On five- and six-sided rational surface patches

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Abstract

We present a new construction of 5- and 6-sided rational Sabin and Hosaka–Kimura like surface patches. They are based on a well known method of algebraic geometry – blowing up base points. Corresponding 5- and 6-sided patches are more efficient as produced by the methods known before. A special attention is paid to the 5-sided patches. A relationship between various approaches and the properties of corresponding 5-sided patches are explained from a point of view of algebraic geometry. We also describe some specific properties of the developed 6-sided patches.

Keywords: multisided surface patch, geometric continuity, control point, base point.

1 Introduction

Sabin (1983, 1991) and Hosaka & Kimura (1984) defined 5- and 6-sided patches, suitable for an inclusion in to the B-spline surfaces. The domains of their patches are nonplanar regions in 5- and 6-dimensional space respectively. The boundary of Sabin patches are conics (Bézier curves of degree $n = 2$) and boundary curves of Hosaka–Kimura patches are cubics (Bézier curves of degree $n = 3$). Zheng & Ball (1997) extended their approach to a case, when boundary contains Bézier curves of arbitrary degree n. Five-sided areas can be parametrizied by the rational functions, so 5-sided Sabin and Hosaka–Kimura patches are rational. For a parametrization of 6-sided area square roots are involved. It is a disadvantage of 6-sided Sabin and Hosaka–Kimura patches – they are nonrational. Loop & De Rose (1989) introduced S-patches. They are rational m-sided patches defined over a planar m-gon for any $m \geq 3$. Later on Loop & De Rose (1990) used S-patches for a construction of rational m-sided

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Sabin (boundary curves are conics) and Hosaka–Kimura (boundary curves are cubics) like patches for arbitrary $m \geq 3$. Karčiauskas (1999) constructed rational m-sided Sabin and Hosaka–Kimura like surface patches over planar domain for any $m \geq 3$, $m \neq 4$, bounded by Bézier curves of arbitrary degree n. This approach produces the patches of lower degree than derived by the previous methods of Sabin (1983), Hosaka-Kimura (1984), Loop & De Rose (1990), Zheng & Ball (1997). Only in case $m = 3$, $n = 2$ they are of degree 4 as the original Sabin (1983) triangular patch, but possess some additional useful properties.

Warren (1992) proposed a new method for creating rational multisided patches. His idea was to use basis functions vanishing simultaneously on some vertices of a domain triangle or rectangle – if we approach by a different lines a vertex, where all basis functions are zero, the limit points draw a space curve. In algebraic geometry this procedure is called blowing-up base points. In (Karčiauskas, 1998) 5- and 6-sided patches were investigated in more details than in the original work of Warren and the essential improvements were made for five-sided patches. One goal of this paper is to construct for arbitrary n the 5- and 6-sided Sabin and Hosaka–Kimura like patches, defined over the blown-up triangle and to show that they are most efficient. This construction relies on the patches developed in (Karčiauskas, 1998). Another goal of this paper is to describe birational geometry of two models of the blown-up plane. Some results from this description are not used right now. But we still present them to make more clear a structure of developed patches. The original paper of Warren (1992) and (Karčiauskas, 1998) can be treated as a quick introduction to the method of blowing-up base points, applied to CAGD. Thoroughly the methods of algebraic geometry are explained for example in the books (Shafarevic, 1974) and (Hartshorne, 1977).

This paper is organized as follows. In Section 2 we define various T-patches and describe their main properties. Section 3 is devoted to birational geometry of blown-up plane. There also the connections between various types of patches are described. In Section 4 the T-patches are used for constructing Sabin and Hosaka–Kimura-like (SHK) patches. These surface patches behave along their boundaries like rectangular Bézier patches and are suitable for an inclusion into B-spline surfaces.

We use following notations. A point $(x_1, \ldots, x_d) = (y_1/y_0, \ldots, y_d/y_0)$ in space $\mathbb{R}^d$ is also represented in a homogeneous form $(y_0 : y_1 : \ldots : y_d)$. Similarlar notations are used for the maps: rational map $(g_1/g_0, \ldots, g_d/g_0)$ is often written in a homogeneous form $(g_0 : g_1 : \ldots : g_d)$. Bernstein polynomials $\binom{n}{j}(1-u)^{n-j}u^j$ are denoted by $B_j^n(u)$. For any set of functions f_q, labeled by a graph $\mathcal{L}$, we set $\dot{f}_q = f_q / \sum_{q' \in \mathcal{L}} f_{q'}$, $q \in \mathcal{L}$.

2 T-patches

2.1 A combinatorial structure and definition

Five- and six-sided patches, considered in this paper, are controlled by the same control point nets as T-patches defined in (Karčiauskas, 1999). Though the basis functions are different we also call them T-patches. This does not lead to confusion since a situation is clear from the context.

Let $W_0, W_1, \ldots, W_{m-1}, W_m = W_0$ be the vertices of a convex m-gon (in this paper $m = 5, 6$); let W be its inner point and let n be a fixed natural number. For each triangle with the vertices W, W_s, W_{s+1}, $0 \leq s \leq m-1$, the points

$$T^s_{ij} = \frac{i}{n}W + \frac{j}{n}W_{s+1} + \frac{n-i-j}{n}W_s, \quad i,\, j \geq 0, i+j \leq n,$$

linked together form its standard triangulation. All together they form a triangulation of m-gon. It is convenient to organize the labeling of this triangulation in the following manner.

Let $\mathcal{T}^n$ be a set of all triples (s,i,j), $0 \leq s \leq m-1$, $0 \leq i \leq n$, $0 \leq j \leq n-i$, where triples $(s,i,n-i)$ and $(s+1,i,0)$ are identified (the first index s is treated in a cyclic fashion). There are $mn(n+1)/2+1$ triples in $\mathcal{T}^n$. Without confusing $\mathcal{T}^n$ can be treated as a graph of the triangulation of m-gon. This labeling (graph) is shown in Figure 1. Notice, index s enumerates the triangles and indices i, j come from a labeling of their standard trangulation.

It follows directly from a definition of $\mathcal{T}^n$ graphs, that they have the same symmetry group as a regular m-gon. More exactly, this means following. A map $\mathrm{rot} : \mathcal{T}^n \mapsto \mathcal{T}^n$, given by the formula $\mathrm{rot}((s,i,j)) = (s+1,i,j)$, corresponds to a rotation of a regular m-gon, when a vertex W_s maps to a vertex W_{s+1}. A map $\mathrm{mir} : \mathcal{T}^n \mapsto \mathcal{T}^n$, given by the formula $\mathrm{mir}((s,i,j)) = (-s-1, i, n-i-j)$, corresponds to a mirror symmetry of a regular m-gon, when a vertex W_0 maps to itself. All maps from a symmetry group are the compositions of the maps rot and mir.

Suppose, there are fixed basis functions f_q, $q \in \mathcal{T}^n$, on a domain D.

Definition 1 *A parametric rational* 5- *or* 6-*sided T-patch of order n over domain D is a map $F^n : D \to \mathbb{R}^3$ defined by the formula*

$$F^n(p) = \frac{\sum_{q\in\mathcal{T}^n} w_q P_q f^n_q(p)}{\sum_{q\in\mathcal{T}^n} w_q f^n_q(p)}. \tag{1}$$

The points P_q are called control points of the patch and the numbers w_q

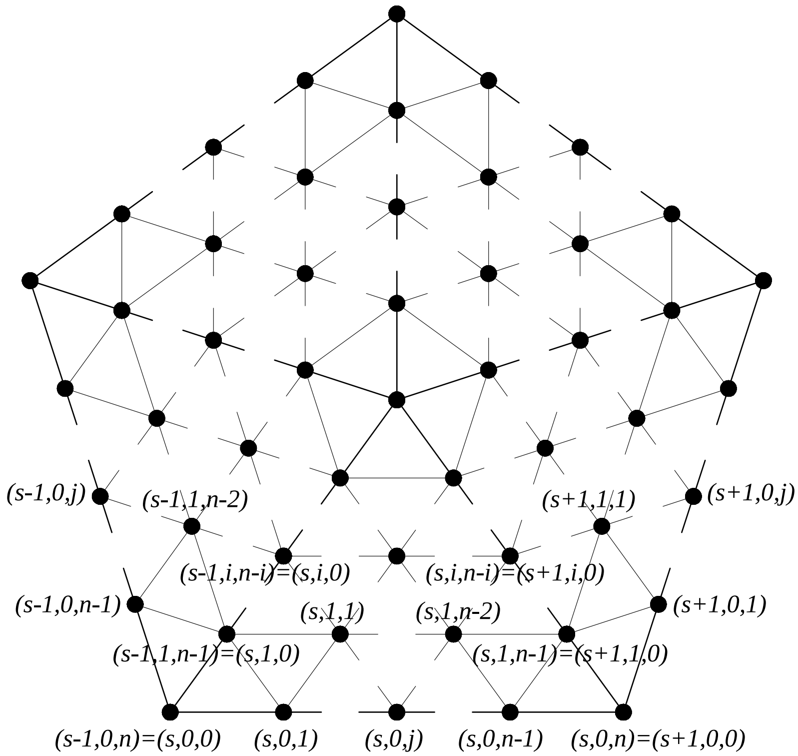

Fig. 1. Combinatorial structure and labeling

are their weights. Geometrically T-patch is understood as the image $F^n(D)$. But without confusing we often consider T-patch as a map (exactly as in Definition 1). It is worth to note we can use the formula (1) also in a case when P_q are the points in $\mathbb{R}^d$, $d \neq 3$. For example, if $d = 2$ we get the maps from a plane to plane, that are useful in a investigation of various properties of T-patches.

2.2 The domains

Blown-up triangle. Let D be a triangle with the vertices V_0, V_1, V_2. For a point p in a plane we denote by $l_0(p)$, $l_1(p)$, $l_2(p)$ its barycentric coordinates respect to the triple V_0, V_1, V_2. If it is convenient to use affine (or homogeneous) coordinates we assume $V_0 = (0,0) = (1:0:0)$, $V_1 = (1,0) = (1:1:0)$, $V_2 = (0,1) = (1:0:1)$. The triangle D is a domain for both 5- and 6-sided patches. Only location of the base points is different: pentagonal patches are defined via basis functions vanishing simultaneously at the vertices V_1, V_2; hexagonal patches are defined via basis functions vanishing simultaneously at all three vertices V_0, V_1, V_2.

Regular domains. We assume $m = 5, 6$ and set $\alpha = 2\pi/m$. By D we denote a regular m-gon in (x,y)-plane with the vertices $R_s = (\cos s\alpha; \sin s\alpha)$, $0 \leq s \leq m-1$. A point of an intersection of the lines $\overline{R_{s-1}R_s}$ and $\overline{R_{s+1}R_{s+2}}$ is denoted by K_s. We set

$$\begin{aligned}
\hat{l}_s &= -x\cos(s\alpha+\tfrac{\alpha}{2}) - y\sin(s\alpha+\tfrac{\alpha}{2}) + \cos\tfrac{\alpha}{2},\ 0\le s\le m-1\ ;\\
\hat{k}_s &= -x\cos(s\alpha) - y\sin(s\alpha) + (1+\cos\alpha)/(2\cos\alpha),\ 0\le s\le m-1.
\end{aligned} \tag{2}$$

It is easy to check that: $\hat{l}_s = 0$ defines a line $\overline{R_sR_{s+1}}$; $\hat{k}_s = 0$ defines a line $\overline{K_{s-1}K_s}$.

2.3 Basis functions and definition

We define various basis functions systems using the following scheme. Suppose there are fixed $m+1$ functions $h_0, h_1, \ldots, h_{m-1}, \bar{h}$. Moreover, suppose for arbitrary n there are fixed positive numbers $k^n_{ij}, 0\le i\le n, 0\le j\le n-i$, satisfying $k^n_{ij} = k^n_{i,n-i-j}$. For $q=(s,i,j)\in\mathcal{T}^n$ the basis functions f^n_q are defined by the formula

$$f^n_q = k^n_{ij} h_s^{n-i-j} h^j_{s+1} \bar{h}^i. \tag{3}$$

This scheme allows to get almost all known basis functions labeled by $\mathcal{T}^n$ graphs: formulas (5) and (7) give the basis functions from (Karčiauskas, 1998); formula (9) give functions from (Karčiauskas, 1997); formula (11) give the functions introduced by Krasauskas (1999); the basis functions of T-patches for $m\ge 5$, introduced by Karčiauskas (1999), can be also defined by the formula (3).

Now we define the systems of basis functions for various type of patches still not specifying the coefficients k^n_{ij}. They are called T^n_5, T^n_6, $\hat{T}^n_5$ and $\hat{T}^n_6$ patches.

- *Five-sided T^n_5-patch over blown-up triangle.* Let

$$\begin{aligned}
&h_0 = l_0^2,\ h_1 = l_0l_1(l_0+l_1),\ h_2 = l_1^2l_2,\ h_3 = l_1l_2^2,\\
&h_4 = l_0l_2(l_0+l_2),\ \bar{h} = l_0l_1l_2.
\end{aligned} \tag{4}$$

  From the formula (3) we get

$$\begin{aligned}
f^n_{0,i,j} &= k^n_{ij} l_0^{2n-i-j} l_1^{i+j} l_2^i (l_0+l_1)^j,\\
f^n_{1,i,j} &= k^n_{ij} l_0^{n-j} l_1^{n+j} l_2^{i+j} (l_0+l_1)^{n-i-j},\\
f^n_{2,i,j} &= k^n_{ij} l_0^{i} l_1^{2n-i-j} l_2^{n+j},\\
f^n_{3,i,j} &= k^n_{ij} l_0^{i+j} l_1^{n-j} l_2^{2n-i-j} (l_0+l_2)^j,\\
f^n_{4,i,j} &= k^n_{ij} l_0^{n+j} l_1^{i} l_2^{n-j} (l_0+l_2)^{n-i-j}.
\end{aligned} \tag{5}$$

- *Six-sided T_6^n-patch over blown-up triangle.* Let

$$h_0 = l_0^2 l_1,\ h_1 = l_0 l_1^2,\ h_2 = l_1^2 l_2,\ h_3 = l_1 l_2^2, \\ h_4 = l_0 l_2^2,\ h_5 = l_0^2 l_2, \bar{h} = l_0 l_1 l_2. \tag{6}$$

From the formula (3) we get

$$f^n_{2r,i,j} = k^n_{ij} l_r^{2n-i-j} l_{r+1}^{n+j} l_{r+2}^i, \quad f^n_{2r+1,i,j} = k^n_{ij} l_r^{n-j} l_{r+1}^{2n-i} l_{r+2}^{i+j}. \tag{7}$$

- *Five-sided $\hat{T}_5^n$-patch over regular pentagon.* Let

$$h_s = \hat{l}_{s+1} \hat{l}_{s+2}^2 \hat{l}_{s+3} \hat{k}_s, \quad s = 0, 1, .., 4,\ \bar{h} = \prod_{s=0}^{4} \hat{l}_s. \tag{8}$$

From the formula (3) we get

$$f^n_{s,i,j} = k^n_{ij} \hat{l}_s^i \hat{l}_{s+1}^{n-j} \hat{l}_{s+2}^{2n-i-j} \hat{l}_{s+3}^{n+j} \hat{l}_{s+4}^{i+j} \hat{k}_s^{n-i-j} \hat{k}_{s+1}^j. \tag{9}$$

- *Six-sided $\hat{T}_6^n$-patch over regular hexagon.* Let

$$h_s = \hat{l}_{s+1} \hat{l}_{s+2}^2 \hat{l}_{s+3}^2 \hat{l}_{s+4}, \quad s = 0, 1, .., 5,\ \bar{h} = \prod_{s=0}^{5} \hat{l}_s. \tag{10}$$

From the formula (3) we get

$$f^n_{s,i,j} = \hat{l}_s^i \hat{l}_{s+1}^{n-j} \hat{l}_{s+2}^{2n-i-j} \hat{l}_{s+3}^{2n-i} \hat{l}_{s+4}^{n+j} \hat{l}_{s+5}^{i+j}. \tag{11}$$

2.4 Basis functions from an algebraic point of view

Many features of T-patches (especially of 5-sided) can be explained using the concept of the basis points from algebraic geometry. Here we give some necessary definitions.

Definition 2 *The function $f(x, y)$ has zero of multiplicity μ at a point $(x_0; y_0)$ if it vanishes at $(x_0; y_0)$ together with all partial derivatives up to the order $\mu - 1$ and at least one partial derivative of order μ does not vanish.*

It is convenient to consider the zeros of the polynomials also at infinity. If we say a polynomial f of degree $\leq n$ has at the infinite point a zero of multiplicity μ, by a definition this means following: polynomial f is represented in the homogeneous coordinates as a homogeneous polynomial of degree n; after projective transformation an infinite point is represented as an affine point and the corresponding polynomial has at this point zero of multiplicity μ. If

$f_i(p) = 0$ for each i then p is called base point of a rational map $(x_0 : x_1 : x_2) \mapsto (f_0(x_0, x_1, x_2), \ldots, f_d(x_0, x_1, x_2))$.

1. T_5^n-patches. The functions defined by the formula (5) are the basis for the polynomials of degree $\leq 3n$ having at the vertices $V_1(1:1:0)$, $V_2(1:0:1)$ and infinity points $(0:1:0)$, $(0:0:1)$ zeros of the multiplicity at least n.
2. T_6^n-patches. The functions defined by the formula (7) are the basis for the polynomials of degree $\leq 3n$ having at the vertices $V_0(1:0:0)$, $V_1(1:1:0)$, $V_2(1:0:1)$ zeros of the multiplicity at least n.
3. $\hat{T}_5^n$-patches. The functions defined by the formula (9) are the basis for the polynomials of degree $\leq 5n$ having at all points K_s zeros of the multiplicity at least $2n$.
3. $\hat{T}_6^n$-patches. The functions defined by the formula (11) are the polynomials of degree $\leq 6n$ having at all points K_s and at three infinite points - the intersections of the pairs of the parallel sides of regular hexagon - zeros of the multiplicity at least n (but are only a part of the basis for these polynomials).

2.5 Properties of T_5^n- and T_6^n-patches

Here we list some important properties of blown-up 5- and 6-sided patches. Most of them were announced in (Karčiauskas, 1998).
1. Linear independence. All systems of basis functions defined in Section 2.3 are linearly independent.
2. Boundary curves. If $k_{0j}^n = \binom{n}{j}$ then a boundary of a patch consist of the rational Bézier curves of degree n with the control points P_{s0j} and the weights w_{s0j}, $0 \leq j \leq n$ (for each s corresponds one boundary curve).
3. Tangent planes along a boundary. Only the functions $f_{s,0,0}^n$, $f_{s,0,1}^n, \ldots, f_{s,0,n}^n$ and $f_{s-1,0,n-1}^n$, $f_{s,1,0}^n, \ldots, f_{s,1,n-1}^n$, $f_{s+1,0,1}^n$ effects a crossderivative along the boundary curve, controlled by the points $P_{s00}, P_{s01}, \ldots, P_{s0n}$.
4. Convex hull property. If all weights w_q are positive the patch lies in a convex hull of the control points. (Sufficient conditions actually are $w_{s00} > 0$, $s = 0, 1, \ldots, m-1$, and $w_q \geq 0$ for the rest $q \in \mathcal{T}^n$.)
5. Implicit degree. Implicit degree of T_5^n-patch is $\leq 5n^2$ and of T_6^n-patch is $\leq 6n^2$.
6. Parametric curves. An image of a general line of a domain is a rational curve of degree n.
7. Order elevation. The coefficients k_{ij}^n can be so recursively defined, that: $k_{0j}^n = \binom{n}{j}$; there exist order elevation procedure; a sequence of the elevated control point nets tends to the patch.
8. Symmetry property. Let $P'_q = P_{\mathrm{rot}(q)}$, $w'_q = w_{\mathrm{rot}(q)}$, $P''_q = P_{\mathrm{mir}(q)}$, $w''_q = w_{\mathrm{mir}(q)}$, $q \in \mathcal{T}^n$. Then the control points and weights P'_q, w'_q and P''_q, w''_q define the same patch as the original control points P_q and weights w_q.

General line in a parameter plane is mapped by a rational map $(g_0 : g_1 : \ldots, g_d)$ of degree k in to a rational curve of degree k. But if a line goes through a base point p of multiplicity at least μ (all polynomials g_i have at p zero of multiplicity at least μ), then an image is a curve of degree $\leq k-\mu$. This simple principle of algebraic geometry is used in (Karčiauskas, 1999) for an efficient plotting of T-patches defined over regular m-gon. So $\hat{T}_5^n$-patch can be represented as a collection of five rational rectangular Bézier patches of bidegree $(3n, 3n)$. For $\hat{T}_6^n$-patch there are two possibilities: standard subdivision of domain hexagon into 6 quadrangles leads to a representation as a collection of six rational rectangular Bézier patches of bidegree $(4n, 4n)$; subdivision of a hexagon into 3 parallelograms gives a collection of three Bézier patches of bidegree $(4n, 4n)$.

For an efficient plotting of blown-up patches we subdivide domain triangle. A subdivision depends if 5- or 6-sided patch is considered. We set $M_s = (V_{s+1} + V_{s+2})/2$, $s = 0, 1, 2$, $M = (V_0 + V_1 + V_2)/3$.

1. Plotting T_5^n-patches. We subdivide domain triangle into three parts: parallelogram $V_0M_2M_0M_1$; triangle $V_1M_0M_2$; triangle $V_2M_1M_0$. A bilinear map bl_0 defined by the formula $\mathrm{bl}_0(u, v) = (V_0(1-u)+M_2u)(1-v)+(M_1(1-u)+M_0u)v$ parametrizies a parallelogram. Moreover, its parametric lines go through the infinite point $(0 : 1 : 0)$ or $(0 : 0 : 1)$. A bilinear map bl_1 defined by the formula $\mathrm{bl}_1(u, v) = V_1(1 - v) + (M_2(1 - u) + M_0u)v$ parametrizies first triangle. Its parametric lines $v = \mathrm{const}$ go through the infinite point $(0 : 0 : 1)$ and the parametric lines $u = \mathrm{const}$ go through the point V_1. Similarly is defined a bilinear map bl_2 parametrizing second triangle. Composing bilinear parametrizations bl_0, bl_1, bl_2 with a map F^n, defining a patch, we get three rectangular Bézier patches of bidegree $(2n, 2n)$. A principle of this presentation is visualized in Figure 2.

2. Plotting T_6^n-patches. We subdivide domain triangle into six triangles: their vertices are V_s, M_{s+2}, M and V_s, M_{s+1}, M, $s = 0, 1, 2$. A rational bilinear map bl_0 defined by the formula

$$\mathrm{bl}_0(u, v) = (V_0(1 - v) + (M_2(1 - u) + (3/2)Mu)v)/(1 + uv/2)$$

parametrizies triangle with the vertices V_0, M_2, M. Moreover, its parametric lines $v = \mathrm{const}$ go through the point V_2 and the parametric lines $u = \mathrm{const}$ go through the point V_0. Similarly are defined the rational bilinear maps bl_s, $s = 1, \ldots, 5$, parametrizing rest five triangles. Composing rational bilinear parametrizations bl_s, $s = 0, 1, \ldots, 5$, with a map F^n, defining a patch, we get six rectangular Bézier patches of bidegree $(2n, 2n)$.

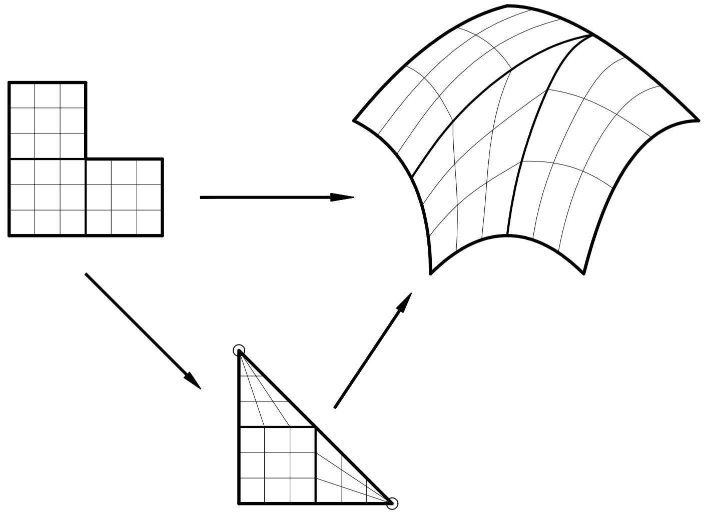

Fig. 2. Plotting T_5^n-patches

3 Connection between blown-up and regular patches

3.1 Connection maps

In this section we show (Proposition 3) that T_5^n- and T_6^n-patches, defined over blown-up triangle, coincide actually with $\hat{T}_5^n$- and $\hat{T}_6^n$-patches, defined over regular pentagon or hexagon. It helps to understand better the geometrical properties of blown-up patches. On the other hand, more complicated definition of T_5^n- and T_6^n-patches leads to some advantages - they can be represented as a collection of rectangular Bézier patches of lower degree (Section 2.6).

It was mentioned after Definition 1 that we will use the formula (1) when $P_q \in \mathbb{R}^d$, $q \in \mathcal{T}^n$, $d \neq 3$. In this case it is sufficient to take $n = 1$. Therefore we simplify some notations: let $P_s = P_{s,0,0}$, $s = 0, 1, \ldots, m-1$, $\bar{P} = P_{0,1,0}$, $k^1 = k^1_{10}$. Moreover, we assume $k^1_{00} = 1$ and all w_q, $q \in \mathcal{T}^1$, are equal to 1. By M we denote barycentric center of domain triangle ($M = (V_0 + V_1 + V_2)/3$). We also set:

if two vertices V_1 and V_2 are blown-up – case of the formula (5) – then

$$\widetilde{V}_0 = V_0,\ \widetilde{V}_1 = V_1,\ \widetilde{V}_2 = V_1,\ \widetilde{V}_3 = V_2,\ \widetilde{V}_4 = V_2; \tag{12}$$

if all vertices V_0, V_1, V_2 are blown-up – case of the formula (7) – then

$$\widetilde{V}_0 = V_0,\ \widetilde{V}_1 = V_1,\ \widetilde{V}_2 = V_1,\ \widetilde{V}_3 = V_2,\ \widetilde{V}_4 = V_2,\ \widetilde{V}_5 = V_0. \tag{13}$$

Now we define some useful connection maps $\mathbb{R}^2 \mapsto \mathbb{R}^2$. They all are defined by the formula (1). Only an input for this formula is different in each case.

- Let: $k^1 = 3(\sqrt{5}+1)/2$; the basis functions are given by the formula (9); $P_s = \widetilde{V}_s$, $s = 0, 1, \ldots, 4$, $\bar{P} = M$, where $\widetilde{V}_s$ are defined by the formula (12).

Corresponding map $\mathbb{R}^2 \mapsto \mathbb{R}^2$ is denoted by H_5.

- Let: $k^1 = 5(\sqrt{5}-1)/2$; the basis functions are given by the formula (5); $P_s = R_s$, $s = 0, 1, \ldots, 4$, $\bar{P}$ is a center of a regular pentagon. Corresponding map $\mathbb{R}^2 \mapsto \mathbb{R}^2$ is denoted by $\tilde{H}_5$.
- Let: $k^1 = 3$; the basis functions are given by the formula (11); $P_s = \widetilde{V}_s$, $s = 0, 1, \ldots, 5$, $\bar{P} = M$, where $\widetilde{V}_s$ are defined by the formula (13). Corresponding map $\mathbb{R}^2 \mapsto \mathbb{R}^2$ is denoted by H_6.

The maps H_5 and $\tilde{H}_5$ are birational transformations of a plane. Moreover, they are inverse to each other. Direct computations give the following formula

$$H_5 = (2\hat{k}_0\hat{k}_2\hat{k}_3 : (\sqrt{5}+3)\hat{l}_3\hat{l}_4\hat{k}_2 : (\sqrt{5}+3)\hat{l}_0\hat{l}_1\hat{k}_3). \tag{14}$$

So H_5 is a rational map of degree 3. The computations also show that $\tilde{H}_5$ is a rational map of degree 2 and H_6 is a rational map of degree 3. A map H_6 is not a birational transformation – generally there are three points (some possibly complex conjugate) mapping to one point.

In order to avoid a confusion we introduce some temporary notations for various kinds of the functions h_s, $\bar{h}$ and the coefficients k_{ij}^n:

$\tilde{h}_s^5 = h_s$, $s = 0, 1, \ldots, 4$, $\tilde{h}_5^5 = \bar{h}$ in a case of the formula (4);
$\tilde{h}_s^6 = h_s$, $s = 0, 1, \ldots, 5$, $\tilde{h}_6^6 = \bar{h}$ in a case of the formula (6);
$\hat{h}_s^5 = h_s$, $s = 0, 1, \ldots, 4$, $\hat{h}_5^5 = \bar{h}$ in a case of the formula (8);
$\hat{h}_s^6 = h_s$, $s = 0, 1, \ldots, 5$, $\hat{h}_6^6 = \bar{h}$ in a case of the formula (10);
$\tilde{k}_{ij}^n = k_{ij}^n$ in case of the formulas (5) or (7);
$\hat{k}_{ij}^n = k_{ij}^n$ in case of the formulas (9) or (11).

Let $a = (\sqrt{5}+1)/2$ (a is known as a *golden section*). Direct computations give the following relations:

$$\frac{\hat{h}_0^5(x,y)}{\tilde{h}_0^5(H_5(x,y))} = \ldots = \frac{\hat{h}_4^5(x,y)}{\tilde{h}_4^5(H_5(x,y))} = \frac{\hat{h}_5^5(x,y)}{a\tilde{h}_5^5(H_5(x,y))}; \tag{15}$$

$$\frac{\hat{h}_0^6(x,y)}{\tilde{h}_0^6(H_6(x,y))} = \ldots = \frac{\hat{h}_5^6(x,y)}{\tilde{h}_5^6(H_6(x,y))} = \frac{\hat{h}_6^6(x,y)}{\tilde{h}_6^6(H_6(x,y))}. \tag{16}$$

From the formulas (15), (16), (3) and (1) we get the following proposition is true.

Proposition 3 *Let $\hat{k}_{ij}^n = a^i\tilde{k}_{ij}^n$, if $m = 5$, and $\hat{k}_{ij}^n = \tilde{k}_{ij}^n$, if $m = 6$. Then T_5^n-patch coincides with the $\hat{T}_5^n$-patch and T_6^n-patch coincides with the $\hat{T}_6^n$-patch.*

This proposition allows to consider blown-up patches as more usual patches over regular domain. It helps to understand easier many features of $\tilde{T}_5^n$- and

$\tilde{T}_6^n$-patches. Take, for example, a third property from Section 2.5: only the functions $f_{s,0,0}^n$, $f_{s,0,1}^n, \ldots, f_{s,0,n}^n$ and $f_{s-1,0,n-1}^n$, $f_{s,1,0}^n, \ldots, f_{s,1,n-1}^n$, $f_{s+1,0,1}^n$ effects a crossderivative along the boundary curve, controlled by the points $P_{s00}, P_{s01}, \ldots, P_{s0n}$. On a regular domain it is obvious: all functions, except of listed above, contain as a factor $\hat{l}_s^\mu$ with $\mu \geq 2$ ($\hat{l}_s$ is defined by the formula(2)).

3.2 Birational geometry of blown-up triangle

It was mentioned in Section 2.5 that T_5^n- and T_6^n-patches are symmetric. It is not obvious since their basis functions are not symmetric in a usual affine plane. $\hat{T}_5^n$- and $\hat{T}_6^n$- patches are symmetric, since their basis functions are symmetric in an euclidian plane (it follows directly from the definitions). So we get from Proposition 3 blown-up patches also are symmetric. Originally it was proved using birational transformations of a domain triangle (in algebraic geometry known as Cremona transformations). By a definition birational transformation means that it has an inverse rational map. Do not forget: blown-up patches are defined not over standard triangle, but over triangle with two or three blown-up vertices. Here we describe the birational transformations of a blown-up plane. They are analog of the usual euclidian transformations of a regular pentagon or hexagon. We additionally assume: $k^1 = 5$ if the functions are defined by the formula (9); $k^1 = 3$ if the functions are defined by the formulas (5) or (7). By O is denoted a center of a regular pentagon or hexagon.

Every symmetry transformation $\widehat{\mathrm{sym}}$ of a regular m-gon is a composition of a rotation of an euclidian plane around O by an angle $2\pi/m$ and of mirror symmetry respect to the line $\overline{R_0 O}$. An analog $\widetilde{\mathrm{sym}}$ of any symmetry transformation $\widehat{\mathrm{sym}}$ is constructed using a following scheme. Let $R_{s'} = \widehat{\mathrm{sym}}(R_s)$. Then $P_s = \widetilde{V}_{s'}$, $\bar{P} = M$, where $\widetilde{V}_s$ are defined by the formula (12) if two vertices V_1, V_2 are blown-up or by the formula (13) if all vertices V_0, V_1, V_2 are blown-up. A birational transformation $\widetilde{\mathrm{sym}}$ of a blown-up plane is defined by the formula (1). The basis functions are given by the formula (5) in a case of two blown-up vertices and by the formula (7) in a case of three blown-up vertices. Generally, $\widetilde{\mathrm{sym}}$ is a birational transformation of degree 2. But some of them are affine transformations (including also identity map). The identity $H_m \circ \widetilde{\mathrm{sym}} = \widehat{\mathrm{sym}} \circ H_m$, $m = 5, 6$, expresses a relationship between standard symmetries and birational symmetries. It is worth to note, that for a pentagon a map $\widehat{\mathrm{sym}}$ can be defined by the formula (1) if $P_s = P_{s'}$, $\bar{P} = O$ and the basis functions are given by the formula (9).

A symmetry of the basis functions f_q^n, defined by the formulas (5) or (7), is expressed by the identities

$$(f_q^n \circ \widetilde{\mathrm{sym}})/f_q^n = (f_{q'}^n \circ \widetilde{\mathrm{sym}})/f_{q'}^n, \tag{17}$$

that are true for any $\widetilde{\mathrm{sym}}$ and $q, q' \in \mathcal{T}^n$. These identities together with the formula (1) gives self contained proof of the symmetry of blown-up patches.

Here we briefly describe some properties of the transformations of a blown-up plane.

Two blown-up points. First let us to note that if we consider projective plane there are four blown-up points – not only the vertices $V_1(1:1:0)$, $V_2(1:0:1)$, but also infinite points $(0:1:0)$ and $(0:0:1)$. The rotation transformation $\widetilde{\mathrm{rot}}$ is given by the formula $\widetilde{\mathrm{rot}} = ((l_0+l_1)(l_2+l_0) : l_0(l_0+l_1) : l_1l_2)$. There are two fixed point of a rotation $\widetilde{\mathrm{rot}}$: $A = (\sqrt{5}-2)V_1 + ((3-\sqrt{5})/2)V_1 + ((3-\sqrt{5})/2)V_2$ and $B = (-\sqrt{5}-2)V_1 + ((3+\sqrt{5})/2)V_1 + ((3+\sqrt{5})/2)V_2$. A point A lies in a domain triangle and $A = H_5(O)$; a map H_5 contracts a circle going through the points K_s to a point B; conversely, a point B is blown-up by the map $\tilde{H}_5$ to that circle. The map H_5 contracts the lines $\overline{R_1R_2}$ and $\overline{R_3R_4}$ to the points V_1 and V_2 respectively; conversely, the vertices V_1 and V_2 are blown-up by the map $\tilde{H}_5$ to those lines. The rotation $\widetilde{\mathrm{rot}}$ contracts a line $\overline{V_0V_1}$ to a point V_1, blows-up V_1 to a line $\overline{V_1V_2}$, contracts that line to a point V_2, blows-up V_2 to a line $\overline{V_2V_0}$, maps that line to a line $\overline{V_0V_1}$. To a standard triangulation of a pentagon, build from the triangles OR_sR_{s+1}, corresponds an invariant respect to $\widetilde{\mathrm{rot}}$ subdivision of a domain triangle (see left scheme in Figure 3). Curves of this subdivision intersect each other at the point A (also at the point B, not displayed in a scheme) and are conics, labeled $1, 2, 3, 4$, or a line, labeled 0. Conic 1 is completely defined by the following conditions: it goes through the points $V_1(1:1:0)$, $(0:0:1)$, A, B and touches a line $\overline{V_0V_1}$ at the vertex V_1. Conic 2 is completely defined by the following conditions: it goes through the points $V_1(1:1:0)$, $(0:1:0)$, $(0:0:1)$, A, B and touches a line $\overline{V_1V_2}$ at the vertex V_1. Similarly are defined the conics 3 and 4.

Three blown-up points. The rotation transformation $\widetilde{\mathrm{rot}}$ is given by the formula $\widetilde{\mathrm{rot}} = (l_0l_1 + l_1l_2 + l_2 + l_0 : l_0l_1 : l_1l_2)$. The point $M = H_6(O)$ is a fixed point of a rotation $\widetilde{\mathrm{rot}}$. The map H_6 contracts the lines $\overline{R_1R_2}$, $\overline{R_3R_4}$, $\overline{R_5R_0}$ to the points V_1, V_2, V_0 respectively. The rotation $\widetilde{\mathrm{rot}}$ contracts a line $\overline{V_0V_1}$ to a point V_1, blows-up V_1 to a line $\overline{V_1V_2}$, contracts that line to a point V_2, blows-up V_2 to a line $\overline{V_2V_0}$, contracts that line to a point V_0, blows-up V_0 to a line $\overline{V_0V_1}$. To a standard triangulation of a hexagon, build from the triangles OR_sR_{s+1}, corresponds an invariant respect to $\widetilde{\mathrm{rot}}$ subdivision of a domain triangle (see right scheme in Figure 3). Curves of this subdivision intersect each other at the point M and are conics, labeled $0, 1, 2, 3, 4, 5$. Conics 0 and 3 are arcs of one conic, completely defined by the following conditions: it goes through the points V_1, V_2, M; touches a line $\overline{V_0V_1}$ at the vertex V_1; touches a line $\overline{V_0V_2}$ at

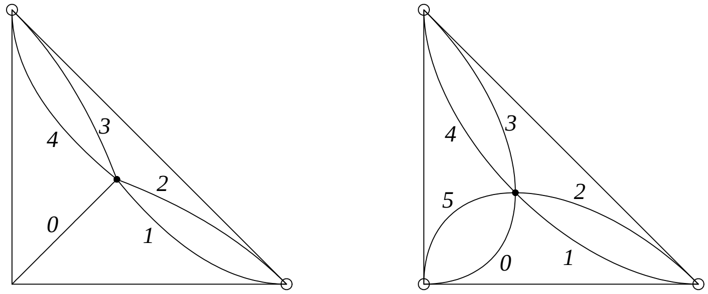

Fig. 3. Symmetric subdivision of blown-up triangles

the vertex V_2. Similarly are defined other conics.

3.3 Relations with the Sabin surface

By SA_5 we denote a surface in $\mathbb{R}^5$ (with coordinates X_1, X_2, X_3, X_4, X_5), defined by equations $X_s - 1 + X_{s+3}X_{s+4} = 0$, $s = 1, 2, \ldots, 5$ (index s is treated in a cyclic fashion). This surface was introduced by Sabin (1983) for a construction of 5-sided patches. A domain of the Sabin patch is a region in SA_5 with $X_s \geq 0$, $s = 1, 2, \ldots, 5$. The same domain was used by Hosaka and Kimura (1984).

Now we define two parametrizations of SA_5. Let $P_0 = (0, 1, 1, 1, 0)$, $P_1 = (0, 0, 1, 1, 1)$, $P_2 = (1, 0, 0, 1, 1)$, $P_3 = (1, 1, 0, 0, 1)$, $P_4 = (1, 1, 1, 0, 0)$, $P_5 = (2/3, 2/3, 2/3, 2/3, 2/3)$ (P_s, $s = 0, 1, \ldots, 4$ are corner points of the Sabin domain). Parametrization maps G_5 and $\tilde{G}_5$ are defined by the formula (1): if basis functions are given by the formula (9), then $k^1 = 3(\sqrt{5}+1)/2$ and a map is denoted by G_5; if basis functions are given by the formula (5), then $k^1 = 3$ and a map is denoted by $\tilde{G}_5$.

Proposition 4 *The maps G_5 and $\tilde{G}_5$ parametrizies the surface SA_5. Moreover, $\tilde{G}_5 \circ H_5 = G_5$.*

Parametrization maps were found using standard methods of algebraic geometry. We also give other explicit formulas for the maps G_5 and $\tilde{G}_5$. They are useful in direct calculations (even with MAPLE package).

$G_5 = (g_0 : g_1 : \ldots : g_5)$, where $g_{s+1} = \hat{h}_s \hat{h}_{s+1} \hat{l}_s (\hat{l}_{s-1} + \hat{l}_{s+1} - \hat{l}_s)^2$, $s = 0, 1, \ldots, 4$, $g_0 = ((5\sqrt{5} - 11)/2) \prod_{i=0}^{4} \hat{h}_i$ ($\hat{l}_s$, $\hat{h}_s$ are defined by the formula (2)).

$\tilde{G}_5 = (g : v(1-v) : (1-u)^2(1-v) : (1-u-v) : (1-u)(1-v)^2 : u(1-u))$, where $g = (1-u)(1-v)$ (assuming the vertices of a domain triangle are $(0, 0)$, $(1, 0)$, $(0, 1)$).

Let us compose a map $\tilde{G}_5$ with a map, defining 5-sided patch on SA_5 (Sabin, 1983). After calculations with MAPLE we get that Sabin patch can be represented as T_5^3-patch or, according to Proposition 3, as $\hat{T}_5^3$-patch. This also means, that basis functions of Sabin patch can be taken symmetric rational cubics in $\mathbb{R}^5$. We do not give here their explicit expressions. Similarly we get that 5-sided Hosaka–Kimura patch can be represented as T_5^4- or $\hat{T}_5^4$-patch.

3.4 Relations with S-patches

S-patches were introduced by Loop and De Rose (1989). Let us consider as their domain only a regular pentagon. In this case an embedding E of a pentagon into hyperplane $X_1 + X_2 + X_3 + X_4 + X_5 = 1$ in $\mathbb{R}^5$, used for a definition of 5-sided S-patch, can be described as follows. Let $g_{s+1} = \hat{l}_{s+1}\hat{l}_{s+2}\hat{l}_{s+3}$, $s = 0, 1, \ldots, 4$, $Q_1 = (1, 0, 0, 0, 0)$, $Q_1 = (0, 1, 0, 0, 0), \ldots, Q_5 = (0, 0, 0, 0, 1)$. Then $E = \sum_{i=1}^5 g_i Q_i / \sum_{i=1}^5 g_i$. An image of E we denote by LR_5.
Let $Y_1 = X_1 - aX_2 + X_3 - aX_4 + X_5 + a - 2$, $Y_2 = X_1 + X_2 - aX_3 + X_4 - aX_5 + a - 2$, $Y_3 = -aX_1 + X_2 + X_3 - aX_4 + X_5 + a - 2$, $Y_4 = X_1 - aX_2 + X_3 + X_4 - aX_5 + a - 2$, $Y_5 = -aX_1 + X_2 - aX_3 + X_4 + X_5 + a - 2$, $Y_0 = (3 - 2a)(X_1 + X_2 + X_3 + X_4 + X_5 + 5a)$, where $a = (\sqrt{5} + 1)/2$. We set $L = (Y_0 : Y_1, \ldots, Y_5)$.

Proposition 5 *A map L maps a surface SA_5 onto a surface LR_5. Moreover, $E \circ \tilde{H}_5 = L \circ \tilde{G}_5$.*

It follows from Proposition 5 that 5-sided S-patches of depth n defined over regular pentagon can be represented as T_5^n- or $\hat{T}_5^n$-patches.

3.5 Model surface for 6-sided patches

Here we describe a surface in $\mathbb{R}^6$ which can be used for a definition of T_6^n- or $\tilde{T}_6^n$-patches. So it is like analog of a surface SA_5.
Let $k^1 = 3$, $P_0 = (1, 0, 0, 0, 0, 0)$, $P_1 = (0, 1, 0, 0, 0, 0), \ldots, P_5 = (0, 0, 0, 0, 0, 1)$, $\bar{P} = (0, 0, 0, 0, 0, 0)$. The maps $G_6, \tilde{G}_6 : \mathbb{R}^2 \mapsto \mathbb{R}^6$ are defined by the formula (1): if the basis functions are given by the formula (11) a map is denoted by G_6; if the basis functions are given by the formulas (7) a map is denoted by $\tilde{G}_6$. Direct computation shows that $G_6 = \tilde{G}_6 \circ H_6$. An image of G_6 (or $\tilde{G}_6$) we denote by W_6. The map $\tilde{G}_6$ is a birational parametrization of W_6. The parametrization G_6 maps generally three points to one. A surface W_6 can be defined by nine quadric equations $g_i = 0$, $i = 1, \ldots, 9$:
$g_i = X_i X_{i+3} - (X_1 + X_2 + X_3 + X_4 + X_5 + X_6 - 1)^2/9,\ i = 1, 2, 3,$
$g_{3+i} = X_i X_{i+2} + X_{i+1}(X_1 + X_2 + X_3 + X_4 + X_5 + X_6 - 1)/3,\ i = 1, 2, \ldots, 6.$
The surface W_6 is a toric manifold. Deeper analysis of this surface from a point of view of toric geometry is given by Krasauskas (1999).

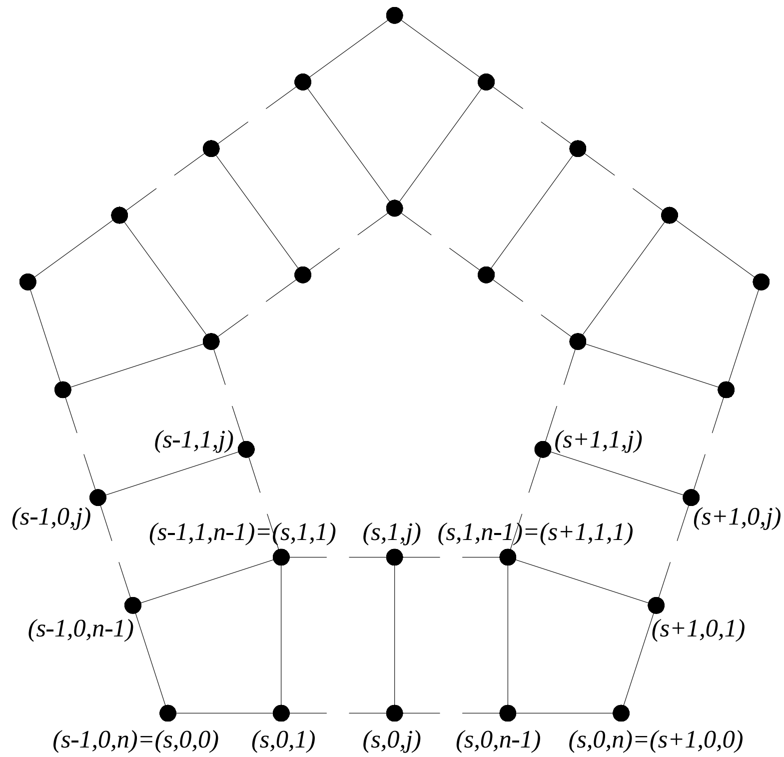

Fig. 4. Combinatorial structure of SHK-patches

4 Sabin and Hosaka–Kimura like patches

4.1 Combinatorial structure and definition

For a simplicity we denote Sabin and Hosaka–Kimura like surface patches as SHK-patches. Control points and weights of m-sided SHK-patches are labeled by the triples $(s, 0, j)$, $(s, 1, k)$, $0 \le s \le m-1$, $0 \le j \le n$, $1 \le k \le n-1$, where $(s-1, 0, n) = (s, 0, 0)$ and $(s-1, 1, n-1) = (s, 1, 1)$ (index s is treated in a cyclic fashion; n is degree of the boundary curves). A graph $\mathcal{H}^n$ is displayed in Figure 4. Note, Sabin pattern has only one inner point $(0, 1, 1) = (1, 1, 1) = \ldots = (m-1, 1, 1)$. The SHK-patches possess following nice property: for each boundary Bézier curve with control points P_{s0j} and weights w_{s0j}, $j = 0, 1, \ldots, n$, a patch can be so reparametrized, that crossderivative along this curve is the same as crossderivative of rectangular Bézier patch with two layers of control points (and corresponding weights) $P_{s,0,0}, P_{s,0,1}, \ldots, P_{s,0,n}$ and $P_{s-1,0,n-1}, P_{s,1,1}, \ldots, P_{s+1,1,1}, P_{s+1,0,1}$. This property is as an unformal definition of SHK-patch. Here we give a definition, that actually outlines a method for a construction of SHK-patches.

Definition 6 *Suppose on the domain D there are fixed the functions g_q, $q \in \mathcal{H}^n$, and there exist the maps* $\mathrm{re}_s : [0,1] \times [0,1] \to D$, $0 \le s \le m-1$, *called* *reparametrization maps, satisfying following conditions:*

(1) $\dot{g}_{s0j}(\mathrm{re}_s(0,u)) = B_j^n(u),\ 0 \le j \le n;$

(2) $\partial\dot{g}_{s0j}(\mathrm{re}_s(t,u))/\partial t\ |_{t=0} = -nB_j^n(u),\ 0 \le j \le n;$

(3) $\partial\dot{g}_{s-1,0,n-1}(\mathrm{re}_s(t,u))/\partial t\ |_{t=0} = nB_0^n(u);$

(4) $\partial\dot{g}_{s+1,0,1}(\mathrm{re}_s(t,u))/\partial t\ |_{t=0} = nB_n^n(u);$

(5) $\partial\dot{g}_{s1j}(\mathrm{re}_s(t,u))/\partial t\ |_{t=0} = nB_j^n(u),\ 1 \le j \le n-1;$

(6) $\dot{g}_q(\mathrm{re}_s(0,u)) = 0$ *for all functions* $\dot{g}_q$, *except of listed* (1), *and* $\partial\dot{g}_q(\mathrm{re}_s(t,u))/\partial t\ |_{t=0}$ *forall* $\dot{g}_q$, *except of listed in* (1) – (5).

A rational SHK-patch of order n is a map $G^n : D \to \mathbb{R}^3$ defined by the formula

$$G^n(p) = \frac{\sum_{q\in\mathcal{H}^n} w_q P_q g_q(p)}{\sum_{q\in\mathcal{H}^n} w_q g_q(p)} \tag{18}$$

Though Definition 6 differs from that in (Karčiauskas, 1999), practically they are equivalent. Obviously we have

$$\partial(B_0^n(t)B_j^n(u))/\partial t\ |_{t=0} = -nB_j^n(u),\ \partial(B_1^n(t)B_j^n(u))/\partial t\ |_{t=0} = nB_j^n(u),$$
$$G^n(p) = \textstyle\sum_{q\in\mathcal{H}^n} w_q P_q \dot{g}_q(p) / \sum_{q\in\mathcal{H}^n} w_q \dot{g}_q(p).$$

These identities implies, that after reparametrization a crossderivative of SHK-patch of order n behaves like a crossderivative of Bézier patch of bidegree (n,n). It allows to join smoothly SHK-patches with adjacent Bézier patches and to build smooth spline surfaces of arbitrary topology.

A following technical lemma from (Karčiauskas, 1999) is essentially used in a proof of the theorem 8.

Lemma 7 *Suppose the functions $h_{ij}(t,u)$,$i = 0,1,\ 0 \le j \le n$, satisfy $h_{0j}(0,u) = B_j^n(u)A(u)$, $h_{1,j}(0,u) = 0$, $\partial h_{0j}/\partial t\ |_{t=0} = B_j^n(u)D(u)$, $\partial h_{1j}/\partial t\ |_{t=0} = nB_j^n(u)A(u)$ for some functions $A(u), D(u)$. Then*

$$\partial(h_{0j}/\sum_{i=0}^{1}\sum_{r=0}^{n} h_{ir})/\partial t\ |_{t=0} = -nB_j^n(u),$$

$$\partial(h_{1j}/\sum_{i=0}^{1}\sum_{r=0}^{n} h_{ir})/\partial t\ |_{t=0} = nB_j^n(u).$$

Basis functions g_q, $q \in \mathcal{H}^n$, for SHK-patches are build as the linear combinations of the basis functions f^n_q, $q \in \mathcal{T}^n$, defined by the formulas (5) or (7). In the main theorem only the functions $f^n_{s,i,j}$, $(s,i,j) \in \mathcal{T}^n$, with $i \leq 1$ are used. We set $k^n_{0j} = \binom{n}{j}$, $k^n_{1j} = 1$. Following notions help to avoid possible confusion indexing basis functions of SHK- and T-patches:
if $q = (s,i,j) \in \mathcal{H}^n$ then $g_{sij} := g_q$;
if $q = (s,i,j) \in \mathcal{T}^n$ then $f_{[s,i,j]} := f^n_q$.

Theorem 8 *Let* $a = n,\ b_j = j\binom{n}{j},\ c_j = (n-j)\binom{n}{j},\ d_j = n\binom{n}{j}$, $1 \leq j \leq n-1$, *and*

- $g_{s00} = f_{[s,0,0]} + af_{[s,1,0]}$;
- $g_{s0j} = f_{[s,0,j]} + b_j f_{[s,1,j-1]} + c_j f_{[s,1,j]},\ 1 \leq j \leq n-1$;
- $g_{s11} = d_1 \sum_{s=0}^{m-1} f_{[s,1,0]}\ if\ n = 2$;

  $g_{s11} = d_1(f_{[s-1,1,n-2]} + f_{[s,1,0]} + f_{[s,1,1]})\ if\ n \geq 3$;

  $g_{s1j} = d_j(f_{[s,1,j-1]} + f_{[s,1,j]}),\ 2 \leq j \leq n-2,\ if\ n \geq 4.$

Then a patch, defined by the formula (18), is a rational SHK-patch of order n.

Proof. Suppose we try to find the functions g_q, $q \in \mathcal{H}^n$, in a form declared in Theorem 8, but with still unknown coefficients a, b_j, c_j, d_j, satisfying symmetry conditions $b_{n-j} = c_j$, $d_{n-j} = d_j$.

Step 1. Let $Q_0 = V_1$, $Q_1 = (V_0 + 2V_1)/3$, $Q_2 = (V_0 + 2V_2)/3$, $Q_3 = V_2$ and $Q(u) = \sum_{k=0}^{3} Q_k B^3_k(u)$. Reparametrization map $\mathrm{re}_2 : [0,1] \times [0,1] \to D$ is defined by the formula $\mathrm{re}_2(t,u) = (V_1(1-u) + V_2 u)(1-t) + Q(u)t$. It follows from the second and third properties of blown-up patches (Section 2.5), that conditions (1) and (6) from Definition 6 are satisfied respect to a map re_2 for arbitrary constants a, b_j, c_j, d_j. We also define the functions $h_{ij}(t,u)$, $i = 0,1$, $j = 0,1,\ldots,n$, by the formulas

$$
\begin{aligned}
&h_{00} = (f_{[2,0,0]} + af_{[2,1,0]}) \circ \mathrm{re}_2,\ h_{0n} = (f_{[2,0,n]} + af_{[2,1,n-1]}) \circ \mathrm{re}_2;\\
&h_{0j} = (f_{[2,0,j]} + b_j f_{[2,1,j-1]} + c_j f_{[2,1,j]}) \circ \mathrm{re}_2,\ 1 \leq j \leq n-1;\\
&h_{10} = (f_{[1,0,n-1]} + c_{n-1} f_{[2,1,0]}) \circ \mathrm{re}_2;\\
&h_{1n} = (f_{[3,0,1]} + b_1 f_{[2,1,n-1]}) \circ \mathrm{re}_2;\\
&h_{1j} = d_j(f_{[2,1,j-1]} + f_{[2,1,j]}) \circ \mathrm{re}_2,\ 1 \leq j \leq n-1.
\end{aligned}
\tag{19}
$$

and set $H = \sum_{i=0}^{1} \sum_{j=0}^{n} h_{ij}$. Second and third properties from Section 2.5 insures, that:

$$\begin{aligned}
&\partial \dot{g}_{20j}(\mathrm{re}_2(t,u))/\partial t \mid_{t=0} = \partial (h_{0j}/H)/\partial t \mid_{t=0}, \;\; 0 \le j \le n; \\
&\partial \dot{g}_{1,0,n-1}(\mathrm{re}_2(t,u))/\partial t \mid_{t=0} = \partial (h_{10}/H)/\partial t \mid_{t=0}; \\
&\partial \dot{g}_{301}(\mathrm{re}_2(t,u))/\partial t \mid_{t=0} = \partial (h_{1n}/H)/\partial t \mid_{t=0}; \\
&\partial \dot{g}_{21j}(\mathrm{re}_2(t,u))/\partial t \mid_{t=0} = \partial (h_{1j}/H)/\partial t \mid_{t=0}, \;\; 1 \le j \le n-1.
\end{aligned} \tag{20}$$

In the next three steps it will be shown, that for the appropriate values of a,b_j, c_j, d_j there exist the functions $A(u), D(u)$, satisfying the assumptions of Lemma 7.

Step 2. Here we compute functions $f_q(\mathrm{re}_2(t,u))$ as the polynomials respect to t. In further computations important are only the restrictions of the functions h_{ij} and their derivatives respect to t on $t = 0$. Therefore we do not include in calculated expressions the terms containing t^k, $k \ge 2$. This really does not lead to a confusion. For a simplicity the same letter f is used for any function f and $f(\mathrm{re}_2(t,u))$.

Simple calculations give $l_0 = (1-u)ut$, $l_1 = (1-u)(1-u^2t)$, $l_2 = u(1-(1-u)^2t)$. The functions f_q, we are interested in, now are the same both for a 5- or 6-sided patches, except $f_{[1,0,n-1]}$ and $f_{[3,0,1]}$. Calculating them we get that in both cases $f_{[1,0,n-1]} = (1-u)^{2n+1}u^n t$, $f_{[3,0,1]} = (1-u)^n u^{2n+1} t$. Therefore there is no difference in a proof between 5- or 6-sided patches. Let $A(u) = (1-u)^n u^n$. Here is a list of the expressions f_q, we are interested in:

$$\begin{aligned}
&f_{[20j]} = A(u)\binom{n}{j}(1-u)^{n-j}u^j(1-((n+j)(1-u)^2 + (2n-j)u^2)t); \\
&f_{[21j]} = A(u)(1-u)^{n-j}u^{j+1}t; \\
&f_{[1,0,n-1]} = A(u)n(1-u)^{n+1}t, \;\; f_{[301]} = A(u)nu^{n+1}t.
\end{aligned} \tag{21}$$

Step 3. Obviously $h_{0j}(0,u) = A(u)B_j^n(u)$. For $j = 1, \ldots, n-1$ simple calculation gives

$$\partial h_{1j}/\partial t \mid_{t=0} = A(u)d_j(1-u)^{n-j}u^j.$$

If $d_j = n\binom{n}{j}$, part of conditions of Lemma 7 are satisfied. Since

$$\partial h_{10}/\partial t \mid_{t=0} = A(u)n(1-u)^n(1-u+(c_{n-1}/n)u),$$

one more condition $\partial h_{10}/\partial t \mid_{t=0} = A(u)nB_0^n(u)$ is true, if $c_{n-1} = n$. If $b_1 = c_{n-1} = n$ it follows from a symmetric definition of the functions h_{ij}, that $\partial h_{1n}/\partial t \mid_{t=0} = A(u)nB_n^n(u)$

Step 4. Calculating in the same fashion as before we get following results:

$$\partial h_{00}/\partial t \mid_{t=0} = D_0(1-u)^n, \text{ where}$$

$D_0 = -A(u)(n(1-u)^2 + 2nu^2 - au)$; if $1 \le j \le n-1$, then

$$\partial h_{0j}/\partial t \mid_{t=0} = D_j \binom{n}{j} (1-u)^{n-j} u^j, \text{ where}$$

$D_j = -A(u)((n+j)(1-u)^2 + (2n-j)u^2 - (b_j/\binom{n}{j})(1-u) - (c_j/\binom{n}{j})u)$. The expressions D_j must be independent of j ($D_j = D(u)$). Simplifying an equation $D_j - D_0 = 0$ we get it can be written in a form $Z_0 + Z_1 u = 0$, where

$$Z_0 = j - b_j/\binom{n}{j}, \; Z_1 = a - 2j + b_j/\binom{n}{j} - c_j/\binom{n}{j}.$$

Solving the system $Z_0 = 0$, $Z_1 = 0$ we get

$$b_j = j\binom{n}{j}, \; c_j = (a-j)\binom{n}{j}.$$

So b_1 is the same as derived in Step 3. Obviously c_{n-1} also must coincide with the c_{n-1} from Step 3. This gives an equation $n(a-n+1) = n$ and we get $a = n$. The final expression for the coefficients c_j becomes $c_j = (n-j)\binom{n}{j}$. It follows from a symmetric definition of the functions h_{ij}, that $\partial h_{0n}/\partial t \mid_{t=0} = D(u)B_n^n(u)$.

Step 5. In the steps $2,3,4$ we have shown, that if $a = n$, $b_j = j\binom{n}{j}$, $c_j = (n-j)\binom{n}{j}$, $d_j = n\binom{n}{j}$, then for the functions from (19) are satisfied the conditions of Lemma 7. Therefore this lemma and the formulas (20) implies, that the conditions $2,3,4,5$ of Definition 6 are satisfied respect to the map re_2 (the conditions 1 and 6 are satisfied for any values of a, b_j, c_j, d_j). Rest of the maps re_s are defined by the formula $\mathrm{re}_s = \widetilde{\mathrm{rot}}^{s-2} \circ \mathrm{re}_2$ (rotation $\widetilde{\mathrm{rot}}$ of a blown-up plane was introduced in Section 3.2). It follows from a symmetry of $\tilde{T}_5^n$- and $\tilde{T}_6^n$-patches (formula (17)), that the conditions of Definition 6 are satisfied for all maps re_s. $\square$

Let us to note, that after proving the conditions of Definition 6 respect to re_2, we can simply lift to a patch, defined over regular domain, and hence with obvious symmetric properties.

4.3 Properties o SHK-patches

Here we consider some features of SHK-patches constructed using Theorem 8.

1. Convex hull property. Since all coefficients a, b_j, c_j, d_j are positive, basis functions g_q, $q \in \mathcal{H}^n$, are nonnegative on a domain D. It implies SHK-patches from Theorem 8 possess convex hull property.

2. Plotting SHK-patches. The basis functions g_q, $q \in \mathcal{H}^n$, of m-sided SHK-patch of order n are linear combinations of the basis functions of T^n_m-patches. It means SHK-patch can be represented as a T^n_m-patch. Therefore: 5-sided patch SHK-patch of order n can be represented as a collection of three rational Bézier patches of bidegree $(2n, 2n)$; 6-sided patch SHK-patch of order n can be represented as a collection of six rational Bézier patches of bidegree $(2n, 2n)$. The corresponding estimations for 5- and 6-sided patches from (Karčiauskas, 1999) are respectively $(2n+1, 2n+1)$ and $(3n+1, 3n+1)$. They are better as the estimations for the patches derived by the methods known before. So Theorem 8 gives most efficient 5- and 6-sided SHK-patches.
In (Loop and De Rose, 1990) SHK-patches are constructed elevating a depth of S-patches. The calculations show, that it is possible to construct 5-sided SHK-patches of order 3 without elevating a depth. Such patches can be represented as a collection of five Bézier patches of bidegree $(6, 6)$. But 5-sided patch of order 3 from Theorem 8 can be represented as a collection of three patches of the same bidegree (so remains a little bit better). We have not done calculations for $n \geq 4$. In any case it is clear that the estimations can not be improved using basis functions of S-patches. The same is true for 6-sided patches.

Remark 9 *We compare various patches using following criteria: a patch is more efficient as another one if it can be represented as a collection of Bézier patches with lower bidegree. Here is possible a confusion. We compare only the methods, that produce multisided patch as an entire algebraic manifold. Subdivision into rectangular patches is important for its representation. The methods producing multisided patch as a collection of smoothly joined rectangular patches may give lower bidegrees. On the other hand, entire patches have, as a rule, better shape.*

3. Extended SHK-patches. In Theorem 8 for a construction of SHK-patches are used only the functions $f^n_{s,i,j}$ with $i \leq 1$. The functions $f^n_{s,i,j}$ with $i \geq 2$ have no effect on the position and on the crossderivatives along the boundary of a patch. Therefore they can be treated as the additional basis functions of SHK-patches. Some features of this additional freedom are discussed in (Karčiauskas, 1999). They remain true with one difference in a favor for the just derived patches: adding new basis functions does not increase a degree of a representation.

4. Twist incompatibility. Suppose we want to fill smoothly with SHK-patch 5- or 6-sided hole, but at the corner $P_{s,0,0}$ there appears "twist incompatibility": a point $P^s_{s,1,1}$, insuring (with some other control points) tangent plane conti-

nuity along boundary curve between $P_{s,0,0}$ and $P_{s+1,0,0}$, does not coincide with a point $P^{s-1}_{s-1,1,n-1}$, insuring tangent plane continuity along boundary curve between $P_{s-1,0,0}$ and $P_{s,0,0}$. This problem is solved using method of variable control points, proposed by Hosaka and Kimura (1984). One possibility is to lift blown-up patch to a patch over regular pentagon or hexagon. In this case the formulas for the variable control points are given in (Karčiauskas, 1999). Another possibility is to work directly with blown-up patches without loosing a symmetry property.
Let $k^1_{00} = 1$, $k^1_{10} = 2$. If $m = 5$, basis functions f^1_q are defined by the formula (5); if $m = 6$, basis functions f^1_q are defined by the formula (7). We set

$$\tilde{l}_s = f^1_{s+2,0,0} + f^1_{s+3,0,0} + f^1_{s+4,0,0} + f^1_{0,1,0}, \text{ if } m = 5;$$
$$\tilde{l}_s = f^1_{s+2,0,0} + f^1_{s+3,0,0} + f^1_{s+4,0,0} + f^1_{s+5,0,0} + f^1_{0,1,0}, \text{ if } m = 6;$$
$$L_s = \tilde{l}_{s+1}\tilde{l}_{s+2} \dots \tilde{l}_{s+m-1}.$$

If $n \geq 3$, a variable control point P_{s11} is defined by the formula

$$P_{s11} = (\tilde{l}_s P^{s-1}_{s-1,1,n-1} + \tilde{l}_{s-1} P^s_{s,1,1})/(\tilde{l}_{s-1} + \tilde{l}_s).$$

If $n = 2$, inner variable control point P_{011} is defined by the formula

$$P_{011} = \sum_{s=0}^{m-1} L_s P^s_{s,1,1} / \sum_{s=0}^{m-1} L_s.$$

From an algebraic point of view plotting blown-up SHK-patches is a repeated blown-up procedure.

5 Conclusion and future work

In this paper we have presented new construction of 5- and 6-sided Sabin and Hosaka–Kimura like surface patches. They are called SHK-patches and can meet surrounding rectangular patches with G^1 continuity. Introduced here method is most efficient compared with other methods, producing a patch as an entire surface. Thoroughly is considered a relationship with the classical pentagonal Sabin and Hosaka-Kimura patches and S-patches from (Loop and De Rose, 1990). We also describe new patches from the point of view of algebraic geometry. Main interest of a current research are rational blown-up 5- and 6-sided patches, meeting surrounding rectangular patches with G^2 continuity.

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