hand, an investor, who has purchases bonds, faces two problems:

1. What is the bond flat price (also known as the acquisition value) when the rate of return (cash value) is known?

2. What is the bond interest rate, if a bond is purchased at a certain price?

The bond, the face value of which is 1000, with the interest rate of 10%, which is converted every six months, is redeemed after four years. What is the bond purchase price now, if the cash value at the time of purchase is 12%, which is converted every six months? The buyer receives two promises:

1. a promises that after four years the amount of 1000 will be paid for the bond;

2. a promise that 50 coupons will be paid every six months.

The focal date, during which the bond is valued, is now. Moreover, the interest rate on the basis of which we will give our evaluation equals 12%, and the interest is converted every six months. We will count the present value of the redemption value of 1000. In this way: $A = 1000, n = 8, i = 0.06$. While discounting this value we obtain the present value $P = 1000(1.06)^{-8} = 627.41$.

The amount of the present values of the interest paid every six months is an ordinary annuity; as a result, when using the usual symbols $R = 50, n = 8, i = 0.06$ we obtain the present value of all the coupons

$$A_n = 50 \cdot a_{8\rceil 0,06} = 50 \cdot 6,21 = 310,5.$$

Consequently, the acquisition price is the sum of the estimated values $FP = P + A_n = 627.41 + 310.5 = 937.91$.

**Note.** Two interest rates are used for the determination of a purchase price

1. A bond rate, which defines the coupon value;
2. A return rate, which is used to define the present value of two liabilities.

The 500000 bond, with the coupons of 10.5% payable every six months, is redeemed after ten years. What is the bond purchase price, if at its selling the interest rate is 9%, which is converted every six months?

We have that

$A = 500000, \quad R = \frac{500000 \cdot 0,105}{2} = 26250, \quad n = 20, \quad i = 0.045.$

Then

$$P = 500000 \cdot (1.045)^{-20} + 262500 \cdot a_{20\rceil 0.045} = 207321 + 341458 = 548779.$$

The face value of the bond is 10000, which is redeemed after 25 years with the redemption value of 106

The bond redemption price is $S = 10000 \cdot 1.06 = 10600$. Kupono vert $R = \frac{10000 \cdot 0.08}{2} = 400, \quad n = 50, \quad i = 0.05$. Then the flat price is

$$P = 10600 \cdot 1.05^{-50} + 400 \cdot a_{50\rceil 0.05} = 924.36 + 7302.37 = 8226.73.$$

## The case of a complex annuity

We have analyzed the issue when an interest payment period and a conversion period of the return rate coincide. When solving a task of the bond value determination we have used the formulas for the calculation of an ordinary annuity. Meanwhile, where the above-mentioned periods are not identical, the formulas of a complex annuity should be used. As well as above, a bond purchase price will be marked by $FP^c$, and a face value by the letter $A$. Let us assume that the bond interest rate is $p$, the coupons are paid $k$ times per year. Then the fixed interest value or, in other words, a coupon value is determined as follows:

$$R = \frac{A \cdot p}{k}.$$

The total number of interest payments is $nT \cdot k$, where $T$ is the time in years until the maturity. Let the cash value or the market interest rate be $r$, and the interest be converted $m$ times per year. Then, the effective interest rate is $i = \frac{r}{m}$. In this case, we have that the lengths of the coupon payment range and the interest conversion interval may not coincide. Hence in this case we have a complex annuity and the efficient rate of the coupon payment range $q$ is determined as follows:

$$q = (1 + i)^{\frac{m}{k}} - 1.$$

Thus, in this case the formula for the calculating of the present bond value will be as follows: 183

1.

$$FP^c = A(1+q)^{-n} + Ra_{n\rceil q}.$$

2. Assume that the bond redemption price is A and the bond is redeemed with the value of K (at a premium or discount); in this case the bond value is determined in the following manner:

$$FP^c = A\frac{K}{100}(1+q)^{-n} + Ra_{n\rceil q}, \quad q = (1+i)^c - 1.$$

We would like to note that the coupon is calculated from the bond value.

Note that $n$ is the number of coupons.

The bonds with the face value of 100000 are redeemed with the value of 103, the interest rate of 11.5%, the interest is paid every six months, are bought eight years until their maturity, the bond return of 10% is compounded quarterly. Determine the bond acquisition cost (Flat price).

We have that

$S = 100000 \cdot 1.03 = 103000, \quad R = 100000 \cdot \frac{0.115}{2} = 5750.$

$n = 16, \; c = \frac{4}{2} = 2, \quad i = \frac{0.1}{4} = 0.025 \quad q = 1.025^2 - 1 = 0.0506.$ Then the present redemption value is

$$S = 103000 \cdot (1.0506)^{-16} = 46738.37.$$

The present value of the semi-annual coupons is

$$A_n^c = 5750a_{16\rceil 0.0506} = 62040.87.$$

Then an acquisition cost is $FP^c = P + A_n^c = 108779.24.$

## 4.8 Bond flat price and market (quoted) price

Earlier we have explored the bond acquisition price at the moment, which coincides with the coupon maturity date. In bond market this acquisition price is called *bond flat price (FP)*. While selling bonds the attention is not usually paid to the fact whether the coupon period is expired at the moment of purchase. In this case there is a problem of how to determine a bond price in the moment of time, which does not coincide with a bond coupon period? We will discuss the methods of setting a flat price of a bond any point in time y. Assume that the bond purchase price is determined for the term $y \in [x, x+u]$, where $x$ and $x+u$ are two successive coupon payment terms, $u$ is the number of the days per coupon period, and $s$ is the number of days between the terms $x$ and $y$.

1. First of all, the bond price $P$ at the last interest term $x$ (before the acquisition) is determined

$$PV = A(1+i)^{-l} + Ra_{l\rceil i},$$

where $i$ is the cash value during the moment of the bond purchase, $l$ is the number of the coupon payment periods from the term x to maturity (see Fig. 4.1).

2. Then the bond acquisition price at the moment $y$ is determined by using the formula for the calculation of the future value in the case of a simple interest, and using the effective (contemporary market) return rate for the number of days $s$, (see Fig. 3.4) between the last

coupon payment term $x$ and the acquisition date y, and by adding the obtained value to the bond value obtained during the term $x$ :

$$FP(i) = P(1 + i\frac{s}{u}).$$

The value $\frac{Pis}{u}$ is usually called the accrued interest. Such the price should be paid for the bond manager by the acquiring entity.

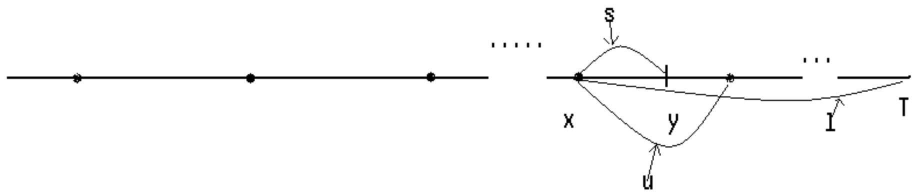



fig 3.4 flat price

Note that in the case of a complex annuity, the actual interest rate $i$ ascribed for the conversion period needs to be changed by the rate of $q = (1+i)^c - 1$, where $c = \frac{m}{k}$ is the number of conversion periods of the market interest rate, while $k$ is the number of the coupon conversions per year.

**Definition.** The actual accrued interest of a bond will be called a percentage share of the face value applied for the number of days s with the bond interest rate. The value of the accrued interest will be marked by the symbol of $AI$.

Suppose that the face value of the bond is $A$. Let the bond interest rate be p, which is paid k times per year. Then the effective interest value is

$$AI = \frac{A \cdot p \cdot s}{ku},$$

$u-$ is the number of days within the coupon period. Note that if$u = s$, in this case the accrued interest coincides with a coupon.

The coupons of the bond with a face value of 500000, plus the interest of 12%, which is compounded every six months, paid on April 1 and October 1, are purchased on August 25 with the value of 104.75. Determine the accrued interest.

We have that the final price is $S = 500000 \cdot 1.0475 = 523750$.

The time interval between April 1 and August 25 is 146 days, the total number of days per conversion period (April 1 to October 1) is 183. Then

$$A = 500000 \qquad i = 0.06; \ \ t = \frac{146}{183}.$$

Thus accrued interest

$$AI = 500000 \cdot (0.06) \cdot \frac{146}{183} = 23934.43.$$

Generally bonds are bought or sold for a variety of purposes. This may be a need to compensate the working capital, to develop activities, with the help of purchasing to make investments in order to receive the profit in the future. Available bonds are offered at a quoted price, which is also sometimes called a market price. A *quoted price* is obtained by subtracting the effective accrued interest (the right to which is acquired by the previous manager of a bond) from the flat price price.

**Definition.** A bond quoted price will be called the difference between the acquisition price and the effective interest. The absolute value of this difference will be marked by the symbol of $QP$. Based on the definition we obtain that:

$$QP = FP - AI, \quad \text{or} \quad FP = QP + AI.$$

In other words, a bond quoted price is a price at the last point of the interest conversion period at the contemporary cash value. In the case of a stable economic situation, bonds are offered at a quoted price plus the added effective interest, this way forming resulting in an acquisition price. Note. A flat price is the price received by the bond manager by selling it. A quoted price is the price at which bonds are traded at the market. If you purchase a bond during the period of interest (coupon) payment, then currently $FP = QP$.

Suppose that the bond with the face value of 1000, the interest of 10% payable every six months, is redeemed on June 1, 2010. What is the bond price on February 18, 2008, if the interest rate of 9% is converted every six months?

The semi-annual coupon is

$$R = 1000\frac{0,1}{2} = 50.$$

Then $n = 5$, the number of days from December 1, 2007 to June 1, 2008, is $u = 183, s = 79$ and $i = 0.045$. The bond price for December 1, 2007, is as follows

$$P = A(1.045)^{-5} + 50a_{5\rceil 0.045} = 1021.95.$$

The flat price for February 18 is as follows

$$FP = 1021.95(1 + 0.045\frac{79}{183}) = 1041.8.$$

Let us calculate the quoted price of the bond. We find the accrued interest, which is

$$AI = R\frac{s}{u} = 50 \cdot \frac{79}{183} = 21.58.$$

Then the quoted price is

$$QP = FP - AI = 1041.8 - 21.58 \approx 1020.$$

**Definition.** The following number is called the bond rate:

$$Q = \frac{QP \cdot 100}{A}.$$

The action of determining the rate is often called a quotation.

Suppose that $Q > 100$, in this case we say that the bond rate is by $Q - 100$ percent higher than the face value and if $Q < 100$, we say that the bond rate is by $100 - Q$ percent smaller than the face value.

In the example above, we see that during the sale of the bond, the rate is by 2

The bond with the face value of 5000 and the coupons of 10.5%, which are paid every six months is redeemed after 6 years and 10 months. Determine: 1) the flat price; 2) the accrued interest rate; 3) the quoted price; 4) the rate.

We assume that at the time of acquisition the cash value is 11.5%, which is converted every six months. The redemption value is A = 5000. The coupon value is $R = 5000 \cdot \frac{0.105}{2} = 262.5$. In addition, $n = 14, i = 0.0575$. The flat price at the point in time, when the interest conversion period coincides with the acquisition period is

$$PV = 5000 \cdot 1.0575^{-14} + 262.5 \cdot a_{14\rceil 0.0575} = 2161.55 + 2591.63 = 4753.18.$$

In order to accumulate the future value after two months ($u = 6, s = 2$) we calculate: $PV = 4753.18$, $i = 0.0575$, $t = \frac{2}{6} = 0.3(3)$. Then the flat price at the set point in time is

$$FP = PV(1 + i \cdot t) = 4753.18(1 + 0.00575 \cdot \frac{1}{3}) = 4844.28.$$

Accrued interest $AI = 5000 \cdot 0.0525 \cdot \frac{1}{3} = 87.50$.

We obtain the quoted price in the following manner $QP = FP - AI = 4844.28 - 87.50 = 4756.78$.

We obtain that at the time of purchase the bond is quoted by approximately 5% less than its face value. During the trading in the market the bond with the value of 95.1 should be offered for sale.

### 4.9 Bond discount and premium

When solving the task of a bond flat price we have faced the problem that a bond quoted price may be both slightly higher and slightly lower than its redemption price (to be more precise, a discounted value of a redemption price with a bond interest rate). If the market price is higher than the redemption price, it is said that the bond is sold with an income and the difference between its flat price and redemption price is referred to as income. Let S be the bond redemption price. Thus, the bond will be sold with an income if

$$PR = QPS > 0,$$

where $QP$ is the quoted price.

If the bond quoted price is lower than the redemption price, we say that the bond is sold at a discount. The difference between the redemption price and the quoted price is called a discount and marked by D. Thus,

$$D = SQP, S > QP.$$

**Note.** We would like to draw the attention of a reader to the fact that the concepts of premium and discounts are applied for a bond manager, and not for an acquiring entity.

Let $p$ be the bond rate and $r$ the market interest rate, $i = \frac{r}{m}$ is the market interest rate. The bond rate is an interest rate payable to the bond manager at the end of each period (in the form of coupons). This rate is stable and does not depend on a market situation. Meanwhile, the market rate is variable and depends on the chosen point in time. Note that these two rates may not necessarily coincide. It is easy to understand, that if the bond rate is lower than the market rate, at that time the bond will be sold for a lower price than its redemption value, in other words, at a discount. Otherwise, on the contrary, it will be sold with the income. If the two rates overlap, the flat price and the face value will coincide (if the bond is redeemed by its face value). All the three situations when the bond is redeemed by its face value can be characterized in the following way:

1) $p = r-$ sold by the face value;

2) $p < r-$ sold at a discount;

3) $p > r-$ sold with an income.

Assume that the interest of the bond, the face value of which is 10000, is 10%, the coupons are paid every six months. Determine the bond flat price ten years before its maturity if the market income is converted every six months and:

a) $r = 10\%$;

b) $r = 12\%$;

c) $r = 8\%$.

We have $A = 10000; R = 100000.05 = 500; n = 20; i = 0.05$.

a) In this case $p = r$ and $i = 0,05$. The quoted (flat) price

$$QP = 10000 \cdot 1.05^{-20} + 500a_{20\rceil 0.05} = 3768.9 + 6231.1 = 10000.$$

Thus, it is sold by au pair.

b) In this case we have that $i = 0.06$; thus $p < r$. Then the quoted (flat) price is

$$QP = 10000 \cdot 1,06^{-20} + 500a_{20\rceil 0.06} = 3118.05 + 5734.96 = 8853.01.$$

In this case the bond is sold at a discount. The discount rate is $D = 100008853.01 = 1146.99$.

c) If $i = 0.04$, then$p > r$. The quoted price is

$$QP = 10000 \cdot 1,04^{-20} + 500a_{20\rceil 0,04} = 4563,87 + 6795,16 = 11359,03.$$

In this case, the price is higher than the face value. Thus, the value includes the premium. The income level is $11359,03 - 10000 = 1359,03$.

As we can see, the premium (discount) is received by analyzing the difference between $\frac{p}{k}$ and $i$. The income is received if $\frac{p}{k} > i$. The income is calculated on the basis of the difference $\frac{p}{k}i$. Let us analyze this fact in further details. Let us assume that the bond face value is $A$, while the redemption price is $S$. Let $k$ be the number of the payment periods per year. Then the periodic bond payments (coupons) are $A \cdot \frac{p}{k}$. If the market interest rate is $r$ and it is converted $m = k$ (in the case of an ordinary annuity) times per year, when at the sale of the bond it is necessary to be determine a share of the redemption value ascribed to the coupon period with the market interest rate. This share equals to $S \cdot \frac{r}{k}$. Let $\Delta$ be the income of a bond interest period. Then, this income can be expressed as the following equation:

$$\Delta = \frac{A \cdot p - S \cdot r}{k}.$$

This income can be both positive and negative. Suppose that this income is positive. Then the amount of the income discounted during all the periods, when $n$ periods remain until the maturity are a general income, or in other words, a premium:

$$PR_n = \Delta a_{n\rceil i} = (A \cdot p - S \cdot r)\frac{a_{n\rceil i}}{k}.$$

Based on what has been said above we obtain that a discount is calculated in the same way as premium, only with the opposite sign:

$$D_n = -\Delta \frac{a_{n\rceil i}}{k} = -PR_n,$$

where $k-$ k is the number of interest payments per year, and $n$ is the number of discounting periods. We will discuss the analogue task in the case of a complex of annuity, i.e. when the interest conversion period does not coincide with the coupon period. Suppose that the coupon rate is $p$, the coupons are paid k times per year, and the effective interest rate is $r$, the interest is converted m times per year. First of all, we determine the efficient rate of the payment period

$$q = \left(1 + \frac{r}{m}\right)^{\frac{m}{k}} - 1.$$

Then the value A used in the formulas for the premium and discount calculation is calculated in the following way:

$$\Delta = A \cdot \frac{p}{k} - S \cdot q.$$

The bond, the face value of which is 25000, was redeemed with the premium of 1004 on July 1, 2001, with the quarter coupons of 11%, purchased on May 20, 2012, when the rate was 10%, compounded quarterly. Set:

1) the income or discount;
2) the flat price;
3) the quoted price?

We have $S = 25000i, 04 = 26000; i = 0.025; p = 0.0275$.

The interest maturity terms are on October 1, January 1, April 1 and July 1. The time interval between the bond acquisition and its redemption is 9 years and 3 months. Thus,$n = 37$. Then the income on April 1, 2012 is:

$$PR = (25000 \cdot 0.0275 - 26000 \cdot 0.025)a_{37\rceil 0.025} = 898.40.$$

The flat price for April 1, 2012 is $P_1 = 26000 + 898.4 = 26898.4$. The time period between April 1 and May 20 includes 49 days; the period of interest between April 1 and July 1 covers 9 days. Thus,

$P_1 = 26898.4, \quad i = 0.025, \quad t = \frac{49}{91}$. The accumulated value for May 20, 2012, is as follows:

$$FP = 26898.40(1 + 0.025\frac{49}{91}) = 27260.49.$$

The accrued interest on May 20 is $AI = 25000 \cdot 0.0275\frac{49}{91} = 370.19$. The quoted price is equal to $QP = 27260.49 - 370.19 = 26890.3$. Thus, on May 20, 2012:

1) the income is 26890.30- 26000=890.3;
2) the flat price is 27260.49;
3) the quoted price is 26890.30.

### 4.10 Periodic, serial and discounted bonds

**Definition.** A bond the face value plus interest of which is redeemed using the method of an ordinary annuity will be called a periodic bond.

Let us consider the periodic bond with the face value $A$, the coupon rate $p$, the coupons are paid $k$ times per year, and the bond is redeemed during n periods. Then the coupon is determined in the following manner: $R = \frac{A}{a_{n\rceil \frac{p}{k}}}$.

If the bond is sold when s coupon payments remain until its maturity at the market rate $r$, in this case the purchase price is determined by the correlation of $QP = R \cdot a_{n\rceil \frac{p}{k}}.$ , while the bond premium (discount) is determined from the correlation of

$$PR = R(a_{s\rceil \frac{p}{m}} - a_{s\rceil \frac{r}{m}}).$$

Assume that the periodic bond of 12% with the face value of 4000000 is redeemed by the even semi-annual payments over ten years. These payments include the face value and interest.

1) Determine the flat price, if at that time the return is equal to 11%, the interest is converted every six months;

2) It is known that after four years the bond will be sold, at that time the cash value will be 13%, the interest is converted every six months. What is the bond sale price?

3) Is the bond sold with a premium or at a discount?

1) We set the size of the half-year payments. We have that $A_n = 4000000, i = 0.06; n = 20$. Solving the equation we get

$$4000000 = Ra_{20\rceil 0.06}$$

Thus we obtain the six-month payments $R = 348738$.

We have that the bond manager will receive the amount of 348738 every six months.

Set the bond flat price. Since $R = 348738; \quad i = 0.055, \quad n = 20$, Hence the flat price at the return rate of 11 % is 4167552.

2) The quoted price after four years is the present value of the remaining payments with a new rate of return. We have that $R = 348738; i = 0.065, n = 12$. Then $A_n = Ra_{12\rceil 0.065} = 2845258$. Consequently, the quoted price after four years at the return of 13% is 2845258.

We obtain the income or discount after the bond sale by calculating the difference between the bond book value and its sale price. Book value is understood as the value received at the time of maturity by discounting all the remaining payments at the original rate of return. Therefore, on the basis of the fact that$R = 348738; i = 0.055, n = 12$we obtain that $A_n = Ra_{12\rceil 0.055} = 3005605$. We obtain that the bonds will be sold at the following discount: $30056052845258 = 160347$.

**Definition.** A bond is called the serial one if its face value or parts of its face value can be redeemed at any time. Assume that the bond the face value of which is $A$, the coupon rate is $p$, the coupons are paid $k$ times per year, is redeemed by the series of $A_1, \ldots, A_t, \quad A = A_1 + \cdots + A_t$.

We will determine a price of the bond at the moment of acquisition when the interest rate is $r$, the number of the paid coupons of the serial members is$n_1, ..., n_t$, respectively. Then the bond flat price is as follows:

$$QP = A_1(1 + \frac{r}{k})^{-n_1} + R_1 a_{n_1\rceil \frac{r}{k}} + \cdots + A_t(1 + \frac{r}{k})^{-n_t} + R_t a_{n_t\rceil \frac{r}{k}},$$

where $R_i = A_i \frac{p}{k}$.

The serial bond with the face value of 2000000, the interest rate of 10%, the interest is paid every six months, is redeemed by two payments: the payment of 1200000 after twelve years and the payment of 500000 after fifteen years. Set the flat value of this bond at the moment of its issue when the annual return of that time is 12%, the money is converted every six months? The flat value of the serial bond at the time of its issue is equivalent to two flat prices, i.e. 1200000 after twelve years and 500000 after fifteen years (at the beginning of the year).

Let us establish the present value $P_1$ of the price 1200000: We have that $A_1 = 1200000, R = A_1 \cdot 0.05 = 60000; n = 24; i = 0,06$. Then $P_1 = 1200000(1.06)^{-24} + 60000a_{24\rceil 0.06} = 1049396$.

We perform the same action with the second price as well:

$$A_2 = 800000, \quad R = A_2 \cdot 0.05 = 40000; \quad n = 30; \quad i = 0.06.$$

In addition $P_2 = 800000(1.06)^{-30} + 40000a_{30\rceil 0.06} = 689681$. Then, the total flat price is the sum of these present values: $QP = P_1 + P_2 = 1739277$.

Similarly to the short-term securities, promissory notes, as well as bonds, can include or exclude coupons. In this case it is considered that the interest is constantly incorporated into the face value of a bond. If the bond excludes the coupons this bonds are called *discounted bonds*. Its present value is obtained by simply discounting the face value by the current market interest rate.

**Definition.** A securities will be called a discounted bond, if the bond is redeemed by its face value, and its price is determined by discounting the face value of the contemporary (agreed) value of money at the time of its purchase.

Just like above, $A$ is the face value of the bond, $i$ is the (compounding period) rate, $n$ is the number of periods with a discounted value; the present value of a bond (a quoted price) is calculated in the following manner:

$$QP = A(1+i)^{-n}.$$

It is obvious that if $i \to 0$, then $QP \to A$.

## Tasks for the Practice

1. The bond with the face value of 10000, with the interest of 9.5%, which are converted every six months, is redeemed according to its face value after 10 years. What is the bond price at the time of the original purchase, provided that:

a) the cash value is 15% converted every six months?

b) the cash value is 5% converted every month?

2. The bond with the face value of 5000 and the interest of 10% converted on a quarterly basis is redeemed with the value of 105. What is the bond purchase price if at the time of its acquisition the interest is 15.15% which is converted every quarter, and the moment of purchase is five years before the maturity.

3. The bond with the face value of 25000 and the coupon interest rate of 13

4. The bond with the value of 10000, and the interest of 9.5%, which is converted every six months, is redeemed according to its face value on March 1, 2014, is purchased on September 19, 2012. At that time the annual interest rate is 12% which is converted every six months. What is the flat price?

5. The bond with the face value of 70000, the coupon interest rate of 1%, the coupons are paid semi-annually, bought with the value of 112, is purchased four years prior to the redemption when the cash value is 12%, which is converted on a quarterly basis. Set the following characteristics of the bond:

a) the premium (discount);

b) specify the flat price;

c) the quoted price;

d) the accrued interest;

e) the effective accrued interest.

6.The periodic bond with the face value of 10000, and the interest of 12%, which is paid on a quarterly basis, shall be paid within 10 years by the equal payments made on a quarterly basis.

a) Determine the bond flat price if at the time of its acquisition the interest rate is 16%, the interest is converted every quarter.

b) Determine the value of the bond after nine years, when the cash value is 10%, the interest is converted every quarter.

c) Determine whether the person will have an income or loss if they sell the bond nine years after the acquisition date, in the case the interest of 9% is converted on a quarterly basis.

7. The serial bond with the face value of 50000, and the coupon rate of 10.5%, which are payable every six months, is redeemed by the payment of 20000 after four years and the remaining amount of 30000 after ten years. Determine the acquisition value of this bond at the time of its issue, if currently the return is 7%, the money is converted every six months.

## 4.11 Return rates of the bond

While analyzing the issue of bond trading, we have solved the following task: what is the bond selling price if the bond is sold before its maturity? As we have seen, this price depends on the market value of money (interest rates) at the time. At the increasing interest rates the value of bonds declines, and vice versa, after the fall in the market interest rates the value of bonds rises.

**Definition.** The bond yield to maturity is called an average return rate during the bond management period. The yield to maturity is usually referred to as the bond return rate or the yield rate.

Let us note that the bond yield to maturity is known if the bond is sold at a price which coincides with the bond quoted price formed according to the contemporary cash value. It is interesting that the subject often purchases a bond under the offered price, which is not always based on the prevailing market interest rate. In this case, the bond yields can be calculated based on the data available at the time of purchase, such as the knowledge of the quoted price $QP$ and the bond parameters (interest rate and face value).

The bond yield measure at the point of its acquisition is the present bond return rate $i_0$, on the basis of which the purchase price is determined given a face value of the bond or a redemption price and interest rate. We will discuss the methods for the establishment of the bond yield to maturity.

Suppose that we know the quoted price of the bond $QP$ when n periods remain until its maturity, and the coupon size is $R$. Based on the calculation formula for the bond quoted value we have that

$$QP = A \cdot (1+i)^{-n} + Ra_{n\rceil i}.$$

Let us note:

$$f(i) = A \cdot (1+i)^{-n} + Ra_{n\rceil i} - QP.$$

Then the zero of this function, i.e. $i_0$, $f(i_0) = 0$ will be the searched return rate (a bond yield rate to maturity), which can be obtained on the basis of, for example, Newtons method or other methods of approximation, according to which the zeros of the function can be determined.

Often bonds are traded at the market by presenting their quoted price, which is determined using a discount or a premium of the bond face value. In this case the need to determine the bond return rate, which is usually not indicated, arises. In this case discussions are simple, if a bond is purchased more expensive than its coupon rate, apparently a certain market situation is formed that the fall in interest rates will be observed in the future, and currently the bond is underestimated. Consequently, it appears that the correction of the real bond returns according to the market behaviour, which will indicate the bond rate of return, is necessary. Often bonds are sold at the quoted price $QP$, which hides the rates of return. The method for the determination of an approximate rate of return to maturity will be presented. In practice this method is used quite often. The method of averages for the determination of a yield rate. Let us note: $S$ a redemption price; $A$ a face value; $K$ the number of coupon payments per year; $PR$ a premium, D a discount of the term to maturity; $\sum I$ the coupon income of the bond for the period to maturity ($\sum I = nR$) $n$ the total number of coupon payments; $\overline{P}$ the size of an average income per the coupon period; $V$ an average bond price to maturity. By using these symbols we obtain that an average income for the coupon period $\overline{P}$ is

$$\overline{P} = \frac{\sum I \pm PR}{n} = R - \frac{PR}{n} -$$

**Note.** We would like to note that $D = PR$. When calculating an average income in the case a bond is purchased at a premium, the value is

$$\overline{P} = R - \frac{PR}{n},$$

if a bond is purchased at a discount, when

$$\overline{P} = R + \frac{D}{n}.$$

An average bond flat price is

$$V = \frac{1}{2}(QP + S).$$

Then, the bond rate of return (per a coupon period) is as follows:

$$\overline{p} = \frac{\overline{P}}{V} = 2 \cdot \left(\frac{A \cdot \frac{p}{k} - \frac{PR}{n}}{QP + S}\right) = 2 \cdot \left(\frac{A \cdot \frac{p}{k} - \frac{QP-S}{n}}{QP + S}\right).$$

Hence the bond rate of return is

$$r_0 = k \cdot \overline{p}.$$

**Definition.** Bond equivalent yield to maturity is called a simple annual interest rate r determined in the following way: $r = k \cdot \overline{p}.$

In other words, bond equivalent yield is an annual rate of return to maturity. In the present case, this is the rate $r_0$.

Bond effective annual yield $e$ is called the bond coupon rate equivalent to the bond coupon yield rate which is converted k times per year:

$$e = (1 + p)^k - 1.$$

The 10-year bond with the face value of 25000 and the interest rate of 11.5

We find that the quoted price is $QP = 25000 \cdot 1.035 = 25875$ and $A = 25000$.

Then

$$V = 0.5(QP + A) = 25437.50.$$

The semi-annual interest is $25000(\frac{0.115}{2}) = 1437.50$. we have that $n = 20$, $\sum I = 20 \cdot 1437.50 = 28750$. Then the premium is $PR = 25875 - 25000 = 875$. The average income for the payment period is

$$\overline{P} = \frac{28750 - 875}{20} = 1393.75.$$

Then

$$\overline{p} = \frac{1393.75}{25437.5} = 0.05479$$

and the approximate nominal rate of return is $r_0 = 2 \cdot \overline{p} = 0.1096$.

The bond with the face value of 8000 and the 10% semi-annual coupons, redeemed after 17 years with the value of 105, at the moment is acquired with the value of 971. Determine the approximate bond rate of return. We have that the quoted price is $QP = 8000 \cdot 0.97375 = 7790$, in addition,$S = 8000 \cdot 1.05 = 8400$. Then an average price is

$$V = 0,5(QP + S) = 8095.$$

The semi-annual interest is$8000 \cdot \frac{0.1}{2} = 400$. We have that $n = 34, \sum I = 34 \cdot 400 = 13600$. The bond discount is $D = 8400 - 7790 = 610$. Then the average income of the interest period is

$$\overline{p} = \frac{417.94}{8095} = 0.05163 = 0.0516.$$

Then the approximate interest rate per interest period is

$$\overline{p} = \frac{417.94}{8095} = 0.05163 = 0.0516.$$

We obtain that the bond equivalent rate of return is two times larger: $r_0 = 2 \cdot \overline{p} = 0.1032$. Meanwhile, the effective annual yield is $e = 1.0516^2 - 1 = 0.1059$.

Let us establish the rate of return of the bond with the face value of 5000 and the semi-annual coupons of 10

While calculating the approximate rate of return of the bond, the value of which is determined at the period of time between the terms of interest payment, we will calculate the number of periods n rounded to the rational number. We can see that December 5 and the next coupon payment period, which is on December 15, is separated by a 13-day period. Then the total coupon payment period is $n = 23 + \frac{13}{183} = 23.07$.

We determine the quoted price

$$QP = 5000 \cdot 1.0375 = 5187.50, \quad A = 5000.$$

We calculate the average price of the bond:

$$V = 0.5(QP + A) = 5093.75.$$

The coupon value is $5000(\frac{0.1}{2}) = 250$. We have that $n = 23.07$ $TI = 23.07 \cdot 250 = 5767.5$. The bond premium is $E_n = 5187.5 - 5000 = 187.5$. Then the average income for the interest period is

$$\overline{P} = \frac{5750 - 187.5}{23.07} = 240.27.$$

The approximate rate of the interest period equals to:

$$\overline{p} = \frac{240.27}{5093,75} = 0.0472.$$

Then the average nominal bond rate of return is $\imath = 2 \cdot \overline{p} = 0.094$.

Then the average income for the interest period is

**Definition.** The bond current yield will be called the ratio of the annual income (coupon payments) to the current price of this bond. Thus:

$$r_0 = \frac{A \cdot p}{QP}.$$

Summing up what has been said above, it should be noted that the bond income received in each period can be reinvested; as a result, generally, the yield rate to maturity is not always an appropriate measure of profitability. In this case, another method to determine the profitability of a bond is used,; this is the concept of an internal rate of return.

**Definition.** A bond internal rate of return is called an nominal interest rate of return on bonds over the life period of a bond considering that every coupon payment is reinvested with the bond yield to maturity.

Assume the bond with the face value of 1000 and the coupon interest rate of 10 percent, which is paid every six months. It was sold three years to maturity. What is the bond internal rate of return?

We have that the coupons are $R = 50$. The first coupon received after six months can be reinvested within five periods, the second within four, etc. The total amount will be equal to:

$$50a_{6\rceil 0.05} + 1000 \approx 1253.$$

We find the internal rate of return iv via the following equation:

$$(i + i_v)^3 = 1.253.$$

Then $i_v$ 0.078.

**Definition.** The effective (annual) interest rate, which is equivalent to the rate of return according to which the coupons have been invested within the same period, will be called the realized return.

**Note.** The realized return can only be determined at the end of an investment period. It is only useful for the determination of the effectiveness of an investment, or for the comparison with other results. In addition, this method is used when modelling interest rates.

Assume that the coupons of the bond with the face value of 1000 value and the interest rate of 10 percent are payable every six months. It was sold three years to maturity. The manager has the option to reinvest the coupons with the interest rate of 12%. What is the realized return?

We have that the coupons are $R = 50$. The first coupon which was received after six months can be reinvested within five periods, the second within four, etc. The total amount will be equal to:

$$50(1.06^5 + 1.06^4 + 1.06^3 + 1.06^2 + 1.06 + 1) + 1000 \approx 1349.$$

We find the realized return via the following equation as well:

$$(i+i_r)^3 \cdot 1000 = i349.$$

Then $i_r = 0.105$.

In literature the bond rate of return to maturity is often considered to be an internal bond rate of return.

### 4.12 Average maturity value

It is easy to understand that all the coupons payable at different points in time affect the total profit differently. Consequently, each of the coupons can be linked to a specific weighting value, which defines the influence of each coupon on the price of a purchased bond. The similar actions have been taken while calculating the acquisition price for a bond portfolio.

Let, as above, $A$ be the face value of a bond (either at a premium or at a discount); $i$ an actual rate (of a conversion and payment periods) $n$ the number of discounting periods (or the number of interest conversion periods), $QP_k$ a quoted price at the beginning of $k-$th outstanding periods before maturity. Let us determine the income weight at the beginning of the k-th payment period in the following way:

$$s_k = \frac{R}{QP(1+i)^k}, \quad k = 1, \dots, n-1; \;\; s_n = \frac{R+A}{QP(1+i)^n}.$$

It is easy to understand that

$$\sum_{i=1}^{n} s_i = 1.$$

Therefore, any value can be regarded as the percentage influence of the k-th income on the general income. By definition $A = R \cdot p$ and the formula

$$QP_k = \frac{A}{(1+i)^k} + A \cdot p(1+i)a_{k\rceil i} = \frac{A}{(1+i)^{k-1}}\Big(\frac{1}{1+i} + \frac{p((1+i)^k-1)}{i}\Big)$$

**Definition.** The sum

$$T = \sum_{i=1}^{n} i \cdot s_i$$

we call an average length of a bond or shortly bond maturity.

Note that the average duration of coupon-zero bonds are the same as the time period during which bonds are redeemed. For example, if the zero-coupon bond is valid 5 years, its maturity will also be equal to 5. Let us examine this situation. Since this is a zero-coupon bond and, if we assume, that it is redeemed after n periods, while the payment period is single, then based on the definition of an average duration, we obtain that

$$T = n\frac{1}{QP} \cdot \frac{A}{(1+i)^n}.$$

Having noticed that $\frac{A}{(1+i)^n} = QP$ we obtain that $T = n$. Assume that the interest $i = 0.12$ is compounded quarterly, and the five-year bond with the face value of 2500 is without interest. Let us find the average maturity of this bond. We have that

$$T = 20\frac{1}{QP} \cdot \frac{2500}{(1.12)^{20}} = 20.$$

It follows that $T = 20$or counting in years  5 year. Meanwhile, an average maturity of the coupon-including bonds is shorter than the redemption period. We present the counting formulas for an average maturity of some periodic payments without any proofs. 1. The coupon-including bond when the duration of the effective market rate i and the bond rate p is equal to:

$$T = \frac{1+i}{i} - \frac{1+i+n(p-i)}{p\big((1+i)^n - 1\big)+i}.$$

2. The average maturity of the bond with coupons when the effective market rate is i and the bond rate p coincide, i.e. the bond is sold under its face value, is equal to:

$$T = \frac{1+i}{i} - \frac{1}{i(1+i)^{n-1}}.$$

3. The average maturity of the infinite-term bond with coupons when the effective market rate i and coupons are constant is equal to:

$$T = \frac{1+i}{i}.$$

4. The average maturity of the periodic payments (annuity), when the effective market rate is i, the number of payments is n is equal to:

$$T = \frac{1+i}{i} - \frac{n}{(1+i)^n - 1}.$$

Let us find the maturity of the bond with coupons, which are payable every six months, and the coupon rate is 10%, provided the bond is purchased 20 years to its maturity, when the effective market interest rate is 8% at maturity; the interest is converted every six months. We have that

$$T = \frac{1.04}{0.04} - \frac{1.04 + 40(0.05 - 0.04)}{0.05(1.04^{40} - 1) + 0.04} \approx 19.8.$$

Let us examine the evolution of the bond price according to the interest rate changes. We have that

$$QP(i) = \sum_{j=1}^{n} \frac{Ap}{(1+i)^j} + \frac{S}{(1+i)^n},$$

where $A$ is a bond face value, $S$ is either a face value or a redemption value at a premium or discount, $i$ and $p$ are market and bond interest rates (for the coupon period), respectively. Having calculated the derivative of the function $QP(i)$ we obtain that

$$\frac{\mathrm{d}QP(i)}{\mathrm{d}i} = -\Big(\sum_{j=1}^{n} \frac{Apj}{(1+i)^{j+1}} + \frac{nS}{(1+i)^{n+1}}\Big).$$

Having divided both sides of this equality by QP(i) we obtain

$$\frac{(QP(i))'}{QP(i)\mathrm{d}i} = -\frac{1}{1+i}\Big(\sum_{j=1}^{n} \frac{Apj}{QP(i)(1+i)^{j+1}} + \frac{nS}{QP(i)(1+i)^{n+1}}\Big) = -\frac{1}{1+i}T.$$

Based on the last equality we can determine for how many percentage points a bond quoted price will change at the rise of an interest rates by one percentage point when the change is made at the point $i$. On the other hand,

$$\frac{\mathrm{d}QP(i)}{\mathrm{d}i} = -\frac{T \cdot QP(i)}{1+i}.$$

This phenomenon is the calculation formula for the bond marginal price at the interest rate $i$. It has the following meaning: how much a bond price changes after the change in the interest rates by one percentage point.

The bond with the face value of 5000 and the coupons of 10.5% payable every six months is redeemed after 6 years and 10 months. Let us establish for how many percent will the bond price change if the cash value is 11.5%, which is converted every six months, changes by one percentage point?

We set the quoted price. It equals to $QP = 4756.78$. We have that $n = 13.66, i = 0.0575, p = 0.0525$. We obtain the maturity of this bond.

$$T = \frac{1.0575}{0.0575} - \frac{1.0575 - 13.66 \cdot 0.005}{0.0525\big((1.0575)^{13.66} - 1\big) + 0.0575} \approx 10.$$

When calculating the maturity in years we obtain that it is equal to 5. Then

$$\frac{\mathrm{d}QP(0.0575)}{QP(0.0575)} = -10\frac{1}{1.0575} = -9.4.$$

We obtain that if the effective rate is increased by one percent point (annual 2), when the bond price will decline by approximately 9.4 percentage points.

### Tasks for the Practice

1. The portfolio of the promissory notes with the face value of 2000000 is paid using the forfeiting method during the period of five years. The portfolio simple interest rate is 6%. Set the value for this portfolio:

a) during the time of its acquisition;

b) three years after the acquisition.

2. The bond with the face value of 10000 and the interest of 11%, which is converted every six months is redeemed on August 1, 2014, at the value of 104, and sold on June, 17 2009, at the value of 95.2. Find the rate of bond return to maturity.

3. The 3-year bond with the face value of 5000 and the semi-annual coupons of 10% is sold for 4700. During the first year the market interest rate was 10

4. The 10-year bond with the face value of 5000 and the semi-annual coupons of 10% is sold for 4000.

1) Calculate the current rate of yield;

2) Calculate the approximate rate of return to maturity;

3) The realized return if the bond is sold 4 years prior to its maturity at the interest rate of 8%.

4) What is the internal rate of return of the bond to maturity?

### Self-control exercises

1. Assume that the five-year demand promissory note with the face value of 10000 is paid after two years by paying 20% of the accumulated amount, after 3 years by paying 1000, after four years by paying 2000, and in the last year by paying the entire loan with interest. The interest had changed in the following way: in the first year the rate was 10%, in the second year the rate dropped to 6Determine how much of the interest is paid for the loan if:

1) At the changed interest rates or after the covering payment, the interest is calculated from the new book value with interest;

2) At the changed interest rates or after the covering payment, the interest is calculated only from the book value of a new loan.

**Ans** : 1)$I = 4214; 2)4115$.

2. The portfolio of 100 promissory notes, the face value of which is 2000 and the interest rate is 10%, is paid by the forfeiting method. Determine the final value of this portfolio, if the maturity is after 4 years. What is the portfolio value after 2.5 years? Calculate the value by using both price calculation strategies.

3. The portfolio of 50 promissory notes, the face value of which is 5000 and the interest rate is 14

Determine the portfolio value after 2.5 years to maturity, if the current interest rate is 16%. How much one would have to pay for this portfolio if it is acquired 1 year to maturity at

a. the interest rate of 8%;

b. the interest rate of 18%;

c. the interest rate of 14%.

4. The bond with the face value of 50000 and the interest of 11.5%, which is converted every six months, is redeemed by its face value after 12 years. What is the bond price at the time of the original purchase, provided that:

(a) the cash value at that time is 10,5%, which is converted every six months?

(b) the cash value at that time is 13%, which is converted every six months?

**Ans** : (a) 53367, (b) 45503.5

5. The four bonds, with the face value of 50000 and the coupon rate of 13

(a) by their face value;

(b) at a premium of 107.

**Ans** : (a) 9295 (premium); 209295 (b) 1487 (premium); 215487.

6. The four bonds, with the face value of 10000 and the interest rate of 16

(a) whether the bond is purchased with an income or at a discount;

(b) the flat price;

(c) the fixed price.

**Ans** : (a) 225.87 (income) (b) 43556.85 (c) 42624.98

7. The bond with the face value of 2500000 and the interest rate of 13%, which is converted every six months, is redeemed on June 15, 2002, at the premium of 107. On May 9, 2002, when the interest rates was 14.5%, converted every six months, a person purchased this bond. Find the flat price.

**Ans** : 107117

8. The bond with the face value of 50000 and the interest of 11%, which is converted every six months, is redeemed by its face value on April 15, 1998. On June 25, 1991, a person purchased this bond at a discount of 92. Find the approximate bond rate of return.

**Ans** : 12. 56%

9. The bond with the face value of 15000 and the interest of 11%, which is converted every six months, was purchased at the discount of 78.1 sixteen years prior to maturity. What is the rate of return?

**Ans** : 12.712%

10. The bond with the face value of 10000 and the interest of 15

**Ans** : 14.990%

11. The periodic bond with the face value of 50000 and the coupons of 14.5%, payable on a quarterly basis is redeemed within 12 years by making equal payments every quarter.

(a) Determine the bond flat price if at the time of its acquisition the interest rate is 16%, the interest is converted on a quarterly basis.

(b) Determine the bond value after nine years.

(c) Determine whether after the sale of the owned bond nine years after the acquisition date, a person will encounter profit or loss if the interest is 17%, which is converted on a quarterly basis?

**Ans** : (a) 46906, 73 (b) 20770, 01 (c) 298, 05 (loss) 12. The serial bond with the face value of 250000 and the interest rate of 15

**Ans** : 256678. 26

13. The bond with the face value of 50000 and the coupons of 14.5%, payable every six months is redeemed on August 1, 2014, was purchased on March 5, 2003, with the value of 95 2. Find the rate of return of this bond.

**Ans** : 15.347%

14. The periodic bond with the face value of 1000000 and the interest rate of 12% payable every quarter is redeemed by the even quarterly payments over 20 years.

a) Determine the bond flat price, if the cash value is 14%, which is converted on a quarterly basis.

b) What is the book value of the bond after seven years?

c) Determine whether after the sale of the owned bond after seven years at the interest 15%, which are converted on a quarterly basis, a person will encounter a loss or a profit?

**Ans** : (a) 885703 (b) 787925 (c) 35127 (loss)

15. The 3-year bond with the face value of 5000 and the semi-annual coupons of 10% is sold for 4700. During the first year the market interest rates was 10

16. The 10-year bond with the face value of 5000 and the semi-annual coupons of 10% is sold for 4000.

1) Calculate the effective yield.

2) Calculate the realized return if the bond is sold 4 years before its maturity at the interest rate of 8 %.

17. Suppose that an average maturity of the 10-year bond with the semi-annual coupons of 12% is 9.5 years. How much will the bond price change if the profit rate will change by 0.75 percentage point?

## Homeworks exercise

1. Assume that the ten-year demand promissory note with the face value of 7000 is paid after three years by paying 10% of the accumulated amount, after 5 years by paying 2000, after four years by paying 600, and in the last year by paying the entire loan with interest. The interest had changed in the following way: in the first year the rate was 4%, in the second year the rate dropped to 8%, in the third year remained 10%, in the fourth year it rose to 7% and in the fifth climbed to 12%. Determine how much of the interest is paid for the loan if:

1) At the changed interest rates or after the covering payment, the interest is calculated from the new book value with interest;

2) At the changed interest rates or after the covering payment, the interest is calculated only from the book value of a new loan.

2. The portfolio of 60 promissory notes, the face value of which is 2500 and the interest rate is 8% , is paid by the forfeiting method. Determine the final value of this portfolio, if the maturity is after 6 years. What is the portfolio value after

a) 2.5 years;

b) 3 years and nine months.

Calculate the value by using both price calculation strategies if value of money is 12% .

3. The portfolio of 50 promissory notes, the face value of which is 5000 and the interest rate is 14%, is paid by the forfeiting method. Determine the final value of this portfolio, if the maturity is after 10 years.

Determine the portfolio value after 4.7 years to maturity, if the current interest rate is 16%. How much one would have to pay for this portfolio if it is acquired 1 year to maturity at

a. the interest rate of 8%;

b. the interest rate of 18%;

c. the interest rate of 14%.

4. The bond with the face value of 1000 and the interest of 11.5%, which is converted every six months, is redeemed by its face value after 12 years. What is the bond price at the time of the original purchase, provided that:

(a) the cash value at that time is 8%, which is converted every six months?

(b) the cash value at that time is 12%, which is converted every six months?

5. The four bonds, with the face value of 60000 and the coupon rate of 9

(a) by au pair;

(b) at a premium of 107.

6. The bond with the face value of 25000 and the interest rate of 13%, which is converted every six months, is redeemed on June 15, 2012, at the premium of 107. On May 9, 2008, when the interest rates was 14.5%, converted every six months, a person purchased this bond. Find the market price.

7. The bond with the face value of 50000 and the interest of 10%, which is converted every six months, is redeemed by its face value on April 15, 2008. On June 25, 2005, a person purchased this bond at a discount of 98. Find the approximate bond rate of return.

**Ans** : 12. 56%

8. The bond with the face value of 5000 and the interest of 6%, which is converted every quarter, was purchased at the discount of 84 six years prior to maturity. What is the rate of return?

9. The periodic bond with the face value of 25000 and the coupons of 10%, payable on a quarterly basis is redeemed within 12 years by making equal payments every quarter.

(a) Determine the bond flat price if at the time of its acquisition the interest rate is 14%, the interest is converted on a quarterly basis.

(b) Determine the bond value after nine years.

(c) Determine whether after the sale of the owned bond nine years after the acquisition date, a person will encounter profit or loss if the interest is 15%, which is converted on a quarterly basis?

10. The serial bond of 12 years with the face value of 15000 and the interest rate of 9

11. The periodic bond with the face value of 20000 and the interest rate of 10% payable every quarter is redeemed by the even quarterly payments over 10 years.

a) Determine the bond flat price, if the cash value is 14%, which is converted on a quarterly basis.

b) What is the book value of the bond after seven years?

c) Determine whether after the sale of the owned bond after seven years at the interest 15%, which are converted on a quarterly basis, a person will encounter a loss or a profit?

12. The 7-year bond with the face value of 10000 and the semi-annual coupons of 8% is sold for 7500. During the first year the market interest rates was 10

13. The 10-year bond with the face value of 8000 and the semi-annual coupons of 10% is sold for 6500.

1) Calculate the effective yield.

2) Calculate the realized return if the bond is sold 4 years before its maturity at the interest rate of 12 %.

**To be able to:** In the case of a simple and compound interest to calculate the future value of promissory notes, to discount promissory notes at the indicated point of time, to calculate the promissory note discount rate, promissory note income and discount, to determine the promissory note interest period, to calculate the promissory note interest rate. To calculate flat and quoted prices of the bonds with coupons, zero-coupon bonds, periodical bonds, and serial bonds, the rate of return to maturity, and the effective and internal rates of return. Apply the bond maturity to determine the bond price changes.