More on p -stable convex sets in Banach spaces

by

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1. Introduction

In the paper we investigate limit distributions for normed sums

(1)
$$b_n^{-1}\sum_{i=1}^{n}\xi_i,$$

where ξ_i , $i\geq 1$, are iid random convex compact (cc) sets in a given separable Banach space and summation is defined in a sense of Minkowski. At present there exists the general theory of random sets summation in the sense of Minkowski developed mainly during decade 1975-85 in large part due to application of methods and results from Probability theory in Banach spaces (see papers [G-H-Z], [G-H] and references therein). ¿From this general theory it follows particularly that the limits in distribution of sums (1) are stable random cc sets, defined in a natural way with respect to Minkowski addition, (see Definition 1 below). It turns out that non trivial stable cc sets exist only with index $0 < p < 1$. Gine and Hahn (see [G-H]) obtained stochastic integral representation of p -stable random cc sets and described their domain of attraction. Davydov and Vershik (see [D-V]) investigated non-trivial probabilistic measures on the space of cc subsets of ^{d}. The particular case of sums (1) where ξ_i are segments was considered. They found that the limiting p-stable random cc set Y_p can be represented as an integral with respect to a random Poisson measure. Due to this representation the geometric structure of the boundary ∂Y_p was determined.

In the present paper we continue the investigation of properties of random p -stable cc sets. The following results are obtained.

(i) For arbitrary random p -stable cc set Y_p in Banach space we prove its representation as an integral with respect to random Poisson measure (Theorem 2) and representation of Y_p by Le Page type series (Theorem 3).

(ii) Using results from (i) we prove the invariance principle for processes

$$S_n(t) = b_n^{-1} \sum_{i=1}^{[nt]} \xi_i, \quad t \in [0,1].$$

We introduce the notion of a set valued process with independent increments and prove the existence of p -stable cc sets Levy motion.

(iii) We show that in the case of segments ξ_i the limits of sums (1) are countable zonotopes, i.e., they can be represented as $\sum_{i=1}^{\infty}[0, x_i]$, where (x_i) are random elements in . Some properties of countable zonotopes are investigated. In particular, singularity of two countable zonotopes Y_{p_1,σ_1} , Y_{p_2,σ_2} in ^{d} (corresponding to values of exponents p_1 , p_2 and spectral measures σ_1, σ_2) is proved in the case $p_1 \neq p_2$ or $\sigma_1 \neq \sigma_2$. This result gives an answer to the question formulated in [D-V]. Here it is appropriate to mention the recent paper [K-M] where Lorenz curves were generalized to the multidimensional setting using random zonoids and zonotopes.

(iv) Some new simple estimates of parameters of stable laws in ^{d} , based on the above mentioned results are suggested.

2. Preliminaries

Let denote a separable Banach space with norm $||\cdot||$. Its dual space then is denoted by * and duality by $f(x)$, $f \in {}^*$, $x \in$. For subsets $A, C \subset$ and a real number $\alpha \geq 0$ Minkowski addition and positive homothetics, respectively, are defined by

$$A + C = \{a + c : a \in A, c \in C\}, \quad \alpha A = \{\alpha a : a \in A\}.$$

Let $\mathcal{K}()$ be the collection of nonempty compact subsets of . It becomes a complete separable metric space when endowed with the Hausdorff distance δ

$$\delta(A, C) = \max\{\sup_{a \in A} \inf_{c \in C} ||a - c||, \sup_{c \in C} \inf_{a \in A} ||a - c||\} =$$
$$= \inf\{\varepsilon > o : A \subset C^{\varepsilon}, C \subset A^{\varepsilon}\},$$

where $D^{\varepsilon} = \{x \in \ : \delta(x, D) < \varepsilon\}$ and $\delta(x, D) = \inf\{||x - y|| : y \in D\}$ for $D \subset$. We also denote $||A|| = \delta(\{0\}, A) = \sup\{||a|| : a \in A\}$ for $A \in \mathcal{K}()$. Two important subsets of $\mathcal{K}()$ are the set $co\mathcal{K}()$ of cc subsets of and the set $co\mathcal{K}_0()$ consisting of convex sets $A \in \mathcal{K}()$ which contain 0.

Throughout the paper $T = ({}^*_1, d^*)$, where ${}^*_1 = \{f \in {}^* : \|f\|^* \leq 1\}$ and

$$d^*(f, g) = \sum 2^{-i}|f(x_i) - g(x_i)|, \quad f, g \in {}^*_1,$$

where $x_1, x_2, \ldots$ is fixed dense set in the unit sphere ${}_1 = \{x \in \ : ||x|| = 1\}$ and $\| \cdot \|^*$ is the norm in * . In what follows we shall suppress the upper subscript and shall write simply $\| \cdot \|$, if it will cause no ambiguity. Then T is the compact metric space and $C(T)$ as usually denotes the Banach space of continuous functions on T with supremum norm $\|g\|_{\infty} = \sup_{x \in T} |g(x)|$.

We recall that the *support function* s_A of a set $A \in \mathcal{K}()$ is a function on ${}^*_1$ defined by

$$s_A(f) = \sup_{x \in A} f(x), \quad f \in {}^*_1.$$

A random compact set K in the Banach space is a Borel measurable function from a probability space $(\Omega, \mathcal{F}, P)$ into $\mathcal{K}()$. If $K \in co\mathcal{K}()$ a.s. then K is called *a random compact convex set* (random cc set).

If $K(\omega)$ is a random compact set then the *support process* $K(f, \omega), f \in {}^*_1$ is defined by

$$K(f, \omega) = s_{K(\omega)}(f), \quad f \in {}^*_1, \quad \omega \in \Omega.$$

The correspondence between random cc sets and support processes is isometry and preserves both addition and multiplication by positive numbers (for details see [G-H]).

Definition 1 ([G-H]). *A random compact convex set* K *is called* p *,* $0 < p \leq 2$ *if for any* K_1, K_2 *independent and distributed as* K *and for all* $\alpha, \beta \geq 0$

$$\alpha K_1 + \beta K_2 \stackrel{\mathcal{D}}{=} (\alpha^p + \beta^p)^{1/p} K.$$

In what follows $\stackrel{\mathcal{D}}{=}$ means equality of distributions.

Definition 2 ([D-V]). *The set* $\sum_{k=1}^{N} [0, \xi_k]$, *where* $\xi_k, k = 1, \ldots, N$ *are random elements of the Banach space* , *is said to be a . A in is the set*

$$\sum_{k=1}^{\infty} [0, \xi_k],$$

where $\xi_k, k = 1, \ldots$ *are random elements of the Banach space and where the series is converging in* $\mathcal{K}()$.

3. Representation of p-stable convex compact sets, $0 < p < 1$

As proved in [G-H] the only interesting strictly stable random cc sets are those with index $0 < p < 1$. Otherwise, if K is strictly p-stable random convex compact set and $1 \leq p \leq 2$, then $K = \{\xi\}$ a.s., where ξ is p-stable random element in .

Let $\mathcal{K}_1 = \{A \in co\mathcal{K}() : ||A|| = 1\}$ and let σ be a finite Borel measure on $\mathcal{K}_1$. Without loss of generality we suppose that σ is a probability distribution, that is $\sigma(\mathcal{K}_1) = 1$. Let θ be a real positive p-stable random variable, $0 < p < 1$.

By M_p we denote *a positive independently scattered random measure* on Borel sets of $\mathcal{K}_1$ such that $M_p(A) \stackrel{\mathcal{D}}{=} (\sigma(A))^{1/\alpha} \theta$ for each Borel set A. Then M_p is called a positive p-stable random measure with controle measure σ.

Theorem 1 ([G-H]). *Let* $0 < p < 1$. *A random set* $K(\omega)$ *is strictly* p*-stable compact convex set in if and only if*

$$K = \int_{\mathcal{K}_1} x M_p(dx) \quad a.s., \tag{2}$$

where M_p *is a positive* p*-stable random measure on* $\mathcal{K}_1$ *with spectral measure* σ.

Representation of p-stable random cc sets via Poison integral is as follows. Earlier for the countable zonotopes in ^{d} such representation was obtained in [D-V].

Let $(S, \mathcal{S}, n)$ be a measure space, and let $\mathcal{S}_0 = \{A \in \mathcal{S} : n(A) < \infty\}$. *A Poison random measure* N on $(S, \mathcal{S}, n)$ is an independently scattered σ-finite random measure such that for each set $A \in \mathcal{S}_0$ the random variable $N(A)$ has a Poisson distribution with mean $n(A)$. Then n is called the control measure of N.

Theorem 2. *Let* Π *be random Poisson measure on* $\mathcal{K}_1 \times {}_+$ *with controle measure* $\sigma \times \gamma$, *where* γ *is the measure on* ${}_+$ *with the density* $x^{-1/(p+1)}$, $0 < p < 1$. *Then*

$$\int_{\mathcal{K}_1} x M_\alpha(dx) \overset{\mathcal{D}}{=} \int_{\mathcal{K}_1 \times {}_+} x \Pi(dx). \tag{3}$$

Proof. The idea of the proof is rather simple: we approximate x on both sides by sequences of some simple functions, show that for these simple functions both sides have the same distribution and finally show that it is possible to take limits. The same construction is used in [G-H] for the construction of stochastic integral in the left-hand side of (3).

Let us take a sequence of simple functions

$$x_n : \mathcal{K}_1 \to \mathcal{K}_1 \tag{4}$$

constructed in the same way as in [G-H], namely for a sequence of partitions $A_{nj}, 0 \le j \le k_n$ of $\mathcal{K}_1$ and some sequence $x_{nj} \in A_{nj}$ we set

$$x_n(z) = x_{nj\,{}_{A_{nj}}}(z),$$

(here ${}_A$ is the indicator function of a set A) and require that for all $z \in \mathcal{K}_1$

$$\delta(x_n(z), z) \le 2^{-n},$$

$$\int_{\mathcal{K}_1} \delta^p(x_n(z), z)\sigma(dz) \xrightarrow[n\to\infty]{} 0.$$

By the definition of stochastic integral we have

$$\int_{\mathcal{K}_1} x_n(z) dM_p(z) = \sum_{j=0}^{k_n} \theta_{nj} [\sigma(A_{nj})]^{1/p} x_{nj}, \tag{5}$$

where θ_{nj} are iid positive stable random variables with the same distribution as θ, defined above.

Now using functions (4) let us defined the sequence of functions

$$\widetilde{x}_n : \mathcal{K}_1 \times {}_+ \to \mathcal{K}_1 \times {}_+$$

as follows. For $(x, c) \in \mathcal{K}_1 \times {}_+$ set

$$\widetilde{x}_n(x, c) = (x_n(x), c).$$

We shall use two simple facts:

1) If μ is a Poisson measure on ${}_+$ with intensity density x^{-1-p}, then

$$\int_0^\infty x\mu(dx) \overset{\mathcal{D}}{=} \theta;$$

2) If the intensity of μ is multiplied by k then corresponding stable random variable is multiplied by $k^{1/p}$.

Using these two facts and the construction of $\widetilde{x}_n$ it is not difficult to see that

$$\int_{\mathcal{K}_1\times_+} \widetilde{x}_n(z)\Pi(dz) \stackrel{\mathcal{D}}{=} \sum_{j=1}^{r_n} \theta_{nj}[\sigma(A_{nj})]^{1/p} x_{nj}. \tag{6}$$

Thus (5) and (6) show that

$$\int_{\mathcal{K}_1} x_n M_p(dx) \stackrel{\mathcal{D}}{=} \int_{\mathcal{K}_1\times_+} \widetilde{x}_n(z)\Pi(dz).$$

In [G-H] it is proved that

$$\int_{\mathcal{K}_1} x_n M_p(dx) \xrightarrow[n\to\infty]{\mathcal{D}} \int_{\mathcal{K}_1} x dM_p,$$

therefore to finish the proof of (3) it remains to check that

$$\int_{\mathcal{K}_1\times_+} \widetilde{x}_n(z)\Pi(dz) \xrightarrow[n\to\infty]{\mathcal{D}} \int_{\mathcal{K}_1\times_+} x\Pi(dx). \tag{7}$$

Let (z_j) be the support of the measure Π. Then

$$\int_{\mathcal{K}_1\times_+} \widetilde{x}_n(z)\Pi(dz) = \sum_j \widetilde{x}_n(z_j),$$

$$\int_{\mathcal{K}_1\times_+} x\Pi(dx) = \sum_j z_j.$$

¿From the construction of $\widetilde{x}_n$ we have

$$\delta(z_j, \widetilde{x}_n(z_j)) \le ||z_j|| 2^{-n}.$$

Therefore

$$\delta\big(\sum_j z_j, \sum_j \widetilde{x}_n(z_j)\big) \le 2^{-n} \sum_j ||z_j||. \tag{8}$$

But $\sum_j ||z_j|| = \int_0^\infty c\hat{\Pi}(dc) < \infty$ a.s. Here $\hat{\Pi}$ is the projection of Π onto $_+$ and since the spectral measure σ on $\mathcal{K}_1$ has unit mass, $\hat{\Pi}$ is the standard Poisson measure on $_+$. Thus from (8) it follows (7) and the proof of (3) is complete.

To state series representation of a p-stable random cc sets we need some preparation. Let (λ_i) be independent random variables with common exponential distribution , that is, $P\{\lambda_i > t\} = e^{-t}$. Set $\Gamma_j = \sum_{i=1}^j \lambda_i$, $j \ge 1$. The sequence (Γ_j) defines the successive times of jumps of a standard Poisson process.

Let η be a *positive* real valued random variable such that $||\eta||_p^p = E\eta^p < \infty$ and let (η_j) be independent copies of η. Throughout

$$c_p = \frac{1-p}{\Gamma(2-p)\cos(\pi p/2)}.$$

Theorem 3. *Assume that* Y_p *is a* p *-stable random convex compact set in* *with* $0 < p < 1$ *and corresponding spectral probability measure* σ. *Let* (ε_j) *be a sequence of independent random elements on* $\mathcal{K}_1$ *having distribution* σ *, and assume that the sequences* $(\Gamma_k), (\eta_k)$ *and* (ε_k) *are independent. Then the series*

$$c_p ||\eta||_p^{-1} \sum_{k=1}^{\infty} \Gamma_k^{-1/p} \eta_k \varepsilon_k, \tag{9}$$

converges almost surely in $\mathcal{K}()$ *and this series is distributed as* Y_p .

Proof. Let $\varepsilon_k(\cdot)$ denote the support function of ε_k, $k \geq 1$. Since the series $\sum_{k=1}^{\infty} \Gamma_k^{-1/p} \eta_k$ converges a.s. (see Theorem 1.4.5 in [S-T]) and

$$\delta\Big(\sum_{k=1}^{M} \Gamma_k^{-1/p} \eta_k \varepsilon_k, \sum_{k=1}^{N} \Gamma_k^{-1/p} \eta_k \varepsilon_k \Big) = \Big\| \sum_{k=N+1}^{M} \Gamma_k^{-1/p} \eta_k \varepsilon_k(\cdot) \Big\|_\infty \leq \sum_{k=N+1}^{M} \Gamma^{-1/p} \eta_k,$$

for $M > N$, it follows that the series (9) converges in $\mathcal{K}()$. Moreover the support function of the series (9) is

$$c_p ||\eta||_p^{-1} \sum_{k=1}^{\infty} \Gamma_k^{-1/p} \eta_k \varepsilon_k(\cdot). \tag{10}$$

According to the series representation of stable elements in Banach spaces (see [R]), this series is distributed as $\int_S x dL'(x)$, where L' is a positive p -stable independently scattered random measure with the distribution of $\varepsilon_1(\cdot)$ as the spectral measure and which is denoted by σ'. Here $S = U \cap \mathcal{V}^+$, $\mathcal{V}^+$ is the closed cone of subadditive positively homogeneous functions on T, $U = \{x \in C(T) : ||x||_\infty = 1\}$. Then, according to [G-H], the series (9) has p -stable distribution with the spectral measure $\widetilde{\sigma}(G) = \sigma'\{s_A(\cdot) : A \in G\}$. It is easy to see that the measure $\widetilde{\sigma}$ coincides with σ and this completes the proof.

4. Invariance principle for random convex compact sets

Consider the process $(X(t), t \in [0,1])$, with values in $co\mathcal{K}()$. That is, for each $t \in [0,1]$, let $X(t)$ be a random cc set. Note, that for the usual meaning of difference of sets, $A + (B - A) \neq B$, thus it is easy to provide examples of cc sets A and B , for which there does not exists cc set C such that $A + C = B$. On the other hand, if such a set exists, it is easy to see that it is unique. This allows us to define the increment of the process in the following way.

Definition 3. *Let* $s < t, s, t \in [0,1]$. *The increment* $X(t) - X(s)$ *of a random cc set process, if it exists, is such a random cc set, that*

$$X(s) + (X(t) - X(s)) = X(t) \quad a.s.$$

¿From [G-H] one deduces the following criterion: in order that for a random cc process $(X(t), t \in [0,1])$ increments could be defined the support process $(X(t)(f), f \in T, t \in [0,1])$ must satisfy the following condition

$$X(t)(f+g) - X(t)(f) - X(t)(g) \leq X(s)(f+g) - X(s)(f) - X(s)(g)$$

with all $f, g \in T$ and $s < t$.

Consider the following examples. Let $\{\xi_k\ , k \geq 1\}$, be random cc sets and $\{U_k,\quad k \geq 1\}$, be another sequence of univariate random variables, independent of $\{\xi_k\}$. Then it is easy to see that for processes

$$X(t) = \sum_k {}_{[0,t]}(U_k)\xi_k, \quad S_n(t) = \sum_{k=1}^{[nt]} \xi_k,$$

the notion of increment is correctly defined:

$$X(t) - X(s) = \sum_k {}_{(s,t]}(U_k)\xi_k, \quad S_n(t) - S_n(s) = \sum_{k=[ns]+1}^{[nt]} \xi_k.$$

Definition 4. *Let* $0 < p < 1$. *A cc set process* $\{Y(t), t \in [0,1]\}$ *is called* p *-stable cc set Levy motion if*

i) $Y(0) = \{0\}$;

ii) *the increments of the process* $(Y(t), t \in [0,1])$ *are well defined and are independent: random compact convex sets* $Y(t_2) - Y(t_1), \dots, Y(t_n) - Y(t_{n-1})$ *are independent for any* $0 \leq t_1 < \dots < t_n \leq 1$;

iii) *if* $0 \leq s < t \leq 1$ *then* $Y(t) - Y(s)$ *has* p *-stable distribution and* $Y(t) - Y(s) \stackrel{\mathcal{D}}{=} (t-s)^{1/p} Y(1)$.

To prove that such processes exist, let $\{U_k\}$ be independent uniformly distributed random variables on $[0,1]$, $\{\varepsilon_k\}$ be iid on $\mathcal{K}_1()$ with distribution function σ. Let $0 < p < 1$ and define

$$Y_p(t) = c_p \sum_{k=1}^{\infty} {}_{[0,t]}(U_k)\Gamma_k^{-1/p}\varepsilon_k, \quad t \in [0,1]. \tag{11}$$

As usually, the sequences $(\Gamma_k), (U_k)$, and (ε_k) are assumed to be independent. The process $(Y_p(t), t \in [0,1])$ is then the p -stable cc set Levy motion. Indeed, the increments of the process $(Y_p(t), t \in [0,1])$ are

$$Y_p(t) - Y_p(s) = c_p \sum_{k=1}^{\infty} {}_{(s,t]}(U_k)\Gamma_k^{-1/p}\varepsilon_k,$$

where $s < t$. It is easy to see that the increments of the process $(Y_p(t))$ are independent on non-intersecting intervals. Let $L^+(\mathcal{K}(),)$ be a set of "positively linear" functions $\phi : \mathcal{K}() \to$ (for a definition of $L^+(\mathcal{K}(),)$ see [V] or [G-H]) such that

$$\sup_{A_1 \neq A_2} \frac{|\phi(A_1) - \phi(A_2)|}{\delta(A_1, A_2)} < \infty.$$

Consider $0 \leq u < s < t \leq 1$. Let $\phi \in L^+(\mathcal{K}(),)$. Then for any $a, b \in$

$$\begin{aligned}
&\exp\{i(a\phi(Y_p(s) - Y_p(u)) + b\phi(Y_p(t) - Y_p(s)))\} = \\
&\exp\{ic_p \sum_{k=1}^{\infty} \phi(\varepsilon_k)(a_{(u,s]}(U_k) + b_{(s,t]}(U_k))\Gamma_k^{-1/p}\} = \\
&\exp\{-(t-u) \int_{\mathcal{K}_1} |\phi(x)|^p \sigma(dx)\} = \\
&\exp\{ia\phi(Y_p(s) - Y_p(u))\} \exp\{ib\phi(Y_p(t) - Y_p(s))\}.
\end{aligned}$$

Since $Y_p(t)-Y_p(s)$ and $Y_p(s)-Y_p(u)$ belongs to $co\mathcal{K}()$ and $L^+(\mathcal{K}(),)$ separates the points of $co\mathcal{K}()$, this yields the independence of $Y_p(t)-Y_p(s)$ and $Y_p(s)-Y_p(u)$.

Furthermore
$$Y_p(t)-Y_p(s) \stackrel{\mathcal{D}}{=} (t-s)^{1/p}Y_p(1),$$
for each $s<t, s,t\in[0,1]$.

Let $D([0,1];\mathcal{K}())$ denote the usual Skorohod space of cadlag functions with values in $\mathcal{K}()$. Let $\xi_1,\dots,\xi_n$ be iid with values in $co\mathcal{K}()$ and let $\xi_1(\cdot),\dots,\xi_n(\cdot)$ be the corresponding support functions which constitute iid random elements in the space $C(T)$. Let $0<p<1$ and assume that ξ_1 belongs to the domain of attraction of a p-stable random cc set (denoted $\xi_1\in DA_p(\mathcal{K}())$). Set
$$S_n(t)=b_n^{-1}\sum_{k=1}^{[nt]}\xi_k,\quad S_n(t)(\cdot)=b_n^{-1}\sum_{k=1}^{[nt]}\xi_k(\cdot),\quad t\in[0,1],$$
where b_n is the norming sequence such that
$$S_n(1)\xrightarrow[n\to\infty]{\mathcal{D}} Y_p$$
and Y_p is a p-stable random compact convex set in .

Theorem 4. *If the assumptions above are satisfied then the sequence of processes $(S_n(t),t\in[0,1])$ converges in distribution in the space $D([0,1];\mathcal{K}())$ to the process $(Y_p(t),t\in[0,1])$, defined by* (13) *.*

Proof. First observe that the processes $(S_n(t),t\in[0,1])$ converge in distribution in the space $D([0,1];\mathcal{K}())$ if and only if $(S_n(t)(\cdot),t\in[0,1])$ converge in distribution in $D([0,1];C(T))$. Let

$$Y_p=c_p\sum_{k=1}^{\infty}\Gamma_k^{-1/p}\varepsilon_k, \tag{12}$$

be the series representation of Y_p and let $\varepsilon_k(\cdot)$ be the support function of ε_k, $k\in N$. Define

$$Y_p(t)=c_p\sum_{k=1}^{\infty}{}_{(0,t]}(U_k)\Gamma_k^{-1/p}\varepsilon_k,\quad t\in[0,1]. \tag{13}$$

Let $(Y_p(t)(\cdot),t\in[0,1])$ be the support process for the process $(Y_p(t),t\in[0,1])$. To show the convergence of the processes $(S_n(t)(\cdot),t\in[0,1])$ in $D([0,1];C(T))$ in distribution to the process $(Y_p(t)(\cdot),t\in[0,1])$, we use the following result.

Lemma 1 ([G-S],Th 5, p435)**.** *Let $\eta_n, n=0,1,\dots$ be a sequence of processes with independent increments defined on $[0,1]$ with values in a Banach space $(\mathcal{X},||\cdot||)$ and with probability one belonging to the space $D([0,1],\mathcal{X})$. If the finite dimensional distributions of η_n converge weakly to those of the process η_0 and for each $\varepsilon>0$*
$$\lim_{h\to 0}\limsup_{n\to\infty}\sup_{|t-s|\le h}\{||\eta_n(t)-\eta_n(s)||>\varepsilon\}=0,$$
then
$$\eta_n\xrightarrow[n\to\infty]{\mathcal{D}}\eta_0 \quad in \quad D([0,1],\mathcal{X}).$$

The convergence of the finite dimensional distributions of $(S_n(t)(\cdot), t \in [0,1])$, to those of $(Y_p(t)(\cdot), t \in [0,1])$ easily follows from the assumption that ξ_1 is in the domain of attraction of p -stable distribution. Indeed,

$$S_n(t) \xrightarrow[n\to\infty]{\mathcal{D}} Z_p(t) \stackrel{\mathcal{D}}{=} t^{1/p} Y_p \stackrel{\mathcal{D}}{=} Y_p(t).$$

For a positive random variable ξ set

$$\Lambda_p(\xi) = \sup_{t\geq 0} t^p P\{\xi > t\}.$$

The following lemma can be proved in much the same way as Lemma .??. in [R].

Lemma 2. *For each* $0 < p < 1$ *there is a constant* c_p *such that if* $\xi_1, \ldots, \xi_n$ *are positive independent random variables then*

$$\Lambda_p(\sum_{k=1}^n \xi_k) \leq c_p \sum_{k=1}^n \Lambda_p(\xi_k).$$

By Lemma 2 we have

$$P\{||S_n(t) - S_n(s)||_\infty > \varepsilon\} = P\Big\{\Big\| \sum_{k=[ns]+1}^{[nt]} \xi_k(\cdot)\Big\|_\infty > \varepsilon b_n\Big\} \leq$$

$$P\Big\{ \sum_{k=[ns]+1}^{[nt]} ||\xi_k|| > \varepsilon b_n\Big\} \leq \frac{[nt]-[ns]}{(b_n\varepsilon)^p} \Lambda_p(||\xi_1||).$$

Thus the tightness conditions of Lemma 1 are satisfied and this completes the proof of the theorem.

5. p -stable countable zonotopes, $0 < p < 1$

- First we consider the limits of zonotopes formed by iid random variables in the domain of attraction of stable law.

Let $\xi_1, \ldots, \xi_n$ be iid random elements with values in . Consider the random zonotope

$$Z_n = \sum_{k=1}^n [0, \xi_k], \quad n \in N.$$

The following result is a generalization of Theorem 6 in [D-V] to the Banach space random cc sets.

Theorem 5. *Let* $0 < p < 1$. *If*

$$b_n^{-1} \sum_{k=1}^n \xi_k \xrightarrow[n\to\infty]{\mathcal{D}} \eta_p,$$

where η_p *is a* p *-stable random element in* *then*

$$b_n^{-1} Z_n \xrightarrow[n\to\infty]{\mathcal{D}} \widetilde{Y}_p,$$

where $\widetilde{Y}_p$ *is a* p *-stable compact convex set in* . *Moreover,* $\widetilde{Y}_p$ *is a countable zonotope, i.e.*

$$\widetilde{Y}_p \overset{\mathcal{D}}{=} \sum_{k=1}^{\infty} [0, \widetilde{\varepsilon}_k],$$

where $(\widetilde{\varepsilon}_k)$ *are random elements in* .

Proof. For $f \in T$ by f^+ we denote the positive part of the functional f, $f^+(x) = f(x)$ if $f(x) \geq 0$ and $f^+(x) = 0$ otherwise. Since the support function of the set $[0, \xi]$ is

$$[0, \xi](f) = f^+(\xi), \quad f \in T,$$

one has to prove that $\{f(\xi_1), f \in T\} \in DA_p(C(T))$, yields $\{f^+(\xi_1), f \in T\} \in DA_p(C(T))$. Set

$$S_n(f) = b_n^{-1} \sum_{k=1}^{n} f^+(\xi_k), \quad f \in T.$$

Since evidently the finite dimensional distributions of $\{S_n\}$ converges it remains to check tightness of this sequence. By Lemma 1 we have

$$\{ \sup_{d^*(f,g)\leq\delta} |S_n(f) - S_n(g)| \geq \varepsilon\} \leq \{ \sup_{d^*(f,g)\leq\delta} b_n^{-1} \sum_{k=1}^{n} |f(\xi_k) - g(\xi_k)| \geq \varepsilon\} \leq$$
$$\{ \sup_{||f||\leq 1} b_n^{-1} \sum_{k=1}^{n} |f(\xi_k)| \geq \varepsilon/\delta\} \leq \{ b_n^{-1} \sum_{k=1}^{n} ||\xi_k|| \geq \varepsilon/\delta\} \leq$$
$$b_n^{-\alpha} \varepsilon^{-\alpha} \delta^{\alpha} n \sup_t t^{\alpha} \{||\xi_1|| \geq t\}.$$

This yields the compactness of $(S_n(\cdot))$ and the first part of the theorem is proved.

Let σ' be the spectral measure of $\widetilde{Y}_p$. By Theorem 2

$$\widetilde{Y}_p \overset{\mathcal{D}}{=} c_p \sum_{k=1}^{\infty} \Gamma_k^{-1/p} M_k,$$

where (M_k) are iid random elements in with distribution σ'. As always, (Γ_k) and (M_k) are independent. On the other hand, according to [G-H]

$$\lim_{t\to\infty} pt^p P\Big\{ \frac{[0, \xi_k]}{||\xi_k||} \in A, ||\xi_k|| \geq t\} = \sigma'(A).$$

But evidently $\sigma'(A) = \sigma(A \cap \mathcal{F})$, where $\mathcal{F} = \{\bigcup_{x\in C} [0, x] : \ C \in \mathcal{B}(_1)\}$. Hence, if $A \cap \mathcal{F} = C$ then $\sigma'(A) = \sigma(C)$, where σ is the spectral measure of η_p. So, $M_1 \overset{\mathcal{D}}{=} [0, \xi]$, where ξ has distribution σ.

This theorem indicates, that the limits in distribution of zonotopes $b_n^{-1} Z_n$, where $\xi_1 \in DA_p()$ are countable stable zonotopes.

•• In this subsection is the d -dimensional Euclidean space ^{d} and $\| \cdot \|$ will stand for the usual Euclidean norm. Consider two countable p_i -stable zonotopes in ^{d} with spectral measures $\sigma_i, \quad i = 1, 2.$ Assume that their series representations are

$$Y_i \overset{\mathcal{D}}{=} \sum_{k=1}^{\infty} \Gamma_k^{-1/p_i} [0, \varepsilon_k^{(i)}],$$

and the corresponding distributions denote by $\mu_i, \quad i = 1, 2.$

Theorem 6. *Assume that neither the measure σ_1 nor the measure σ_2 has atoms. Then if either $p_1 \neq p_2$ or $\sigma_1 \neq \sigma_2$ the distributions μ_1 and μ_2 are singular.*

Remark. Here it is appropriate to note, that situation with respect to equivalency or singularity of zonotopes in ^{d} is quite different comparing with the same problem for usual stable measures in ^{d} : since all stable measures have densities (with respect to Lebesgue measure) it is clear that there can be equivalence of stable measures corresponding to different exponents $p_1 \neq p_2$. The requirement that the measures σ_i have no atoms is essential, as the following example shows.

Example. Consider the space ^{2} and set

$$\sigma = \frac{1}{2} \delta_{e_1} + \frac{1}{2} \delta_{e_2},$$

where $e_1 = (1, 0), e_2 = (0, 1)$ and δ_a denotes Dirac measure. Let $\mathcal{P}_i$ be the distribution of Z_i, where

$$Z_i = \sum_{k=1}^{\infty} \Gamma_k^{-1/p_i} \varepsilon_k, \quad i = 1, 2$$

and $\{\varepsilon_k\}$ are iid random vectors with distribution σ. Evidently

$$Z_i \overset{\mathcal{D}}{=} [0, e_1] \Sigma_1^i + [0, e_2] \Sigma_2^i,$$

where

$$\Sigma_1^i = \sum_{k=1}^{\infty} \Gamma_k^{-1/p_i} {}_{\{\varepsilon_k = e_1\}}, \quad \Sigma_2^i = \sum_{k=1}^{\infty} \Gamma_k^{-1/p_i} {}_{\{\varepsilon_k = e_2\}}, \quad i = 1, 2.$$

It is easy to see that the random variables Σ_1^i and Σ_2^i are independent. Indeed, for any $a, b \in$

$$\begin{aligned}
&\exp\{i(a\Sigma_1^i + b\Sigma_2^i)\} = \\
&\exp\{i \sum_{k=1}^{\infty} (a_{\{\varepsilon_k = e_1\}} + b_{\{\varepsilon_k = e_2\}}) \Gamma_k^{-1/p_i}\} = \\
&\exp\{-c_{p_i}^{-1/p_i} |a_{\{\varepsilon_1 = e_1\}} + b_{\{\varepsilon_1 = e_2\}}|^p\} = \\
&\exp\{ia\Sigma_1^i\} \exp\{ib\Sigma_2^i\}.
\end{aligned}$$

Moreower, both random variables Σ_1^i and Σ_2^i have bounded positive densities on $_+$. Hence, the vector (Σ_1^i, Σ_2^i) have bounded positive density on $^2_+$. This yields, that the measures $\mathcal{P}_1$ and $\mathcal{P}_2$ are equivalent for any $p_1, p_2 \in (0, 1)$.

Proof of Theorem 6.

The idea of the proof is rather simple. Having zonotopes Y_i, $i = 1, 2,$ we can restore the sequences $\{\Gamma_k^{-1/p_i}\}$ and $\{\varepsilon_k^{(i)}\}$. Now, if $p_1 \neq p_2$, then μ_1 and μ_2 will be supported by disjoint sets defined by relations $\lim_k \gamma_k k^{1/p_i} = 1, \quad i = 1, 2$. In case of $\sigma_1 \neq \sigma_2$ the singularity of measures μ_1 and μ_2 will be implied by singularity of product-measures $\sigma_1 \times \sigma_1 \times \ldots$ and $\sigma_2 \times \sigma_2 \times \ldots$. Unfortunately, we were not able to prove directly the measurability of the map, sending a countable zonotope to sequences $\{\Gamma_k^{-1/p}\}$ and $\{\varepsilon_k\}$, therefore we must use more lengthy argument.

Consider the set for $0 < p < \max(p_1, p_2)$

$$\Gamma = \{\gamma = (\gamma_k) \in \ell_1 : \gamma_k \geq \gamma_{k+1} \geq 0, \text{ for all } k \geq 1, \text{ and } \sup_k \gamma_k k^{1/p} < \infty \quad \} \subset \ell_1,$$

and the set

$$E = \{e = (e_k) : e_k \in S^{d-1} \text{ for all } k \geq 1\}$$

equiped with the distance $d(e, f) = \sum_k 2^{-k} ||e_k - f_k||$. Consider the function $Z : \Gamma \times E \to co\mathcal{K}(^d)$,

$$Z(\gamma, e) = \sum_{k=1}^{\infty} \gamma_k [0, e_k].$$

Claim 1. *The function Z is continuous (we consider $\Gamma \times E$ as a metric space with l_1 metric on Γ).*

Proof. Since

$$d(Z(\gamma, e), Z(\gamma', e')) \leq \sum_{k=1}^{\infty} \delta(\gamma_k[0, e_k], \gamma'_k[0, e'_k]) \leq \sum_{k=1}^{\infty} |\gamma_k - \gamma'_k| + \sum_{k=1}^{\infty} \gamma_k |e_k - e'_k|$$

and both terms tends to zero provided $\gamma \to \gamma'$ in ℓ_1 and $e \to e'$ in E (noting also that $\gamma_k \leq C k^{-1/p}$).

Set $\mathbf{Z}_0 = Z(\Gamma \times E)$ and for $0 < c < 1$ consider the function $\Pi_{N,c} : \mathbf{Z}_0 \to \mathbf{Z}_0$,

$$\Pi_{N,c}\Big(\sum_{k=1}^{\infty} \gamma_k [0, e_k] \Big) = \sum_{k=1}^{\infty} \gamma_{N,c,k} [0, e_k],$$

where

$$\gamma_{N,c,k} = \begin{cases} \gamma_k & \text{if } k \leq N \\ (1-c)\gamma_k & \text{if } k > N. \end{cases}$$

Claim 2. *For each fixed N and $0 < c < 1$ the function $\Pi_{N,c}$ is continuous.*

Proof. The set $\Gamma \times E$ is a countable union of compact sets $\Gamma_A \times E,$ where the set

$$\Gamma_A = \{(\gamma_k) \in \Gamma : \sup_k \gamma_k k^{1/p} \leq A\}.$$

Evidently, both functions Z and $Z \circ \Pi_{N,c}$ are continuous, on each set $\Gamma_A \times E.$ We need the following simple fact, and since we have not found relevant reference, we provide it with the proof.

Lemma 3. *Assume, that* X, Y, Z *are metric compact spaces and* $f : X \to f(X) = Y$ *is continuous. If a function* $g : Y \to Z$ *is such that* $g \circ f$ *is continuous, then* g *is continuous.*

Proof of Lemma 3. Let $y_n \to y_0$ in Y. We prove, that $g(y_n) \to g(y_0)$ in Z. To this aim it suffices to prove, that any subsequence $(g(y_{n'}))$ has a subsequence such that $g(y_{n''}) \to g(y_0)$. Let our selected subsequence be (y_n). Since $y_n = f(x_n)$ and X is compact there exists a subsequence $(x_{n'})$ such that $x_{n'} \to x_0$. But continuity of f yields $f(x_{n'}) \to f(x_0)$. So $f(x_0) = y_0$ and we have $g(y_{n'}) = g(f(x_{n'})) \to g(f(x_0)) = g(y_0)$.

Lemma 3 yields the continuity of $\Pi_{N,c}$ on each set $Z(\Gamma_A \times E)$ and therefore on $\mathbf{Z}_0 \subset \cup_{A \in N} Z(\Gamma_A \times E)$.

We need the following notations. P_i and Q_i stand for the distributions of the sequences $\{\Gamma_k^{-1/p_i}\}$ and $\{\epsilon_k^{(i)}\}$,respectively, $i = 1, 2$. By P_i^N and Q_i^N we denote the distributions of the finite sequences $\{\Gamma_1^{-1/p_i}, \dots, \Gamma_N^{-1/p_i}\}$ and $\{\epsilon_1^{(i)}, \dots, \epsilon_N^{(i)}\}$, respectively, $i = 1, 2$. We denote by μ_i^N the distribution of the finite zonotope

$$\sum_{k=1}^{N} \Gamma_k^{-1/p_i}[0, \varepsilon_k^{(i)}], \quad i = 1, 2.$$

It is clear that $(P_i \times Q_i)Z^{-1} = \mu_i,\ i = 1, 2.$

Note, that both measures $\mu_i \Pi_{Nc}^{-1}$ converge weakly as $c \to 1$ to $\mu_i^N,\ i = 1, 2.$ Then (see e.g. [D-L-S], Th. 2.7)

$$||\mu_1^N - \mu_2^N|| \le \liminf_{c \to 1} ||\mu_1 \Pi_{Nc}^{-1} - \mu_2 \Pi_{Nc}^{-1}|| \le ||\mu_1 - \mu_2||. \tag{14}$$

Consider the function

$$f : (\gamma_1, \dots \gamma_N, \varepsilon_1, \dots, \varepsilon_N) \to \sum_{k=1}^{N} \gamma_k [0, \varepsilon_k],$$

defined on the set

$$B_N = \{(\gamma, \varepsilon) : \gamma_1 > \dots > \gamma_N, \varepsilon_1 \ne \dots \ne \varepsilon_N\}.$$

The set B_N is open in $R_+^N \times (S^{d-1})^N$ and therefore is a countable union of increasing sequence of compact sets, say K_m. Let $X_m = f(K_m)$. The function f is continuous on K_m. Applying Theorem 3.5.3 from [S] ,it is possible to prove that the inverse function $f^{-1} = g$ exists on X_m and by Lemma 3 is continuous. Hence, the function

$$g : \sum_{k=1}^{N} \gamma_k [0, \varepsilon_k] \to (\gamma_1, \dots, \gamma_N, \varepsilon_1, \dots, \varepsilon_N)$$

is continuous on the set $\cup_m X_m$.

Evidentelly, $P_i^N \times Q_i^N = \mu_i^N g^{-1},\ i = 1, 2.$ Therefore, by (14)

$$||P_1^N \times Q_1^N - P_2^N \times Q_2^N|| \le ||\mu_1 - \mu_2||. \tag{15}$$

Now consider two cases separately.
First case, $p_1 \neq p_2$ **.** In this case

$$||P_1^N - P_2^N|| \le ||P_1^N \times Q_1^N - P_2^N \times Q_2^N||$$

and (15) yields

$$||\mu_1 - \mu_2|| \ge \liminf_N ||P_1^N - P_2^N|| \ge ||P_1 - P_2|| = 2.$$

The last equality easily follows by the fact, that the distribution P_1 is concentrated on the sequences (γ_k) such that $\lim_k \gamma_k k^{1/p_1} = 1$, whereas the second measure sits on the sequences (γ_k) such that $\lim_k \gamma_k k^{1/p_2} = 1$.
Second case, $\sigma_1 \neq \sigma_2$. In this case we have

$$||Q_1^N - Q_2^N|| \le ||P_1^N \times Q_1^N - P_2^N \times Q_2^N||$$

and (15) yields

$$||\mu_1 - \mu_2|| \ge \liminf_N ||Q_1^N - Q_2^N|| \ge ||Q_1 - Q_2|| = 2$$

by Kakutani theorem (see, for example, [J-S]. p. 217).

6. Estimation of parameters of p-stable laws, $0 < p < 1$

Any stable law in ^{d} (or more general Banach space) G_p is completely determined by an exponent $p \in (0,2)$, and a spectral measure ν on Borel sets of unit sphere. We propose rather simple estimators of these parameters of a stable laws in the case $0 < p < 1$, based on Le Page type representation of stable vectors. There is a vast literature on the estimation of parameters of stable laws or some other parameters characterizing the tail behaviour of distributions, a reader can consult , for example, the survey paper [MC] and references in it.

Let $\xi_1, \xi_2, \ldots, \xi_N$ be a sample of size N from a stable distribution with unknown parameter $p < 1$ and unknown spectral measure ν. For our purposes we shall assume that $N = n^2, n \ge 1$.

- Firstly we estimate the parameter p. Let us divide the sample into n groups with n elements in each group in the following way. If $V_{n1}, \ldots, V_{nn}$ denote n groups, each containing of n vectors from the sample of size $N = n^2$, then passing to the size $N_1 = (n+1)^2$ we use the following rule:

$$V_{n+1,i} = V_{n,i} \cup \{\xi_{n^2+i}\}, i = 1, \ldots, n,$$
$$V_{n+1,n+1} = \{\xi_{n^2+n+1}, \ldots, \xi_{(n+1)^2}\}.$$

Let

$$M_{ni}^{(1)} = \max\{||\xi|| : \xi \in V_{ni}\}$$

and let $M_{ni}^{(2)}$ denote the second norm maximal element in the same group.

¿From Le Page type representation of a stable vectors it follows that for each i

$$\Big(\frac{M_{ni}^{(1)}}{b_n}, \frac{M_{ni}^{(2)}}{b_n}\Big) \xrightarrow[n\to\infty]{\mathcal{D}} \big(\Gamma_1^{-1/p}, \Gamma_2^{-1/p}\big), \tag{16}$$

where $b_n = n^{1/p}$ in the case of a sample from stable distribution. In the case of sample from distribution in the domain of attraction of a stable law b_n will be norming constants in limit theorem. Here $\Gamma_i = \sum_{j=1}^{i} \lambda_j$, $\lambda_1, \lambda_2, \dots$ being iid random variables with standard exponential distribution. Set

$$\kappa_{ni} = M_{ni}^{(2)}/M_{ni}^{(1)}, \quad S_n = \sum_{i=1}^{n} \kappa_{ni}.$$

Theorem 7. *For any* $d \geq 1$, *as* $n \to \infty$

$$n^{-1} S_n \xrightarrow[n\to\infty]{\text{a.s.}} \frac{p}{1+p}. \tag{17}$$

Therefore the quantity $S_n/(n-S_n)$ presents consistent and asymptotically unbiased estimator of parameter p, $0 < p < 1$.

Remark. The proposed estimator does not depend neither on the dimension d nor on spectral measure ν.

Proof. From (16) it follows that for all i

$$\kappa_{ni} \xrightarrow[n\to\infty]{\mathcal{D}} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{1/p}.$$

Since $0 \leq \kappa_{ni} \leq 1$ then for any integer m

$$E\kappa_{ni}^m \xrightarrow[n\to\infty]{} E\Big(\frac{\lambda_1}{\lambda_1+\lambda_2}\Big)^{m/p}. \tag{18}$$

Simple calculations show that

$$E\Big(\frac{\lambda_1}{\lambda_1+\lambda_2}\Big)^{1/p} = \frac{p}{p+1}. \tag{19}$$

Random variables $\kappa_{n1}, \dots, \kappa_{nn}$ are independent and identically distributed. Let $a_n = E\kappa_{n1}$. Since $0 \leq \kappa_{ni} \leq 1$ there exists an absolute constant C such that

$$\begin{aligned} &P\big\{|n^{-1}S_n - a_n| > \varepsilon\big\} \leq \varepsilon^{-4} n^{-4} E|S_n - na_n|^4 \leq \\ &C\varepsilon^{-4} n^{-2} E|\kappa_{n1}|^4 \leq C\varepsilon^{-4} n^{-2}. \end{aligned}$$

Taking $\varepsilon = \varepsilon_n = n^{-1/8}$ and applying Borel-Cantelli lemma we get that with probability 1 for all sufficiently large n

$$|n^{-1}S_n - a_n| \leq n^{-1/8}.$$

This means that $n^{-1}S_n - a_n \to 0$ a.s. and this together with (18) and (19) proves (17). The theorem is proved.

•• Next we consider an estimator of the spectral measure ν. We shall assume that ν is normed so that $\nu\{||x|| = 1\} = 1$. Set

$$\xi_{ni} = \xi_j = \xi_{j(n,i)},$$

where $j(n,i)$ is such that $M_{ni}^{(1)} = ||\xi_{j(n,i)}||$ and set

$$\theta_{ni} = \frac{\xi_{ni}}{||\xi_{ni}||}, \quad i = 1, \dots, n.$$

Random vectors $\theta_{n1}, \dots, \theta_{nn}$ are iid. Again from the representation of stable law by Le Page type series it follows that for each i

$$\theta_{ni} \xrightarrow[n\to\infty]{\mathcal{D}} \nu.$$

Therefore for any Borel set B on unit sphere S^{d-1} such that $\nu(\partial B) = 0$ we have

$${}_B(\theta_{ni}) \xrightarrow[n\to\infty]{\mathcal{D}} {}_B(\gamma),$$

where γ is a random vector with the distribution ν. This yields

$$(20) \qquad {}_B(\theta_{ni}) \xrightarrow[n\to\infty]{} \nu(B).$$

The last step is again to apply strong law of large numbers, thus we get that with probability one

$$(21) \qquad n^{-1}\sum_{i=1}^{n} {}_B(\theta_{ni}) - {}_B(\theta_{n1}) \xrightarrow[n\to\infty]{} 0.$$

(20) and (21) proves the following result.

Theorem 8. *The empirical distribution based on sample* $\theta_{n1}, \dots, \theta_{nn}$ *is consistent estimator for the spectral measure* ν, *that is, for any* $B \in \mathcal{B}(S^{d-1})$ *such that* $\nu(\partial B) = 0$ *we have*

$$n^{-1}\sum_{i=1}^{n} {}_B(\theta_{ni}) \xrightarrow[n\to\infty]{\text{a.s.}} \nu(B).$$

••• Now we return to Theorem 7 and show that with appropriate normalization our estimator of the parameter p is asymptotically normal (in the case of samples from stable distribution). Again, we shall deal not with estimator $S_n/(n - S_n)$ but with more convienent statistics $n^{-1}S_n$. It is easy to see that having confidence interval for $p(1+p)^{-1}$ say

$$a_n - b_n < \frac{p}{1+p} < a_n + b_n$$

we easily get confidence interval for p

$$\frac{a_n - b_n}{1 - a_n + b_n} < p < \frac{a_n + b_n}{1 - a_n - b_n}.$$

Here, of course, we assumed that $a_n + b_n < 1$. Now we shall prove the following result.

Theorem 9. *Let* S_n *be defined as in Theorem 7. Then*

$$n(n^{-1}S_n - p(p+1)^{-1})\Big(\sum_{j=1}^{n}\kappa_{nj}^2 - n^{-1}S_n^2\Big)^{-1/2} \xrightarrow[n\to\infty]{\mathcal{D}} N(0,1). \tag{22}$$

Remark. One can take more simple expression on the left-hand side of (22), namely $n^{-1/2}(n^{-1}S_n - p(p+1)^{-1})$, but then the limit normal law will have variance $\sigma^2 = p((p+1)^2(p+2))^{-1}$. This means that limit law would be dependent on unknown parameter p. Therefore we used in (22) self-normalized type sum.

Proof of Theorem 9. From the proof of Theorem 7 we have that

$$a_n = \kappa_{n1} = \frac{p}{p+1} + \gamma_n,$$

where $\gamma_n \to 0$ as $n \to \infty$. Since

$$\sqrt{n}(n^{-1}S_n - p(p+1)^{-1}) = n^{-1/2}\sum_{i=1}^{n}(\kappa_{ni} - a_n) + \sqrt{n}\gamma_n,$$

$$n^{-1}\sum_{i=1}^{n}\kappa_{ni}^2 - (n^{-1}S_n)^2 = n^{-1}\sum_{i=1}^{n}(\kappa_{ni} - a_n)^2,$$

it is easy to see that in order to prove (22) we need to show the following three relations:

$$n^{-1/2}\sum_{i=1}^{n}(\kappa_{ni} - a_n) \xrightarrow[n\to\infty]{\mathcal{D}} N(0,\sigma^2); \tag{23}$$

$$n^{-1/2}\gamma_n \to 0; \tag{24}$$

$$n^{-1}\sum_{i=1}^{n}(\kappa_{ni} - a_n)^2 \xrightarrow[n\to\infty]{P} \sigma^2. \tag{25}$$

(23) follows from the CLT applied to triangular array $\{\kappa_{ni}, 1 \le i \le n\}$ of iid in each row random variables, taking into account the limit relation

$$\sigma_n^2 := (\kappa_{ni} - a_n)^2 \to \sigma^2 = p((p+1)^2(p+2))^{-1},$$

which follows from (19) and limit value is obtained evaluating $(\lambda_1/(\lambda_1+\lambda_2))^{2/p}$.

(25) follows from LLN for the triangular array $\{(\kappa_{ni} - a_n)^2, 1 \le i \le n\}$, applying, for example, criterion (4.8.4) given in [H]. Having (23) and (25) by the well known result (see, for example, [B]) we get convergence of the joint distributions:

$$\Big(n^{-1/2}\sum_{i=1}^{n}(\kappa_{ni} - a_n), n^{-1}\sum_{i=1}^{n}(\kappa_{ni} - a_n)^2\Big) \xrightarrow[n\to\infty]{\mathcal{D}} (N(0,\sigma^2), \sigma^2).$$

This relation together with (24) proves the theorem. Therefore it remains to prove (24).

Let $G(x) = P\{||\xi_1|| > x\}$. It is well-known (see, for example, [Br] or [L-W-Z]) that

$$\kappa_{n1} \overset{\mathcal{D}}{=} G^{-1}\big(\frac{\Gamma_2}{\Gamma_{n+1}}\big)\Big(G^{-1}\big(\frac{\Gamma_1}{\Gamma_{n+1}}\big)\Big)^{-1},$$

where G^{-1} is inverse function for (continuous) function G. In [F] it is given asymptotic expansion for the distribution density of the norm of a strictly p -stable random vector, which allows us to write

$$G(x) = c_1 x^{-p} + c_2 x^{-2p} + O(x^{-3p}), \quad \text{as} \quad x \to \infty.$$

Here $c_i, \quad i = 1, 2$ are constants depending on p and on $\nu\{||x|| = 1\}$. Using Lagrange theorem on expansions of inverse function (see, for example, [M] p.340) we can write

$$G^{-1}(t) = b_1 t^{-1/p} + b_2 t^{1-1/p} + O(t^{2-1/p}), \quad \text{as} \quad t \to 0,$$

where $b_1 = c_1^{1/p}, \quad b_2$ is some function of $c_i, \quad i = 1, 2$. This implies that for sufficiently small δ and some constants C_3, C_4

$$1 - C_4 t \le G^{-1}(t) b_1^{-1} t^{1/p} \le 1 + C_3 t, \quad \text{if} \quad 0 < t < \delta. \tag{26}$$

Now we are ready to estimate the quantity

$$\begin{aligned} &|\gamma_n| = |a_n - p(p+1)^{-1}| = \\ &\Big|\Big(G^{-1}\big(\frac{\Gamma_2}{\Gamma_{n+1}}\big)\Big(G^{-1}\big(\frac{\Gamma_1}{\Gamma_{n+1}}\big)\Big)^{-1}\Big) - \frac{p}{p+1}\Big|. \end{aligned} \tag{27}$$

Let $R_+^{n+1} = \{\overline{x} = (x_1, \dots, x_{n+1}) : x_i \ge 0, i = 1, \dots, n+1\}, \quad \Sigma_n = x_1 + \dots + x_n;$

$$A_{2,n} = \{\overline{x} \in R_+^{n+1} : (x_1 + x_2)/\Sigma_n \ge \delta\}, \quad A_{2,n}^c = R_+^{n+1} \backslash A_{2,n},$$

where δ is from (26). Now

$$\begin{aligned} &\Big(G^{-1}\big(\frac{\Gamma_2}{\Gamma_{n+1}}\big)\Big(G^{-1}\big(\frac{\Gamma_1}{\Gamma_{n+1}}\big)\Big)^{-1}\Big) = \\ &\int_{R_+^{n+1}} G^{-1}\big(\frac{x_1 + x_2}{\Sigma_{n+1}}\big)\Big(G^{-1}\big(\frac{x_1}{\Sigma_{n+1}}\big)\Big)^{-1} e^{-\Sigma_{n+1}} d\overline{x} = I_1 + I_2, \end{aligned} \tag{28}$$

where

$$I_1 = \int_{A_{2,n}}, \quad I_2 = \int_{A_{2,n}^c}.$$

Since

$$G^{-1}\big(\frac{x_1 + x_2}{\Sigma_{n+1}}\big)\Big(G^{-1}\big(\frac{x_1}{\Sigma_{n+1}}\big)\Big)^{-1} \le 1,$$

then

$$I_1 \le \Big\{\frac{\lambda_1 + \lambda_2}{\lambda_1 + \dots + \lambda_{n+1}} > \delta\Big\} \le \delta^{-1} \frac{\lambda_1 + \lambda_2}{\Gamma_{n+1}} = \frac{2}{\delta(n+1)}. \tag{29}$$

Taking into account that for $\overline{x} \in A^c_{2,n}$ it holds $x_1\Sigma^{-1}_{n+1} \le (x_1+x_2)\Sigma^{-1}_{n+1} < \delta,$ we can apply (26) to the integrand and we get

$$a_1 \le G^{-1}\big(\frac{x_1+x_2}{\Sigma_{n+1}}\big)\Big(G^{-1}\big(\frac{x_1}{\Sigma_{n+1}}\big)\Big)^{-1} \le a_2, \tag{30}$$

where

$$a_1 = \Big(\frac{x_1}{x_1+x_2}\Big)^{1/p}\frac{\Sigma_{n+1} - C_4(x_1+x_2)}{\Sigma_{n+1}+C_1x_1},$$
$$a_2 = \Big(\frac{x_1}{x_1+x_2}\Big)^{1/p}\frac{\Sigma_{n+1} + C_3(x_1+x_2)}{\Sigma_{n+1}-C_4x_1},$$

We recall that
$$\int_{R^{n+1}_+}\Big(\frac{x_1}{x_1+x_2}\Big)^{1/p}e^{-\Sigma_{n+1}}d\overline{x} = \Big(\frac{\lambda_1}{\lambda_1+\lambda_2}\Big)^{1/p} = \frac{p}{p+1},$$

therefore, using (30) it is not difficult to get the following estimates from above and below:

$$I_2 \le \frac{p}{p+1} + I_2^{(1)}, \tag{31}$$

$$I_2 \ge \frac{p}{p+1} - I_2^{(2)} - I_2^{(3)}, \tag{32}$$

where

$$I_2^{(1)} = \int_{A^c_{2,n}}\Big(\frac{x_1}{x_1+x_2}\Big)^{1/p}\frac{\Sigma_{n+1}+C_3(x_1+x_2)}{\Sigma_{n+1}-C_4x_1}e^{-\Sigma_{n+1}}d\overline{x},$$
$$I_2^{(2)} = \int_{A_{2,n}}\Big(\frac{x_1}{x_1+x_2}\Big)^{1/p}e^{-\Sigma_{n+1}}d\overline{x},$$
$$I_2^{(3)} = \int_{A^c_{2,n}}\Big(\frac{x_1}{x_1+x_2}\Big)^{1/p}\frac{\Sigma_{n+1}+C_4(x_1+x_2)}{\Sigma_{n+1}+C_3x_1}e^{-\Sigma_{n+1}}d\overline{x},$$

$I_2^{(2)}$ can be estimated in the same way as in (29). For $x \in A^c_{2,n}$ we have $x_1 < \delta\Sigma_{n+1}$ and $\Sigma_{n+1} - C_4x_1 \ge \Sigma_{n+1}(1-C_4\delta),$ therefore $I_2^{(1)}$ and $I_2^{(3)}$ can be estimated by $(\Gamma_2/\Gamma_{n+1}) = 2/(n+1)$ multiplied by some constant, depending on $\delta,$ C_3 and C_4. Thus we have

$$I_2^{(j)} = O(n^{-1}), \quad j = 1,2,3. \tag{33}$$

Collecting (27)–(29), (31)–(33) we get $\gamma_n = O(n^{-1})$. Now (24) follows and the theorem is proved.

In Theorems 7 - 9 we took $n = \sqrt{N},$ but it is possible to consider more general function $n = n(N),$ if at the beginning we group all sample of size N into $n(N)$ groups, each containing $m = m(N)$ elements, where $nm = N$. Then denoting the groups by $V_{m1}, \dots, V_{mn},$ we define

$$M^{(1)}_{mi} = \max\{||\xi|| : \xi \in V_{mi}\}, \quad i = 1, 2, \dots, n$$

and similarly we define $M^{(2)}_{mi}$ and $\kappa_{mi} = M^{(2)}_{mi}(M^{(1)}_{mi})^{-1}$. Choosing m as independent variable we get n as function of $m,$ so let us denote $l_m = Nm^{-1}$ and define

$$S_m = \sum_{i=1}^{l_m}\kappa_{mi}.$$

Assuming that $m = m(N) \to \infty$ and $m/N \to 0$ as $N \to \infty$, we can generalize (17) and get

$$l_m^{-1} S_m \xrightarrow[N\to\infty]{P} \frac{p}{p+1}.$$

In a similar way we can generalize Theorem 8. More complicated is generalization of Theorem 9. This and some other problems, not discussed in this section are beyond the scope of the present paper and will be addressed in detail elswhere.

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