CENTRAL LIMIT THEOREM IN $\mathbf{D}[0,1]$

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Abstract. We survey recent results on the central limit theorem for stochastically continuous processes having sample paths in the Skorokhod space $D[0,1]$

Introduction

Let $D \equiv D[0,1]$ denote the space of real-valued functions on $[0,1]$ which are right-continuous on $[0,1)$ with left limits on $(0,1]$, which is endowed with the Skorokhod topology. The background for the theory of the weak convergence of stochastic processes in $D[0,1]$ was laid in papers of Yu.V. Prokhorov (1956), A.V. Skorokhod (1956) and N.N. Chentsov (1956 b). Later on Billingsley's book (1968) became the most popular reference book for the weak convergence of measures on metric spaces, in particular on the space D.

Let $X_n, n = 1,2,\dots$ be a sequence of random processes with sample paths in D. It is well-known that such a sequence converges weakly in D to a limiting process X (denoted by $X_n \Longrightarrow X$) if the finite-dimensional distributions of X_n converge to those of X and if the sequence is tight. In Section 1 we consider refined tightness criteria in the space D. In Section 2 these criteria are applied to establish sufficient conditions for the central limit theorem (CLT) in D. Two applications of the CLT in D are given in Section 3: the weak convergence of weighted empirical processes and the asymptotic distribution of the fiber bundle strength in the fiber bundle model introduced by H.Daniels (1945). In Section 4 an estimate of the rate of convergence in the CLT in D is presented. Results from the sections 1 and 2 are extended to the space D_k (the Skorokhod space of k-variate real càdlàg functions on $[0,1]^k$) in Section 5.

1.Tightness criteria

Assume that the process X is continuous at the point 1, i.e., $\mathbf{P}\{X(1) = X(1-0)\} = 1$ and that finite-dimensional distributions $\mathcal{L}(X_n(t_1),\dots,X_n(t_k))$ converge to $\mathcal{L}(X(t_1),\dots,X(t_k))$ as $n \to \infty$, for each $t_1,\dots,t_k \in [0,1]$ such that $\mathbf{P}\{X(t_i) = X(t_i - 0)\} = 1,\ i = 1,2,\dots,k,\ k = 1,2,\dots$. Suppose that there exist $\delta > 0, \gamma > 0$, increasing continuous function F on $[0,1]$a nd increasing function f such that

$$(1)\qquad \mathbf{P}\{|X_n(s) - X_n(t)| \wedge |X_n(t) - X_n(u)| \geq \lambda\} \leq \lambda^{-\gamma} |F(u) - F(s)| f(F(u) - F(s))$$

for all $\lambda > 0$, for all $n \geq 1$, for all $0 \leq s \leq t \leq u \leq 1$ such that $u - s < \delta$.

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Theorem 1.1. *(Theorem 15.6 of Billingsley (1968)).* $X_n \Longrightarrow X$ *in* $D[0,1]$ *if (1) holds with* $f(u) = cu^\alpha$ *for some* $c > 0$ *and* $\alpha > 0$.

The proof of Theorem 15.6 from Billingsley (1968) is based on the estimate for the fluctuations of partial sums of random variables, see Theorem 1.2 below.

Let $\xi_1, \dots, \xi_m$ be a sequence of real random variables and let $S_i = \xi_1 + \dots + \xi_i$ ($S_0 = 0$). By $u_1, \dots, u_m$ we denote a sequence of positive numbers and write $u = u_1 + \dots + u_m$. Assume that $u < 1$ and consider the following two conditions.

Condition A: *there exist* $c_1, \gamma > 0$ *and increasing function* f_1 *such that*

$$\mathbf{P}\{|S_j - S_i| \wedge |S_k - S_j| \geq \lambda\} \leq c_1 \lambda^{-\gamma} f_1\left(\sum_{i \leq r \leq k} u_r\right) \sum_{i \leq r \leq k} u_r, \tag{2}$$

for each $0 \leq i \leq j \leq k \leq m$ *and for all* $\lambda > 0$;

Condition B: *there exist* $c_2, \gamma > 0$ *and increasing function* f_2 *such that for each* $\lambda > 0$

$$\mathbf{P}\left\{\max_{0 \leq i \leq j \leq k \leq m} |S_j - S_i| \wedge |S_k - S_j| \geq \lambda\right\} \leq c_2 \lambda^{-\gamma} f_2(u) u. \tag{3}$$

Theorem 1.2. *(Theorem 12.5 of Billingsley (1968)). If for some* $\gamma > 0$ *and* $\alpha > 0$ *the condition* **A** *is satisfied with* $f_1(u) = u^\alpha$*, then the condition* **B** *is satisfied with the same parameter* γ *and the function* $f_2(u) = u^\alpha$.

Analyzing the proof of Theorem 15.6 from Billingsley (1968) one can see that an improvement of the estimate for the fluctuation of sums given in Theorem 1.2 leads to corresponding improvement of the tightness criteria (Theorem 1.1). In what follows we present a new estimate for the fluctuations of sums.

Proposition 1.1. (Bloznelis and Paulauskas (1993)). *If for some positive numbers* α *and* γ *the condition* **A** *is satisfied with the function* $f_1(u) = \log^{-\gamma-1-\alpha}(u^{-1})$*, then the condition* **B** *is satisfied with the same* γ *and with the function* $f_2(u) = \log^{-\alpha}(u^{-1})$.

An application of this proposition gives the following refinement of Theorem 15.6 from Billingsley (1968).

Proposition 1.2. (Bloznelis and Paulauskas (1993)). $X_n \Longrightarrow X$ *in* $D[0,1]$ *if (1) holds with* $f(u) = c|\log u|^{-\gamma-1-\alpha}$*, for some* $\gamma > 0$ *and some* $\alpha > 0$.

Further improvements of Theorem 12.5 from Billingsley (1968) and Proposition 1.1 would imply improvements of Theorem 1.1. Analysis of the proof of Theorem 15.6 ibidem shows that in order to obtain the tightness criterion with a function f in (1), it suffices to show that the condition **A** with the function $f_1 = f$ in (2) implies the condition **B** with arbitrary increasing, but vanishing at zero function f_2. Given $\gamma > 0$, it would be interesting to determine the class of such functions f. So far this question seems to be open. Another question of interest would be to determine the class of those functions f for which (1) is sufficient for the weak convergence $X_n \Longrightarrow X$. It is worthwhile to mention here that the condition (1) with the function $f(u) = c|\log u|^{-\gamma}$ is not sufficient for the weak convergence $X_n \Longrightarrow X$, see Bloznelis and Paulauskas (1993).

Further extensions and refinements of Theorem 15.6 from Billingsley are given in Bloznelis and Paulauskas (1994 b) and in the recent paper by Genest et all (1996). In these papers the following generalization of the condition (1) was considered:

$$\mathbf{P}\{|X_n(s) - X_n(t)| \wedge |X_n(t) - X_n(u)| \geq \lambda\} \leq \lambda^{-\gamma} h_n(s,u), \quad 0 \leq s \leq t \leq u \leq 1$$

where the function $h_n(s,u)$ is defined on $\{(s,u) : 0 \le s \le u \le 1\}$ and is decreasing in the first argument and increasing in the second one. Genest et all (1996) also give refined versions of Theorem 12.5 from Billingsley and Proposition 1.1. In particular, from their result it is not difficult to derive the analogous statement as in Proposition 1.1, but with the functions $f_1(u) = c|\log u^{-1}|^{-\gamma-1}|\log\log u^{-1}|^{-\gamma-1-\alpha}$ and $f_2(u) = |\log\log u^{-1}|^{-\alpha}$, for u small enough.

The tightness criteria given in Theorem 15.6 of Billingsley (1968) and its improvements considered here are formulated in terms of the tails of increments, see (1). P.H. Bezandry and X.Fernique (1990, 1992) proposed a new tightness criterion in $D[0,1]$. This criterion is formulated in terms of the moments of increments

$$\Delta_X(s,t,u) := |X(s) - X(t)| \wedge |X(t) - X(u)|$$

instead of the tail probabilities.

Let m be an integer and let $\delta_1, \dots, \delta_m$ and $\theta_1, \dots, \theta_m$ be increasing continuous real functions on $[0,1]$ which vanish at the origin. We assume that the functions $\theta_1, \dots, \theta_m$ are concave and

$$\int_0^1 u^{-2}\delta_i(u)\theta_i(u)du < \infty, \qquad i = 1,2,\dots,m.$$

Let C be a family of random processes $\{\{X(t), t \in [0,1]\}\}$ with sample paths in D such that for all $0 \le s \le t \le u \le 1$ and for all $M > 0$

$$\mathbf{E}\Delta_X(s,t,u)\mathbb{I}\{\Delta_X(s,t,u) \ge M\} \le \sum_{i=1}^m \delta_i(u-s)\theta_i(\mathbf{P}\{\Delta_X(s,t,u) \ge M\}).$$

Theorem 1.3. (Bezandry and Fernique (1992)). *The family of distributions $\{\mathcal{L}(X), X \in C\}$ is tight in D if and only if the following two conditions hold:*

1. *for each $t \in [0,1]$, the family of one-dimensional distributions $\{\mathcal{L}(X(t)), X \in C\}$ is tight in R;*
2. *the family of random functions $\{t \to |X(t) - X(0)| + |X(1) - X(1-t)|, X \in C\}$ is equicontinuous in probability at the point $t = 0$.*

2. Central limit theorem

Let $X = \{X(t), t \in [0,1]\}$ be a centered stochastically continuous random process with sample paths in D such that $\mathbf{E}X^2(t) < \infty$, for all $t \in [0,1]$. Let $X_1, X_2, \dots$ be independent copies of X and $S_n = n^{-1/2}(X_1 + \dots + X_n)$. The process X is said to satisfy the CLT in D if the distributions $\mathcal{L}(S_n)$ of S_n converge weakly to a Gaussian distribution on D.

Probably the first general central limit theorem in D was proved in Hahn (1978), see also Phoenix and Taylor (1973). Later on, sufficient conditions for a random process X to satisfy the CLT in D were obtained by Juknevičiene (1985), Paulauskas and Stieve (1990), Bezandry and Fernique (1992), Bloznelis and Paulauskas (1993), (1994 a,b,c), Fernique (1994), etc.

Since tightness criteria in D usually are formulated in terms of two consecutive increments of processes, sufficient conditions for the CLT in D are typically formulated it terms of moments of these increments. Consider the following conditions. There exist a continuous increasing function F on $[0,1]$ and increasing functions f, g such that for all $0 \le s \le t \le u \le 1$

$$\mathbf{E}(X(s) - X(u))^2 \le |F(u) - F(s)|^{1/2} f(F(u) - F(s)), \tag{4}$$

$$\mathbf{E}(X(s)-X(t))^2(X(t)-X(u))^2 \le |F(u)-F(s)|g(F(u)-F(s)), \tag{5}$$

$$\mathbf{E}(|X(s)-X(t)|\wedge 1)^2(X(t)-X(u))^2 \le |F(u)-F(s)|g(F(u)-F(s)). \tag{6}$$

Hahn (1978) showed that X satisfies the CLT in D if conditions (4) and (5) hold with $f(u)=g(u)=cu^\alpha$, for some $\alpha>0$.

Note that condition (5) requires the finiteness of the fourth order moments and this is not natural in the CLT setting. Paulauskas and Stieve (1990) proved the CLT in D under second moment assumption only. They introduced condition (6) and showed that if (4) is satisfied with $f(u)=cu^\alpha$, for some $\alpha>1/6$, and (6) is satisfied with $g(u)=cu^\beta$, for some $\beta>0$, then X satisfies the CLT in D. In their proofs Hahn (1978) and Paulauskas and Stieve (1990) used the tightness criterion given in Theorem 15.6 of Billingsley (1968). Refinements of this criterion given in Section 1 lead to more precise sufficient conditions.

Theorem 2.1. (Bloznelis and Paulauskas (1993)). *Assume that, for some $\alpha>0$, X satisfies (4) with the function $f(u)=\log^{-2,5-\alpha}(u)$ and that (5) is satisfied with the function $g(u)=\log^{-5-\alpha}(u)$. Then X satisfies the CLT in D.*

Theorem 2.1 improves the result of Hahn (1978). The following theorem provides more precise sufficient conditions in terms of truncated moments (6).

Theorem 2.2. (Bloznelis and Paulauskas (1993)). *Assume that, for some $\alpha>0$, X satisfies (4) with the function $f(u)=\log^{-4,5-\alpha}(u)$ and that (6) is satisfied with the function $g(u)=\log^{-5-\alpha}(u)$. Then X satisfies the CLT in D.*

Bloznelis and Paulauskas (1993) construct examples which show that the power of the logarithmic term in (5) and (6) in Theorems 2.1 and 2.2 is close to the optimal one, since this power being less or equal to 4 does not provide the CLT.

An application of the tightness criterion due to Bezandry and Fernique (1992) gives stronger results.

Theorem 2.3. *(Bezandry and Fernique (1992)). Let $X=\{X(t),t\in[0,1]\}$ be a centered stochastic process defined on the probability space $(\Omega,\mathbf{P})$. Assume that there exist real continuous increasing functions δ,η and θ on $[0,1]$ such that $\delta(0)=\eta(0)=\theta(0)=0$, θ is concave,*

$$\mathbf{E}(X(s)-X(t))^2 \le \delta^2(t-s),\ \mathbf{E}X^2(0)<\infty,$$

$$\mathbf{E}(X(s)-X(t))^2\wedge(X(t)-X(u))^2\mathbb{I}_A \le \eta^2(u-s)\theta(\mathbf{P}(A)),$$

for all $0\le s\le t\le u\le 1$ and all measurable $A\in\Omega$, and

$$\int_0^1 u^{-5/4}[\log(1+1/u)]^{1/4}\delta(u)du<\infty,$$

$$\int_0^1 u^{-3/2}\theta^{1/2}(u\log(1+1/u)/\log 2)\eta(u)du<\infty.$$

Then X has a version, say X', with sample paths in D and X' satisfies the CLT in D.

The integral conditions of the theorem contain logarithmic factors which in some cases (e.g., where θ is a power function) are superfluous. In what follows we present the central limit theorem which was proved independently by Bloznelis and Paulauskas (1994a) and Fernique (1994).

Theorem 2.4. *Assume that $p, q \geq 2$. Let f, g be nonnegative functions on $[0, +\infty)$ which are nondecreasing near 0 and let F be increasing continuous function on $[0,1]$. Let $X(t)$ be a random process with mean 0, finite second moments, and sample paths in D satisfying*

$$\mathbf{E}|X(s) - X(t)|^p \wedge |X(t) - X(u)|^p \leq f(F(u) - F(s)), \tag{7}$$

$$\mathbf{E}|X(s) - X(t)|^q \leq g(F(u) - F(s)), \tag{8}$$

for all $0 \leq s \leq t \leq u \leq 1,\ u - s$ small and

$$\int_0 f^{1/p}(u) u^{-1-1/p} du < \infty, \tag{9}$$

$$\int_0 g^{1/q}(u) u^{-1-1/(2q)} du < \infty. \tag{10}$$

Then X satisfies CLT in D.

Bloznelis and Paulauskas (1994 a) provides an example, see also Hahn (1977 a,b), Hahn and Klass (1977) and Fernique (1994), which demonstrates that the condition (9) is optimal in the following sense. Let f be a continuous nondecreasing positive function on $(0,1]$ satisfying the condition:

$$\int_0 f^{1/p}(u) u^{-1-1/p} du = \infty$$

and very weak additional condition: there exist positive constants K and α such that for all $0 < x < y \leq 1$,

$$f(x) x^{-\alpha} \leq K f(y) y^{-\alpha}.$$

Then there exists a stochastically continuous process X with sample paths in D such that (7) holds, but X does not satisfy the CLT in D. An interesting question (the answer to which is unknown to the authors) is if the condition (10) is optimal too.

We shall briefly mention the major steps of the proof of Theorem 2.4. Firstly, we apply the tightness criteria due to Bezandry and Fernique (1992), see Theorem 1.3 above, to prove the weak convergence of $\mathcal{L}(S_n)$ in the case where the processes $X_1, X_2, \ldots$ have symmetric distribution. Then we apply an analogue of the desymmetrization lemma from Jain and Marcus (1975). Observe that the desymmetrization lemma from Jain and Marcus (1975) cannot be applied immediately, since the addition is not continuous in D.

Lemma 2.1. (Bloznelis and Paulauskas (1994a)). *Let $\{X_i, i \geq 1\}$ be a sequence of stochastically continuous random processes with sample paths in D and let $\overline{X}_i$ be independent copies of $X_i, i = 1, 2, \ldots$ Assume that the finite-dimensional distributions of $\{X_i, i \geq 1\}$ converge to those of some continuous random process. Assume that $\{X_n^* = X_n - \overline{X}_n, n \geq 1\}$ converges weakly to a sample continuous random process. If, moreover,*

$$\forall \varepsilon > 0, \forall \eta > 0, \exists \delta > 0 : |t - s| < \delta \Longrightarrow \mathbf{P}\{|X_i(t) - X_i(s)| > \varepsilon\} < \eta, \forall i \in N,$$

then the sequence $\{X_n, n \geq 1\}$ is weakly convergent.

Another application of the tightness criterion from Theorem 1.3 is the CLT in D in the case where the limiting process is stable. It is well-known, see, e.g. Feller (1971), that if a random variable Z is in the domain of normal attraction of a p-stable law, then the weak p-th moment

$$\Lambda_p(Z) = \sup_{t>0} t^p \mathbf{P}(|Z| > t)$$

is finite, whereas $\mathbf{E}|Z|^p = +\infty$. Therefore, we shall formulate sufficient conditions in terms of weak moments of increments $\Lambda_p(X(u) - X(s))$ and $\Lambda_p(|X(s) - X(t)| \wedge |X(t) - X(u)|)$.

Let f, g be nonnegative increasing functions such that $f(0) = g(0) = 0$. Assume that

$$\Lambda_p(X(s) - X(u)) \le g(u - s), \qquad 0 \le s \le u \le 1, \tag{11}$$

$$\Lambda_p(|X(s) - X(t)| \wedge |X(t) - X(u)|) \le f(u - s), \qquad 0 \le s \le t \le u \le 1. \tag{12}$$

Theorem 2.5. (Bloznelis (1996)). *Let $1 < p < 2$. Suppose that $X = \{X(t), t \in [0, 1]\}$ is random process such that*

(i) the k-dimensional finite joint distributions of X are in the domain of normal attraction of a strictly p-stable measure on R^k for all $1 \le k < \infty$;

(ii) condition (12) holds with f satisfying (9) and (11) holds with g such that

$$\int_0 u^{-1-1/(2p)} g^{1/p}(u) du < \infty. \tag{13}$$

Then the process X has a version X' with sample paths in $D[0, 1]$ and the sequence $\mathcal{L}(n^{-1/p}(X'_1 + \cdots + X'_n))$ converges weakly in D to the distribution of a stable process. Here $X'_i, i = 1, 2, \ldots$ denote independent copies of X'.

In Bloznelis (1996) an example is given which shows that condition (9) is close to the optimal for $1 < p < 2$. Again the answer to the question whether (13) is optimal is unknown.

3. Two examples

Example 1. Theorem 2.4 can be applied to study the asymptotic strength distribution of fiber bundle in the classical fiber bundle model introduced by H.Daniels (1945). It is interesting to note that at that time there was no rigorous theory of weak convergence of stochastic processes and only three decades later it was realized that the asymptotic normality of the strength of a fiber bundle can be obtained as a consequence of the CLT in the Skorokhod space D.

A classical fiber bundle consists of n parallel continuous fibers. The bundle is clamped at both ends and elongated by increasing the distance between the clamps. In the model equal load sharing is assumed, which means that all remaining fibers, at any stage, equally share the total applied load. Let t be the nominal bundle strain, i.e., $t = (L - L_0)/L_0$, where L_0 is the length of the unstretched and unloaded bundle and L is the bundle length after elongation. The problem is to characterize the maximum tensile load that the bundle will sustain in terms of the probabilistic and mechanical characteristics of the individual fibers.

Let $Y_i(t)$ denote the force carried by the i-th fiber when the bundle strain is t. Assume that $Y_i(t)$ is given by the expression

$$Y_i(t) = q(t, \theta_i)\mathbb{I}\{\xi_i > t\},$$

where q is a fixed load strain function depending on a random vector θ_i, ξ_i is a non-negative random variable denoting the bundle strain at which fiber i breaks.

The total force supported by the bundle is the sum of the forces carried by the individual fibers. The bundle load is this total force divided by the number of fibers

$$Q_n(t) = n^{-1} \sum_{i=1}^{n} Y_i(t), \quad t \geq 0.$$

Two characteristics are important for physical applications: the bundle strength which is defined as

$$Q_n^*(t) = \sup\{Q_n(t) : t \geq 0\}$$

and the standardized bundle strength

$$W_n = n^{1/2}(Q_n^* - \mu_{\max}),$$

where $\mu_{\max} = \sup\{\mu(t) : t \geq 0\}$ and $\mu(t) = \mathbf{E}Y_i(t), t \geq 0$. In the first step of the asymptotic study of the quantities Q_n^* and W_n one proves the weak convergence of normalized and centered sums

$$S_n = n^{-1/2} \sum_{i=1}^{n} (Y_i(t) - \mu(t))$$

to a limiting Gaussian process. The second step is to apply the continuous mapping theorem. We shall deal only with the weak convergence of S_n. For the second step we refer to Phoenix and Taylor (1973) and Harlow and Yukich (1993).

We assume that the function q satisfies the Lipschitz condition in the first argument, i.e., there exists a constant $c > 0$ such that

$$|q(t_1, \theta) - q(t_2, \theta)| \leq c|t_1 - t_2|,$$

for all $\theta \in R$ and all $t_1, t_2 \geq 0$ and that $q(0, \theta) \equiv 0$. It is easy to show that if the distribution function of ξ is continuous, then the processes $\{S_n(t), t \geq 0\}, n = 1, 2, \dots$ converge weakly in $D[0, \infty]$ to a sample continuous Gaussian process, say Z, see, e.g., Bloznelis and Paulauskas (1994 c). Since, in fact, we are interested in the limiting behaviour of the distributions of the supremum type functionals, the limit theorem in $D[0, \infty)$ is not an appropriate tool to treat such problems because the supremum functional is not continuous on $D[0, \infty)$ and thus the continuous mapping theorem cannot be applied. This theorem would be applicable if, e.g., we prove that the following statement **C** holds:

C *: there exists a monotone increasing continuous function* $V : [0, \infty] \to [0, 1]$ *with* $V(0) = 0$ *and* $V(\infty) = 1$ *such that the process* $YV := \{Y(V^{-1}(t)) : t \in [0, 1]\}$ *satisfies the CLT in* D.

Applying Theorem 2.4 we obtain the following sufficient condition for **C** to hold.

Proposition 3.1. (Bloznelis and Paulauskas (1994 c)). *Assume that the distribution function of* ξ *is continuous and that for some* $\alpha > 0$,

$$\lim_{t \to +\infty} \mathbf{P}\{\xi > t\} t^2 \log t (\log \log t)^{3+\alpha} = 0.$$

Then **C** *holds.*

If conditions of the proposition are satisfied, then an application of the continuous mapping theorem, see Proposition 4 in Phoenix and Taylor (1973), gives the weak convergence $\mathcal{L}(W_n) \Longrightarrow \mathcal{L}(W)$, where the random variable $W := \sup_{t:\mu(t)=\mu_{\max}} Z(t)$. This improves a result of Harlow and Yukich (1993) who showed the weak convergence of $\mathcal{L}(W_n)$ under a bit stronger condition $\mathbf{E}\xi^{2+\alpha} < \infty, \alpha > 0$.

Example 2. We consider the weak convergence in D of weighted empirical processes

$$F_n(t) = n^{-1/p} \sum_{i=1}^{n} w(t) V_i \mathbb{I}\{U_i \le t\}, \quad n \ge 1.$$

Here $U_1, U_2, \dots$ are i.i.d. random variables uniformly distributed in $[0,1]$, $V_1, V_2, \dots$ are i.i.d. random variables from the domain of the normal attraction of a p-stable distribution, $1 < p < 2$ and $w : [0,1] \to [0, +\infty)$, $w(0) = 0$ is a weight function. We shall assume that the sequences $\{U_i, i = 1, 2, \dots\}$ and $\{V_i, i = 1, 2, \dots\}$ are independent. The problem is to describe those functions w for which the sequence $\mathcal{L}(F_n), n = 1, 2, \dots$ converges weakly in D to the distribution of a stable process (weighted p-stable Lévy motion). An application of Theorem 2.5 gives the following result.

Proposition 3.2. (Bloznelis (1996)). *Assume that $w(t) = t^{-1/p} m(t)$, where m is positive, continuous and nondecreasing function. If, for some $\delta > 1/p$, $m(t) = O(\log^{-\delta}(t^{-1}))$ as $t \to 0$, $t > 0$ then the sequence $\mathcal{L}(F_n), n = 1, 2, \dots$ is weakly convergent.*

4. Rate of convergence

In this section we consider the estimation of the rate of convergence in the CLT in D on the class of sets $\{x \in D : \sup_t |x(t)| \le a\}, a \ge 0$. There is a number of papers devoted to the estimates of the rate of convergence in the CLT for Banach space valued random variables on different classes of sets, e.g., on the class of balls with a fixed center, etc., see, e.g., Paulauskas and Račkauskas (1989), Bentkus et al (1991) and references therein. The rate of convergence in the CLT in D was considered by Paulauskas and Juknevičiene (1988), Paulauskas and Stieve (1990), Paulauskas (1990), Bloznelis (1997).

Define the functional $\| \cdot \| : D \to R$, by $\|x\| = \sup_{t \in [0,1]} |x(t)|, x \in D$. Limit theorems for this functional are important for statistical applications. For example, if $X(t) = \mathbb{I}\{U \le t\} - \mathbf{E}\mathbb{I}\{U \le t\}$, where U is a random variable uniformly distributed in $[0,1]$, then $\|S_n\|$ is the Kolmogorov-Smirnov statistics. The following result estimates the uniform (Kolmogorov's) distance between the distributions $\mathcal{L}(\|S_n\|)$ and $\mathcal{L}(\|Y\|)$ in the case where X is a D-valued random variable (càdlàg process), $X_i, i = 1, 2, \dots$ are i.i.d. copies of X and $S_n = n^{-1/2}(X_1 + \dots + X_n) \Longrightarrow Y$. Here Y denotes the limiting Gaussian process.

Theorem 4.1. (Bloznelis (1997)). *Let $p, q \ge 2$. Let X be a centered stochastically continuous càdlàg process. Assume that for some $\alpha > 0$, the process X satisfies conditions (7) and (8) with the functions $f(u) = c \cdot u^{1+\alpha}$ and $g(u) = c \cdot u^{1/2+\alpha}$, respectively. Then there exists a constant $C = C(\mathcal{L}(Y), p, q, \alpha)$ such that for each $r \ge 0$*

$$|\mathbf{P}(\|S_n\| \le r) - \mathbf{P}(\|Y\| \le r)| \le C(1 + \mathbf{E}\|X\|^3) n^{-1/6} \log^{2/3}(n).$$

More general result as well as the non-uniform estimate,

$$\sup_{r>0} (1 + r^3) |\mathbf{P}(\|S_n\| \le r) - \mathbf{P}(\|Y\| \le r)| = O(n^{-1/6} \ln^2 n),$$

are given in Bloznelis (1997). We only mention that the proof of Theorem 4.1 is based on the finite-dimensional approximation method developed by E.Gine, V.Bentkus, V.Paulauskas, Račkauskas and others, see e.g., Paulauskas and Račkauskas (1989), V.Bentkus et al. (1991). This method in combination with a useful lemma given in Sakhanenko (1988) allows us to obtain an estimate of the convergence rate in Theorem 2.5 in a special case where $X(t) = V \mathbb{I}\{U \le t\}$ and where V is a random variable from the domain of normal attraction of a strictly p-stable random variable, say Z, for $1 < p < 2$. Let $X_1', X_2', \dots$ be independent copies of the process X and put $S_n = n^{-1/p}(X_1' + \dots + X_n')$.

Proposition 4.1. (Bloznelis (1991)). *Assume that* $\mathbf{E}V = 0$, *and that*

$$\int_R x^2(\mathcal{L}(V) - \mathcal{L}(Z))(dx) = 0, \qquad \nu := \int_R |X|^{1+p}|\mathcal{L}(V) - \mathcal{L}(Z)|(dx) < \infty.$$

Then $\mathcal{L}(S_n)$ *converges weakly in* D *to the distribution of* p*-stable Lévy motion,* $W = \{W(t),\ t \in [0,1]\}$ *such that* $\mathcal{L}(W(1)) = \mathcal{L}(Z)$ *and for all* $r \geq 0$,

$$|\mathbf{P}(\|S_n\| \leq r) - \mathbf{P}(\|W\| \leq r)| \leq C_2 \max\{n^{-1/(2p+1)}, \nu^{1/(p+1)} n^{-1/(p(p+1))}\},$$

where C_2 *is a constant which depends on* $\mathcal{L}(Z)$.

5. Central limit theorem in D_k

Let $D_k \equiv D_k[0,1]$ denote the Skorokhod space of k-variate càdlàg functions on $[0,1]^k$. For details about the space D_k endowed with the Skorokhod topology we refer to Neuhaus (1971) and Straf (1972).

For a random process $X = \{X(\mathbf{t}), \mathbf{t} \in [0,1]^k\}, k \geq 1$ define

$$\Delta^{(i)}_{(a,b]} X(\mathbf{u}) = X(u_1, \dots, u_{i-1}, b, u_{i+1}, \dots, u_k) - X(u_1, \dots, u_{i-1}, a, u_{i+1}, \dots, u_k),$$
$$\mathbf{u} = (u_1, \dots, u_k) \in [0,1]^k,\ 1 \leq i \leq k,\ a, b \in [0,1].$$

A rectangle B in the unit cube $T \equiv [0,1]^k$ is a subset of T of the form

$$(s, \mathbf{t}] = \prod_{i=1}^{k} (s_i, t_i],\ \mathbf{s} = (s_1, \dots, s_k),\ \mathbf{t} = (t_1, \dots, t_k) \in T;$$

the i-th face of B is $\prod_{j \neq i}(s_j, t_j]$. Disjoint rectangles B and C are neighbours if they abut and have the same i-th face for some i. For a rectangle $B = (s, \mathbf{t}]$ let

$$X[B] = \Delta^{(1)}_{(s_1,t_1]} \dots \Delta^{(k)}_{(s_k,t_k]} X(\mathbf{u})$$

be the increment of X around B; $X[\cdot]$ is a random finitely additive function on rectangles. The set $LB(T) = \{(t_1, \dots, t_k) \in T : t_i = 0 \text{ for some } i\}$ is called the lower boundary of T, and $UB(T) = \{(t_1, \dots, t_k) \in T : t_i = 1 \text{ for some } i\}$ is called the upper boundary of T.

We shall give an extension to the k-variate case of the tightness criterion due to Bezandry and Fernique (1992).

Let $(\Omega, \mathcal{F}, P)$ denote a probability space on which the random elements under consideration are defined. We say that random process $X = \{X(\mathbf{t}), \mathbf{t} \in T\}$ satisfies *condition* $(\mathbf{D})$ on the unit cube T if there exist positive nondecreasing functions $f_1, \dots, f_m, \theta_1, \dots, \theta_m$, additionally θ_i is convex, $1 \leq i \leq m$, and measures $F_1, \dots, F_m$ on T with continuous marginal distributions such that for all neighbouring rectangles $B, C \subset T$ and each $A \in \mathcal{F}$

$$\mathbf{E}(|X[B]| \wedge |X[C]|)\mathbb{I}_A \leq \sum_{i=1}^{m} f_i(F_i(B \cup C))\theta_i(P(A)) \tag{14}$$

and functions f_i, θ_i satisfy

$$\int_0^{\varepsilon} (u)^{-2} \log^{k-1}(1 + u^{-1}) \sum_{i=1}^{m} f_i(u)\theta_i(u) du < \infty. \tag{15}$$

Condition (**D**) is a multidimensional analogue of the corresponding one formulated in Bezandry and Fernique (1992).

We say that the sequence $\{X_n, n \geq 1\}$ satisfies *condition* (**D**) if each X_n satisfies (**D**) with the same functions f_i, θ_i and measures $F_i, 1 \leq i \leq m$.

Random process X with sample paths in D_k is said to be continuous at the upper boundary of T if for each $i = 1, 2, \dots, k$,

$$\lim_{a \uparrow 1} \sup_{s \in [0,1]^k} \Delta^{(i)}_{(a,1]} X(\mathbf{s}) = 0 \text{ with probability 1.} \tag{16}$$

Let $\mathcal{U}$ be a collection of subsets of T of the form $\mathcal{U} = U_1 \times \dots \times U_k$, where each $U_i \subset [0,1]$ contains zero and one and has a countable complement.

Let $X_1, X_2, \dots$ be a sequence of random processes with sample paths in D_k. We shall assume that X_n vanishes along the lower boundary of T, $n = 1, 2, \dots$, i.e., that

$$P(X_n(\mathbf{t}) = 0) = 1 \text{ for all } \mathbf{t} \in LB(T), \quad n = 1, 2, \dots. \tag{17}$$

Theorem 5.1. (Bloznelis and Paulauskas (1994 d) *Let $X_n, n \geq 1$ and X be random processes with sample paths in D_k and suppose that X is continuous at the upper boundary of T. Assume that the sequence $\{X_n, n \geq 1\}$ satisfies condition* (**D**) *and each X_n satisfies (17). If for some $U \in \mathcal{U}$ and all choices $\mathbf{t}^1, \dots, \mathbf{t}^r \in U, r \geq 1$,*

$$\mathcal{L}(X_n(\mathbf{t}^1), \dots, X_n(\mathbf{t}^r)) \to \mathcal{L}(X(\mathbf{t}^1), \dots, X(\mathbf{t}^r)) \text{ as } n \to +\infty,$$

then $X_n \Longrightarrow X$ (converges weakly in D_k).

Condition (16) appears in Bickel and Wichura (1971) and may be viewed as a multidimensional analogue of the condition $P\{X(1) = X(1-)\} = 1$ of Theorems 15.4, 15.6 of Billingsley (1968).

Let $X, X_1, X_2, \dots$ be centered independent identically distributed random processes with sample paths in D_k. Denote $S_n = n^{-1/2}(X_1 + \dots + X_n)$. A random process X is said to satisfy the CLT in D_k if the distributions of S_n converge weakly to a Gaussian distribution on D_k.

Applying Theorem 5.1 we obtain the following sufficient conditions for X to satisfy the CLT in D_k.

Theorem 5.2. (Bloznelis and Paulauskas (1994 d)). *Let $p, q \geq 2$ and $k \geq 1$. Let $X = \{X(\mathbf{t}), \mathbf{t} \in T\}$ be a random process with $\mathbf{E}X(\mathbf{t}) = 0$, $\mathbf{E}X^2(\mathbf{t}) < \infty$ for each $\mathbf{t} \in T$. Assume that X vanishes along the lower boundary of T, i.e., that it satisfies (17). Assume that there exist nondecreasing non-negative functions f, g and finite measures F, G on T with continuous marginals such that for all neighbouring rectangles $B, C \subset T$*

$$\mathbf{E}(|X[B]| \wedge |X[C]|)^p \leq f(F(B \cup C)), \tag{18}$$

$$\mathbf{E}|X[B]|^q \leq g(G(B)) \tag{19}$$

and for some $\varepsilon > 0$

$$\int_0^\varepsilon (u)^{-1-1/p} f^{1/p}(u) \log^{k-1}(u^{-1}) du < \infty, \tag{20}$$

$$\int_0^\varepsilon (u)^{-1-1/(2q)} g^{1/q}(u) \log^{k-1}(u^{-1}) du < \infty. \tag{21}$$

Then X has a version X' with sample paths in D_k and X' satisfies the CLT in D_k.

Note that the random process X is stochastically continuous by (19), (21). For one-parameter processes ($k = 1$) Theorem 5.1 coincides with Theorem 2.4. Condition (17) which has appeared yet in Chentsov (1956 a) and Bickel and Wicura (1971) (and is restrictive for $k \geq 2$) could be explained by the fact that (18) and (19) do not control the behaviour of X on the boundary of T because the measures F and G have continuous marginals.

As in the case $k = 1$ condition (20) is optimal in the following sense. Let f be a nondecreasing positive function on $[0,1]$ for which (20) fails and such that there exist positive constants K and α such that for all $0 < x < y \leq 1$

$$x^{-\alpha} f(x) \leq K y^{-\alpha} f(y).$$

Then there exists a stochastically continuous process X with sample paths in D_k such that (17) and (18) hold but X does not satisfy the CLT in D_k.

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