Rational m-sided Sabin and Hosaka–Kimura like surface patches

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Abstract

We present a new control point scheme of non-rectangular rational m-sided surface patches ($m \geq 3,\ m \neq 4$). They are called T-patches and are used for a construction of rational m-sided Sabin and Hosaka–Kimura like patches. There are no restrictions on m and on degree of boundary Bézier curves. General construction deals with a case $m \geq 5$. Triangular patches ($m = 3$) have some specific properties. Main features of the introduced m-sided patches are described.

Keywords: multisided surface patch, geometric continuity, control point.

1 Introduction

There are various methods for constructing m-sided patches with $m \neq 4$. They can be categorized as recursive subdivision, surface splitting, data blending and control point schemes. Overviews of those various methods are given, for example, by Varady (1987) and Malraison (1998). We present a new control point scheme for m-sided patches.

The original surface patches of Sabin (1983, 1991) and Hosaka & Kimura (1984) are defined over nonplanar 3-,5- and 6-sided areas in 3-, 5- and 6-dimensional space respectively. They are suitable for a smooth joining to rectangular Bézier patches. The boundary of Sabin and Hosaka–Kimura patches are Bézier curves of degree 2 and 3 respectively. Zheng & Ball (1997) extended their approach to a case, when boundary contains Bézier curves of arbitrary degree n. The 3- and 5-sided areas can be parametrizied by the rational functions, so 3- and 5-sided Sabin and Hosaka–Kimura patches are rational. For a

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parametrization of 6-sided area square roots are involved. It is a disadvantage of 6-sided Sabin and Hosaka–Kimura patches – they are nonrational. Loop & De Rose (1989) introduced S-patches. They are rational m-sided patches defined over planar m-gon for any $m \geq 3$. Later on Loop & De Rose (1990) used S-patches for a construction of rational m-sided Sabin (boundary curves are conics) and Hosaka–Kimura (boundary curves are cubics) like patches for arbitrary $m \geq 3$.

In this paper we construct rational m-sided Sabin and Hosaka–Kimura like surface patches over planar domain for any $m \geq 3$, $m \neq 4$, bounded by Bézier curves of arbitrary degree n. Our approach produces the patches of lower degree as derived by the previous methods of Sabin (1983), Hosaka-Kimura (1984), Loop & De Rose (1990), Zheng & Ball (1997). Only in case $m = 3$, $n = 2$ they are of degree 4 as the original Sabin (1983) triangular patch, but possess some additional useful properties. This paper is organized as follows. In Section 2 we define T-patches and describe their main properties. Then T-patches are used for a construction of surface patches, that behave along their boundaries like rectangular Bézier patches. We call them SHK-(Sabin–Hosaka–Kimura like) patches. Formal definition of SHK-patches and some important technical results are given in Section 3. Later on a case $m \geq 5$ is considered: in Section 3.2 main theorem is formulated; in Section 3.3 main properties of derived SHK-patches are described. A proof of the main theorem is given in Section 4. Section 5 is devoted to the triangular patches. In Section 6 spline surfaces are briefly discussed.

We use following notations: $u_0 = 1 - u$, $u_1 = u$; $B_j^n(u) = \binom{n}{j} u_0^{n-j} u_1^j$ ($B_j^n(u)$ are Bernstein polynomials). For any set of functions f_q, labeled by a graph $\mathcal{L}$, we set $\dot{f}_q = f_q / \sum_{q' \in \mathcal{L}} f_{q'}$, $q \in \mathcal{L}$.

2 T-patches

2.1 *A combinatorial structure and definition*

Let $W_0, W_1, \ldots, W_{m-1}, W_m = W_0$ be the vertices of a convex m-gon; let W be its inner point and let n be a fixed natural number. For each triangle with the vertices W, W_s, W_{s+1}, $0 \leq s \leq m-1$, the points

$$T_{ij}^s = \frac{i}{n} W + \frac{j}{n} W_{s+1} + \frac{n-i-j}{n} W_s, \quad i,\, j \geq 0, i + j \leq n,$$

linked together, form its standard triangulation. All together they form a triangulation of m-gon. It is convenient to organize the labeling of this triangulation in the following manner.

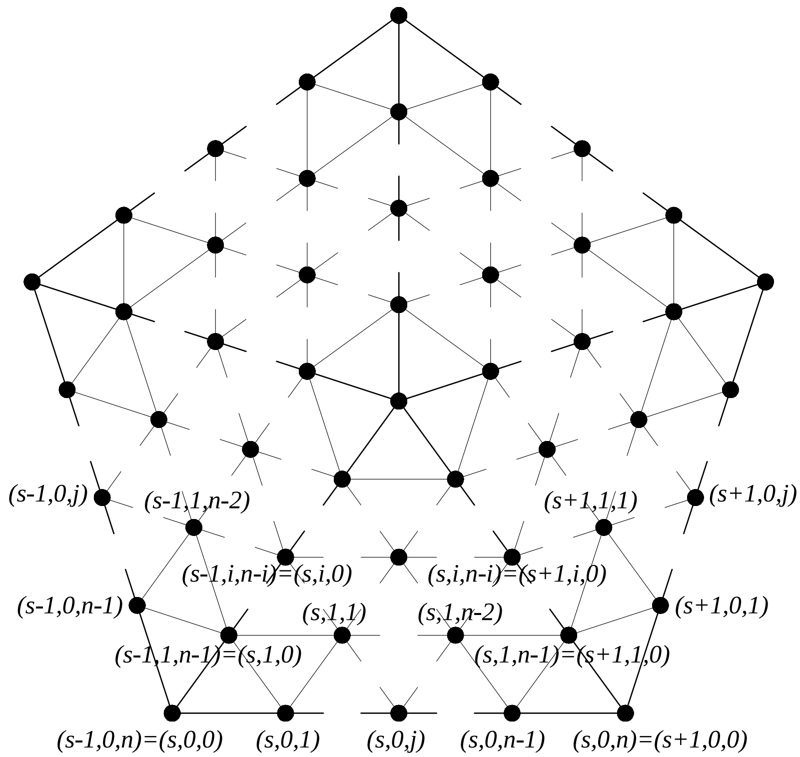

Fig. 1. Combinatorial structure and labeling

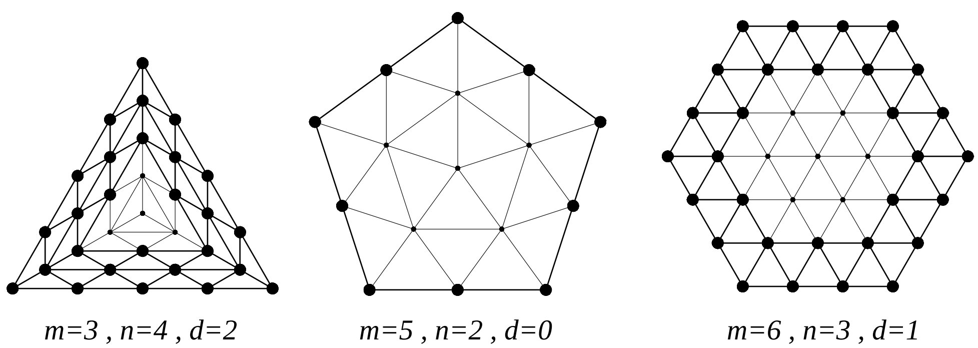

Fig. 2. Various cases of $\mathcal{T}_d^n$ subgraphs

Let $\mathcal{T}^n$ be a set of all triples (s, i, j), $0 \le s \le m-1$, $0 \le i \le n$, $0 \le j \le n-i$, where triples $(s, i, n-i)$ and $(s+1, i, 0)$ are identified (the first index s is treated in a cyclic fashion). There are $mn(n+1)/2+1$ triples in $\mathcal{T}^n$. Without confusing $\mathcal{T}^n$ can be treated as a graph of the triangulation of m-gon. This labeling (graph) is visualized in Figure 1.

We also set $\mathcal{T}_d^n = \mathcal{T}^n \setminus \{(s,i,j) \mid i \ge d+1\}$, $0 \le d \le n$, (obviously $\mathcal{T}_n^n = \mathcal{T}^n$); In Figure 2 there are shown some cases of $\mathcal{T}_d^n$ subgraphs. The vertices, included in $\mathcal{T}_d^n$, are plotted as the bigger disks.

Now we assume $m \ge 5$ and set $\alpha = 2\pi/m$. By D we denote a regular m-gon in (x,y)-plane with the vertices $V_s = (\cos s\alpha; \sin s\alpha)$, $0 \le s \le m-1$. Polygon D is a domain for m-sided rational patches still to be defined (see Figure 3).

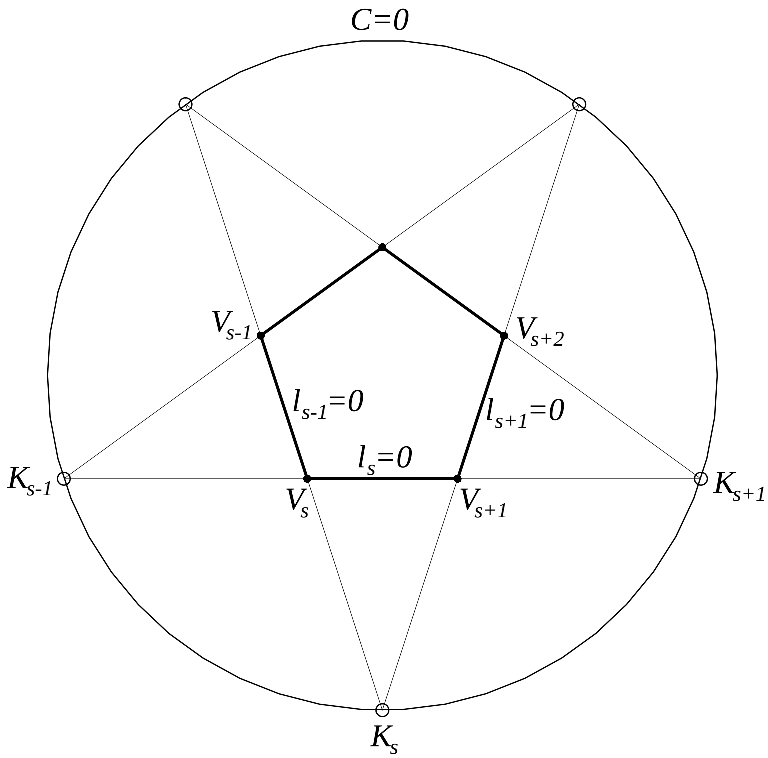

Fig. 3. Domain of m-sided patches

A point of an intersection of the lines $\overline{V_{s-1}V_s}$ and $\overline{V_{s+1}V_{s+2}}$ is denoted by K_s. We also set

$$l_s(x,y) = -x\cos(s\alpha + \frac{\alpha}{2}) - y\sin(s\alpha + \frac{\alpha}{2}) + \cos\frac{\alpha}{2},\ 0 \le s \le m-1;$$

$$C(x,y) = \frac{\cos^2\frac{\alpha}{2}}{\cos^2\alpha} - x^2 - y^2.$$

It is easy to check that:

- l_s is zero on the edge $\overline{V_sV_{s+1}}$ and takes positive values on the other points of a domain D;
- $C = 0$ defines a circle through the points K_s, $0 \le s \le m-1$, and C takes positive values on D.

Suppose there are fixed positive numbers $k^n_{ij}, 0 \le i \le n, 0 \le j \le n-i$, satisfying $k^n_{ij} = k^n_{i,n-i-j}$. Let $R_s = \{0, 1, \ldots, m-1\} \setminus \{s-1, s, s+1\}$, $0 \le s \le m-1$. The functions $f^{n,d}_q, q = (s,i,j) \in \mathcal{T}^n_d$ are defined by the formula

$$f^{n,d}_q = k^n_{ij} C^{d-i} l^{i+j}_{s-1} l^i_s l^{n-j}_{s+1} (\prod_{r\in R_s} l_r)^n. \tag{1}$$

Definition 1 *A parametric rational m-sided T-patch of order n and of depth d is a map $F_d^n : D \to \mathbb{R}^3$ defined by the formula*

$$F_d^n(p) = \frac{\sum_{q\in\mathcal{T}_d^n} w_q P_q f_q^{n,d}(p)}{\sum_{q\in\mathcal{T}_d^n} w_q f_q^{n,d}(p)}. \tag{2}$$

The points P_q are called control points of the patch and the numbers w_q are their weights. Geometrically T-patch is understood as the image $F_d^n(D)$. But without confusing we often consider T-patch as a map (exactly as in Definition 1). It follows directly from the formula (2) that

$$F_d^n(p) = \frac{\sum_{q\in\mathcal{T}_d^n} w_q P_q \dot{f}_q^{n,d}(p)}{\sum_{q\in\mathcal{T}_d^n} w_q \dot{f}_q^{n,d}(p)}. \tag{3}$$

Remark 2 *Since $f_q^{n,n} = C^{n-d} f_q^{n,d}$, $q \in \mathcal{T}_d^n$, obviously is true*

$$F_d^n(p) = \sum_{q\in\mathcal{T}_d^n} w_q P_q f_q^{n,n}(p) / \sum_{q\in\mathcal{T}_d^n} w_q f_q^{n,n}(p).$$

So we can use in a definition of a map F_d^n only the functions $f_q^{n,n}, q \in \mathcal{T}_d^n$. But the formula (2) is more convenient since the functions $f_q^{n,d}$, $q \in \mathcal{T}_d^n$, do not have a common factor.

2.2 Main properties of T-patches

Proposition 3 *The functions $f_q^{n,d}$ have the properties:*
(1) $f_q^{n,d}(p) \geq 0$ and $\sum_{q\in\mathcal{T}_d^n} f_q^{n,d}(p) > 0$ for $p \in D$;
(2) only the functions $f_{s00}^{n,d}, f_{s01}^{n,d}, \ldots, f_{s0n}^{n,d}$ are nonzero on the edge $\overline{V_sV_{s+1}}$;
(3) only the functions $f_{s,0,0}^{n,d}, f_{s,0,1}^{n,d}, \ldots, f_{s,0,n}^{n,d}$ and $f_{s-1,0,n-1}^{n,d}, f_{s,1,0}^{n,d}, \ldots, f_{s,1,n-1}^{n,d}$, $f_{s+1,0,1}^{n,d}$ have nonzero crossderivatives along the edge $\overline{V_sV_{s+1}}$;

Proof. The first property follows directly from the formula (1) and a positivity of the functions l_s and C on D. The second property follows from a fact, that all functions, except the of listed in statement *(2)*, contain as a factor l_s (l_s is zero on $\overline{V_sV_{s+1}}$). The third property is true, since all functions, except of listed in statement *(3)*, contain as a factor l_s^μ with $\mu \geq 2$. □

Proposition 4 *(1) If $k_{0j}^n = \binom{n}{j}$ the map F_d^n restricted on the edge $\overline{V_sV_{s+1}}$ defines a rational Bézier curve of degree n with the control points P_{s0j} and the weights w_{s0j}, $0 \leq j \leq n$.*
(2) If all weights w_q are positive a patch lies in a convex hull of the control points P_q.

Proof. (1) It follows from Proposition 3*(2)*, that

$$\dot{f}_{s0j}^{n,d}(V_s u_0 + V_{s+1} u_1) = f_{s0j}^{n,d}(V_s u_0 + V_{s+1} u_1) / \sum_{j'=0}^{n} f_{s0j'}^{n,d}(V_s u_0 + V_{s+1} u_1).$$

Assuming $k_{0j}^n = \binom{n}{j}$, using the identities (derived by direct calculation)

$$\begin{aligned} l_{s-1}(V_s u_0 + V_{s+1} u_1) &= 2 \sin\alpha \sin\tfrac{\alpha}{2} u_1, \\ l_{s+1}(V_s u_0 + V_{s+1} u_1) &= 2 \sin\alpha \sin\tfrac{\alpha}{2} u_0 \end{aligned} \tag{4}$$

and well known identity $\sum_{r=0}^{n} B_r^n(u) = 1$ we get $\dot{f}_{s0j}^{n,d}(V_s u_0 + V_{s+1} u_1) = B_j^n(u)$. Together with the formula (3) and Proposition 3(2) it implies first statement is true.

(2) We denote $g_q(p) = w_q f_q^{n,d}(p) / \sum_{q' \in \mathcal{T}_d^n} w_{q'} f_{q'}^{n,d}(p)$. If $w_q \geq 0$ then $g_q(p) \geq 0$. Since $F_d^n(p) = \sum_{q \in \mathcal{T}_d^n} g_q(p) P_q$ and $\sum_{q \in \mathcal{T}_d^n} g_q(p) = 1$, it follows convex hull property is true for T-patches. $\square$

Proposition 5 *The functions* $f_q^{n,d}$, $q \in \mathcal{T}_d^n$, *are linearly independent.*

Proof. Let $d = n$. Since a statement of this proposition does not depend of a choice of numbers k_{ij}^n, we simply take $k_{ij}^n = 1$. Let $G = \sum_{q \in \mathcal{T}^n} \lambda_q f_q^{n,n}$ and assume $G = 0$. We will show by an induction all λ_q are zero.
From (1), (4) and the assumption $G = 0$ we get for $s = 0, 1, \ldots, m-1$

$$G(V_s u_0 + V_{s+1} u_1) = E_s^n (2 \sin\alpha \sin\frac{\alpha}{2})^n (\sum_{j=0}^{n} \lambda_{s0j} u_0^{n-j} u_1^j) = 0,$$

where $E_s = C(V_s u_0 + V_{s+1} u_1) \prod_{r \in R_s} l_r(V_s u_0 + V_{s+1} u_1)$.

Since $E_s \neq 0$ and the functions $u_0^{n-j} u_1^j$ are linearly independent, all λ_{s0j} must be zero. It means $G = \sum_{q \in \mathcal{T}^n \setminus \mathcal{T}_0^n} \lambda_q f_q^{n,n} = 0$. From (1) we get for $i \geq 1$

$$f_{s,i,j}^{n,n} = (l_0 l_1 \ldots l_{m-1}) f_{s,i-1,j}^{n-1,n-1}.$$

Identifying $\mathcal{T}^n \setminus \mathcal{T}_0^n$ with $\mathcal{T}^{n-1}$ (a triple (s,i,j) as a label from $\mathcal{T}^n \setminus \mathcal{T}_0^n$ is considered as a label $(s, i-1, j)$ from $\mathcal{T}^{n-1}$) we can write

$$G = (l_0 l_1 \ldots l_{m-1}) (\sum_{q \in \mathcal{T}^{n-1}} \lambda_q f_q^{n-1,n-1}) = 0.$$

This implies $\sum_{q \in \mathcal{T}^{n-1}} \lambda_q f_q^{n-1,n-1} = 0$. Carrying on we get all λ_q are zero.
Since $f_q^{n,n} = C^{n-d} f_q^{n,d}$, $q \in \mathcal{T}_d^n$, we get that the functions $f_q^{n,d}$, $q \in \mathcal{T}_d^n$, are also linearly independent. $\square$

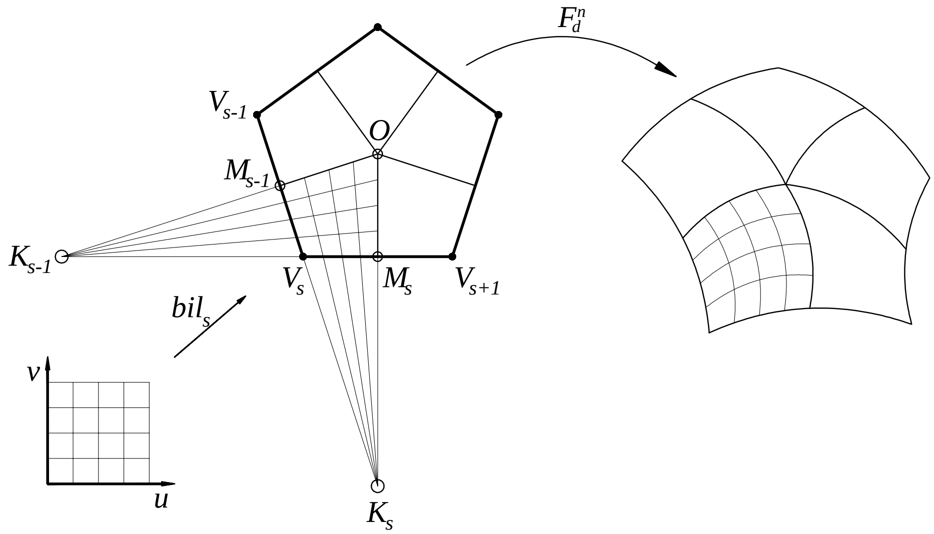

Fig. 4. Plotting T-patches

Proposition 6 *(Plotting T-patches). m-sided T-patch of order n and depth d can be represented as a collection of m rational Bézier patches of bidegree $((m-3)n+d, (m-3)n+d)$.*

Proof. From a formula (1) we get degree of $f_q^{n,d}$ is $(m-2)n+2d$. This means a general line L is mapped by F_d^n into a rational curve of degree $(m-2)n+2d$. The functions $l_{s-1}(x,y),\ l_{s+1}(x,y),\ C(x,y)$ are zero at the point K_s. From the formula (1) we get a minimum of the sum of their powers in $f_q^{n,d}$, $q \in \mathcal{T}_d^n$, is $n+d$. This means every function $f_q^{n,d}$ has at the point K_s zero of multiplicity $\geq n+d$. (By a standard definition a function $f(x,y)$ has at a point $(x_0;y_0)$ zero of multiplicity μ if it vanishes at $(x_0;y_0)$ together with all partial derivatives up to the order $\mu-1$ and at least one partial derivative of order μ does not vanish.) It implies a restriction of every function $f_q^{n,d}$ on a fixed line L, going through K_s, has the same linear factor Z into a power $\geq n+d$. Canceling factor Z^{n+d} from a numerator and denominator of the restriction of $F_q^{n,d}$ on L we get a degree of a curve is $(m-2)n+2d-(n+d)=(m-3)n+d$. For $s=0,1,\ldots,m-1$ we define the bilinear maps $\mathrm{bil}_s : [0,1]\times[0,1] \to D$ by the formula

$$\mathrm{bil}_s(u,v) = \frac{V_s(1-u)(1-v) + M_s u(1-v) + M_{s-1}(1-u)v + Ouv}{(1-u)(1-v) + u(1-v) + (1-u)v + u\sin^2\alpha},$$

where M_s is a middle point of the edge $\overline{V_sV_{s+1}}$ and O – a center of D. The lines $\mathrm{bil}_s(\mathrm{const}, v)$ go through the point K_s and the lines $\mathrm{bil}_s(u, \mathrm{const})$ go through the point K_{s-1}. It follows from the above considerations a map $F_d^n(\mathrm{bil}_s(u,v))$ represents two parameter family of the rational curves of degree $(m-3)n+d$. So we have almost finished a proof. A rest is only the technical details. Carefully caring on we get an efficient algorithm for plotting T-patches. A principle of this presentation is visualized in Figure 4. $\square$

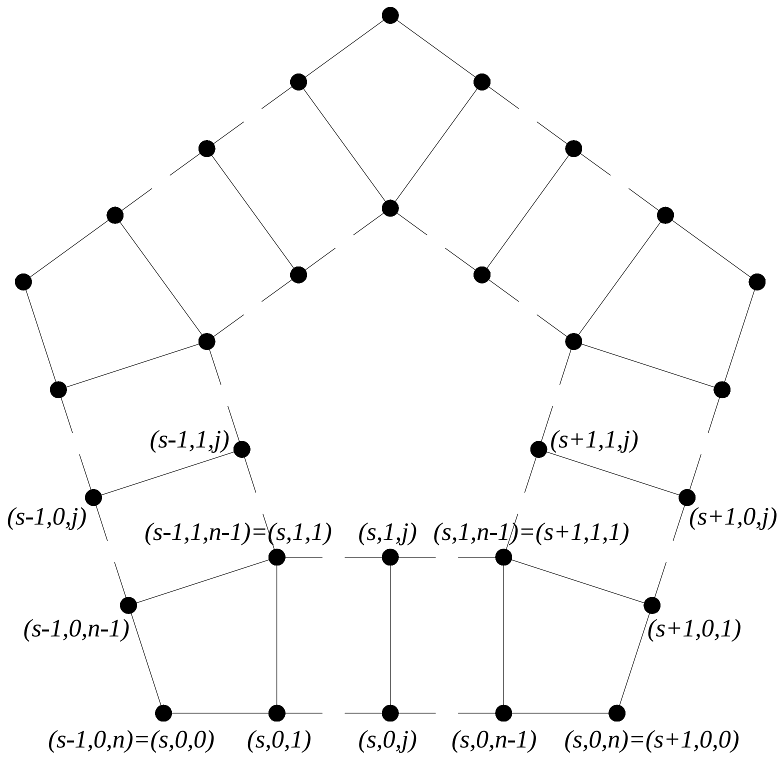

Fig. 5. Combinatorial structure of SHK-patches

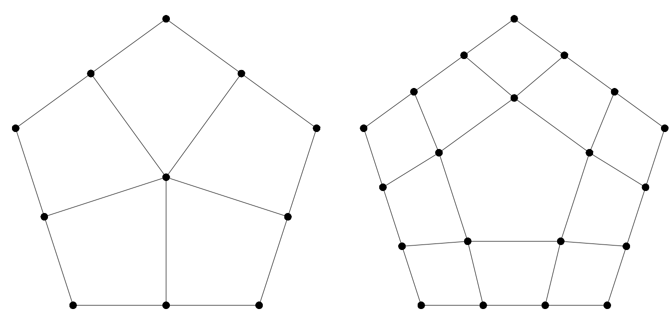

Fig. 6. Sabin and Hosaka–Kimura patterns

3 SHK-patches

3.1 Minimal combinatorial structure and definition

We assume here $m \geq 3$. For a fixed natural number m by $\mathcal{H}^n$ we denote a set of triples $(s, 0, j)$, $(s, 1, k)$, $0 \leq s \leq m-1$, $0 \leq j \leq n$, $1 \leq k \leq n-1$, where $(s-1, 0, n) = (s, 0, 0)$ and $(s-1, 1, n-1) = (s, 1, 1)$ (index s is treated in a cyclic fashion). A graph $\mathcal{H}^n$ is shown in Figure 5. The cases $n = 2$ (Sabin pattern) and $n = 3$ (Hosaka–Kimura pattern) are shown in Figure 6. Note Sabin pattern has only one inner point $(0, 1, 1) = (1, 1, 1) = \ldots = (m-1, 1, 1)$.

Rational m-sided SHK-patch will be constructed using various sets of basis functions $g_q \in \mathcal{H}^n$. Control points of a SHK-patch we denote by P_q and the weights by w_q, $q \in \mathcal{H}^n$. Control points of a rational Bézier patch of bidegree $n \times n'$ we denote by $B_{rr'}$ and corresponding weights by $w_{rr'}$, $0 \leq r \leq n$, $0 \leq r' \leq n'$. (Using the same letter w for both type patches does not lead to a confusion since their sets of labels are different.) For a control point $Q \in \mathbb{R}^3$ of any patch and its weight w we set $\underline{Q} = (wQ, w)$. A point $\underline{Q} \in \mathbb{R}^4$ is considered as a homogeneous control point of a patch.

Definition 7 *A m-sided patch with the control points P_q and the weights w_q, $q \in \mathcal{H}^n$, is a rational SHK-patch of order n if the following conditions are satisfied:*
(1) boundary curves are rational Bézier curves of degree n with the control points P_{s0j} and the weights w_{s0j}, $0 \leq j \leq n$, $0 \leq s \leq m-1$;
(2) it is tangent plane continuous to a adjacent rational rectangular Bézier patch of bidegree (n, n') with $\underline{P}_{s0j} = \underline{B}_{j0}$, $0 \leq j \leq n$, if
$\underline{P}_{s,1,j} + k\underline{B}_{j1} = (k+1)\underline{P}_{s,0,j}$, $1 \leq j \leq n-1$,
$\underline{P}_{s-1,0,n-1} + k\underline{B}_{01} = (k+1)\underline{P}_{s,0,0}$, $\underline{P}_{s+1,0,1} + k\underline{B}_{n1} = (k+1)\underline{P}_{s,0,n}$
for some fixed $k > 0$.

In practice as a rule we have $n = n'$, $k = 1$ and all weights are equal to 1. In this case the condition (*2*) simply means the control points of a common boundary curve are middle points of the edges, formed by the corresponding control points of the next layer of each patch.

Proposition 8 *Suppose on the domain D there are fixed the functions g_q, $q \in \mathcal{H}^n$, and there exist the maps $\mathrm{re}_s : [0,1] \times [0,1] \to D$, $0 \leq s \leq m-1$, called reparametrization maps, satisfying following conditions:*
(1) $\mathrm{re}_s(0, u) = V_s u_0 + V_{s+1} u_1$;
(2) $\dot{g}_{s0j}(\mathrm{re}_s(0, u)) = B_j^n(u)$, $0 \leq j \leq n$;
(3) $\partial \dot{g}_{s0j}(\mathrm{re}_s(t, u))/\partial t \mid_{t=0} = -nB_j^n(u)$, $0 \leq j \leq n$;
(4) $\partial \dot{g}_{s-1,0,n-1}(\mathrm{re}_s(t, u))/\partial t \mid_{t=0} = nB_0^n(u)$;
(5) $\partial \dot{g}_{s+1,0,1}(\mathrm{re}_s(t, u))/\partial t \mid_{t=0} = nB_n^n(u)$;
(6) $\partial \dot{g}_{s1j}(\mathrm{re}_s(t, u))/\partial t \mid_{t=0} = nB_j^n(u)$, $1 \leq j \leq n-1$;
(7) $\dot{g}_q(\mathrm{re}_s(0, u)) = 0$ *for all functions* $\dot{g}_q$*, except of listed in (2), and*
$\partial \dot{g}_q(\mathrm{re}_s(t, u))/\partial t \mid_{t=0}$ *for all* $\dot{g}_q$*, except of listed in (2)-(6).*
Then the map $G^n : D \to \mathbb{R}^3$ defined by the formula

$$G^n(p) = \frac{\sum_{q \in \mathcal{H}^n} w_q P_q g_q(p)}{\sum_{q \in \mathcal{H}^n} w_q g_q(p)} \tag{5}$$

is a rational SHK-patch of order n.

Proof. Formally self contained proof would be longer. Unformally it is very simple. Obviously we have

$$\partial(B_0^n(t)B_j^n(u))/\partial t\ |_{t=0} = -nB_j^n(u),\ \ \partial(B_1^n(t)B_j^n(u))/\partial t\ |_{t=0} = nB_j^n(u),$$
$$G^n(p) = \sum_{q\in\mathcal{H}^n} w_q P_q \dot{g}_q(p) / \sum_{q\in\mathcal{H}^n} w_q \dot{g}_q(p).$$

These identities and the assumptions of the proposition implies, that after reparametrization a crossderivative of m-sided patch behaves like a cross-derivative of Bézier patch of bidegree (n, n). The standard arguments of the theory of Bézier patches show the proposition is true. $\square$

A following technical lemma is essentially used in the proofs of the theorems 10 and 13.

Lemma 9 *Suppose the functions* $h_{ij}(t, u)$ *,*$i = 0, 1,\ 0 \le j \le n$*, satisfy* $h_{0j}(0, u) = B_j^n(u)A(u)$*,* $h_{1,j}(0, u) = 0$*,* $\partial h_{0j}/\partial t\ |_{t=0} = B_j^n(u)D(u)$*,* $\partial h_{1j}/\partial t\ |_{t=0} = nB_j^n(u)A(u)$ *for some functions* $A(u), D(u)$*. Then*

$$\partial(h_{0j}/\sum_{i=0}^{1}\sum_{r=0}^{n} h_{ir})/\partial t\ |_{t=0} = -nB_j^n(u),$$

$$\partial(h_{1j}/\sum_{i=0}^{1}\sum_{r=0}^{n} h_{ir})/\partial t\ |_{t=0} = nB_j^n(u).$$

A proof of lemma is very simple: we calculate the derivatives respect to t and use the assumptions of lemma together with the well known identity $\sum_{r=0}^{n} B_r^n(u) = 1$.

3.2 Main theorem

We again assume $m \ge 5$. Basis functions g_q, $q \in \mathcal{H}^n$, for SHK-patches are build as the linear combinations of the basis functions $f_q^{n,d}$, $q \in \mathcal{T}_d^n$, defined by the formula (1). In the main theorem we use only the functions $f_q^{n,1}$, $q \in \mathcal{T}_1^n$, and set $k_{0j}^n = \binom{n}{j}$, $0 \le j \le n$, $k_{1r}^n = 1$, $0 \le r \le n-1$. In order to avoid confusion indexing these two type of functions the following notions are used:
if $q = (s, i, j) \in \mathcal{H}^n$ then $g_{sij} := g_q$;
if $q = (s, i, j) \in \mathcal{T}_1^n$ then $f_{[s,i,j]} := f_q^{n,1}$.

Theorem 10 *Let* $a = \dfrac{n}{\sin 2\alpha \sin\alpha}$*,* $b_j = \dfrac{j\binom{n}{j}}{\sin 2\alpha\sin\alpha}$*,* $c_j = \dfrac{(n-j)\binom{n}{j}}{\sin 2\alpha \sin\alpha}$*,* $d_j = \dfrac{n\binom{n}{j}(2\cos\alpha + 1)}{\sin^2 2\alpha}$*,* $1 \le j \le n-1$*, and for* $s = 0, 1, \ldots, m-1$

- $g_{s00} = f_{[s,0,0]} + af_{[s,1,0]}$;
- $g_{s0j} = f_{[s,0,j]} + b_j f_{[s,1,j-1]} + c_j f_{[s,1,j]}$, $1 \leq j \leq n-1$;
- $g_{s11} = d_1 \sum_{s=0}^{m-1} f_{[s,1,0]}$ *if* $n = 2$;
  $g_{s11} = d_1(f_{[s-1,1,n-2]} + f_{[s,1,0]} + f_{[s,1,1]})$ *if* $n \geq 3$;
  $g_{s1j} = d_j(f_{[s,1,j-1]} + f_{[s,1,j]})$, $2 \leq j \leq n-2$, *if* $n \geq 4$.

Then a patch, defined by the formula (5), is a rational SHK*-patch of order* n.

3.3 Properties of SHK*-patches*

Here we consider some features of SHK-patches constructed using main theorem.

1. Convex hull property. Since all coefficients a, b_j, c_j, d_j in the main theorem are positive the functions g_q, $q \in \mathcal{H}^n$, are nonnegative on D. The same arguments, as in Proposition 4, show a patch lies in a convex hull of the control points P_q if all weights w_q are positive.

2. Plotting SHK*-patches.* Since basis functions g_q, $q \in \mathcal{H}^n$, are linear combinations of the functions f_q, $q \in \mathcal{T}_1^n$, SHK-patch of order n can be represented as a T-patch of order n and depth 1. So it follows from Proposition 6 m-sided SHK-patch of order n can be represented as a collection of m rational Bézier surface patches of bidegree $((m-3)n+1, (m-3)n+1)$.

3. Comparing with S*-patches.* It follows from a definition (see Loop & De Rose (1989)) the basis functions of m-sided S-patch of a depth d are of degree $d(m-2)$ and have at the points K_s zeros of multiplicity $\geq d$. (*Warning: depth of* T*-patches has a quite different meaning as a depth of* S*-patches.*) The same arguments as in Proposition 6 show S-patch can be represented as a collection of m rational Bézier patches of bidegree $(d(m-3), d(m-3))$. Loop & De Rose (1990) constructed SHK-patches of orders 2 and 3 as S-patches of depth 5 and 6 respectively. So the bidegrees of the Bézier patches are $(5(m-3), 5(m-3))$ and $(6(m-3), 6(m-3))$ respectively. The estimations from a previous paragraph are correspondingly $(2m-5, 2m-5)$ and $(3m-8, 3m-8)$. So SHK-patches constructed in this paper are more efficient (in a sense of degree of representation).

4. Twist incompatibility. Suppose we want to fill smoothly with SHK-patch m-sided hole, but at the corner $V_s \in D$ there appears a "twist incompatibility": a point $P^s_{s,1,1}$, insuring (with some other control points) tangent plane continuity along the edge $\overline{V_s V_{s+1}}$, does not coincide with a point $P^{s-1}_{s-1,1,n-1}$, insuring tangent plane continuity along the edge $\overline{V_{s-1} V_s}$. This problem of incompatibility is solved using method of variable control points, proposed by Hosaka and Kimura (1984).

If $n \geq 3$ a variable control point P_{s11} is defined by the formula

$$P_{s11} = \frac{l_s P^{s-1}_{s-1,1,n-1} + l_{s-1} P^s_{s,1,1}}{l_{s-1} + l_s}. \tag{6}$$

If $n = 2$ we set $L_s = l_{s+1} l_{s+2} \ldots l_{s+m-1}$, $0 \leq s \leq m-1$, and define variable inner control point P_{011} by the formula

$$P_{011} = \frac{\sum_{s=0}^{m-1} L_s P^s_{s,1,1}}{\sum_{s=0}^{m-1} L_s}. \tag{7}$$

5. Extended SHK-patches. In Theorem 10 for a construction of SHK-patches there are used only the functions f_q, $q \in \mathcal{T}^n_1$. It follows from Proposition 3 the functions f_q, $q \in \mathcal{T}^n \setminus \mathcal{T}^n_1$, have no effect on the position and on the crossderivatives along the boundary of T-patch. So we can add to the graph $\mathcal{H}^n$ any label from $\mathcal{T}^n \setminus \mathcal{T}^n_1$. No doubt, we do it keeping combinatorial symmetry of a patch. Straightforward approach - to add entire subgraph $\mathcal{T}^n \setminus \mathcal{T}^n_1$ - is shown on the left examples in Figure 8 (cases $n = 2, 3$). Corresponding control point nets fascinates, probably, only a spiders community. From a designers point of view better approach would be to combine symmetrically the functions f_q, $q \in \mathcal{T}^n \setminus \mathcal{T}^n_1$, and only those combinations consider as the additional basis functions. A simplest possibility is to add only one new basis function. Corresponding patterns are displayed on the right in Figure 8. But it follows from Proposition 6, that adding new basis functions increases a degree of representation. So in practice we have to find a compromise between additional shape control and increasing degree of extended SHK-patch.

The SHK-patch of order 2 (Sabin like patch) needs a special consideration. In this case there is only one additional function $f_{[0,2,0]}$. Adding a new control point would destroy a natural structure of Sabin net. Better approach is to add the function $f_{[0,2,0]}$ in a symmetric fashion to the original basis functions from the main theorem. Therefore the modified basis functions for SHK-patches of order 2 are redefined by the formulas

$$\begin{aligned} &g_{s00} := g_{s00} + a_0 f_{[0,2,0]}, \;\; g_{s01} := g_{s01} + a_1 f_{[0,2,0]}, \quad 0 \leq s \leq m-1, \\ &g_{s11} := g_{s11} + a_2 f_{[0,2,0]}, \end{aligned} \tag{8}$$

where a_0, a_1, a_2 are any real (in practice nonnegative) numbers. A method of symmetric adding the functions f_q, $q \in \mathcal{T}^n \setminus \mathcal{T}^n_1$, to the original basis functions is also valid for SHK-patches of order ≥ 3.

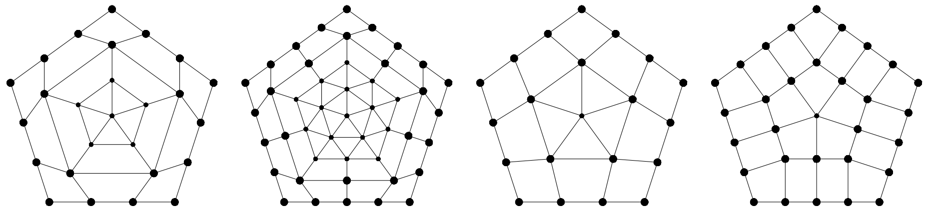

Fig. 7. Examples of the extended SHK-patterns

4 Proof of the main theorem

4.1 Reparametrization maps

Let $z_0 = \dfrac{2}{3}$, $z_1 = \dfrac{8\cos^3\alpha - 4\cos^2\alpha - 1}{3 + 6\cos\alpha}$. For $s = 0, 1, \ldots, m-1$ we set

$$Q_0^s = V_{s-1},\ Q_1^s = z_0 V_{s-1} + z_1 V_s + (1 - z_0 - z_1)O,$$
$$Q_2^s = z_0 V_{s+2} + z_1 V_{s+1} + (1 - z_0 - z_1)O,\ Q_3^s = V_{s+2},$$

where O is a center of D. For $s = 0, 1, \ldots, m-1$ we define the cubic curves mac_s and reparametrization maps $\mathrm{re}_s : [0,1] \times [0,1] \to D$ by the formulas

$$\begin{aligned} &\mathrm{mac}_s(u) = \textstyle\sum_{k=0}^{3} Q_k^s B_k^3(u), \\ &\mathrm{re}_s(t,u) = (V_s u_0 + V_{s+1} u_1)(1-t) + \mathrm{mac}_s(u) t. \end{aligned} \tag{9}$$

On a left in Figure 7 there are displayed a cubic mac_s together with its control points and the (full) lines $\mathrm{re}_s(t, \mathrm{const})$. On a right picture there are also shown the (dashed) lines, connecting a point K_s with the points $V_s(1 - \mathrm{const}) + V_{s+1}\mathrm{const}$. We set:

$$\bar{l}_{-2} = l_{s-2}(\mathrm{re}_s(0,u)),\ \bar{l}_{-1} = l_{s-1}(\mathrm{re}_s(0,u)),\ \bar{l}_1 = l_{s+1}(\mathrm{re}_s(0,u)),$$
$$\bar{l}_2 = l_{s+2}(\mathrm{re}_s(0,u)),\ \overline{C} = C(\mathrm{re}_s(0,u)),$$
$$\tilde{l}_{-1} = \partial l_{s-1}(re_s(t,u))/\partial t\ |_{t=0},\ \tilde{l}_0 = \partial l_s(\mathrm{re}_s(t,u))/\partial t\ |_{t=0},$$
$$\tilde{l}_1 = \partial l_{s+1}(\mathrm{re}_s(t,u))/\partial t\ |_{t=0},\ \widetilde{C} = \partial C(\mathrm{re}_s(t,u))/\partial t\ |_{t=0}.$$

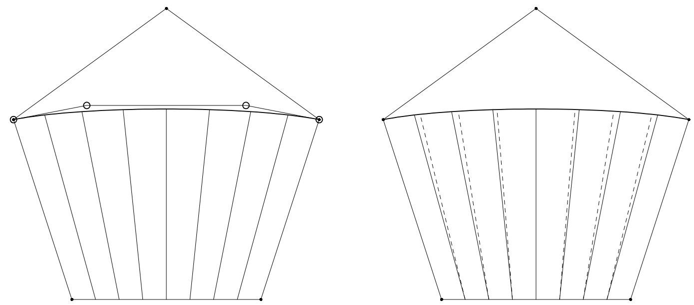

Fig. 8. Reparametrization with cubic mac_s

Direct computation gives the following important identities:

$$
\begin{aligned}
&\bar{l}_{-1} = 2\sin\alpha \sin\frac{\alpha}{2} u_1,\ \ \bar{l}_1 = 2\sin\alpha\sin\frac{\alpha}{2}u_0,\\
&\bar{l}_{-2} = 2\sin\alpha\sin\frac{\alpha}{2}(2u_1\cos\alpha+1),\ \ \bar{l}_2 = 2\sin\alpha\sin\frac{\alpha}{2}(2u_0\cos\alpha+1),\\
&\tilde{l}_0 = \frac{\sin^2 2\alpha}{2\sin\alpha\sin\frac{3\alpha}{2}}\overline{C},\ \ \bar{l}_{-2}\bar{l}_2 = \sin^2 2\alpha\overline{C},\\
&\tilde{l}_{-1} = \frac{2\sin 2\alpha\sin^2\frac{\alpha}{2}}{\sin\frac{3\alpha}{2}}u_1(2\cos\alpha+(2\cos 2\alpha+5)u_1-(2\cos 2\alpha+4)u_1^2),\\
&\tilde{l}_1 = \frac{2\sin 2\alpha\sin^2\frac{\alpha}{2}}{\sin\frac{3\alpha}{2}}u_0(2\cos\alpha+(2\cos 2\alpha+5)u_0-(2\cos 2\alpha+4)u_0^2).
\end{aligned}
\tag{10}
$$

Remark 11 *At the moment the author knows only experimental justification of cubics* mac_s *(consider* mac *as a "magic cubic"):*
(1) the main theorem was first proved using MAPLE package for the most important values $n = 2, 3, 4$*;*
(2) it was found in the "sledgehammered" cases $n = 2, 3, 4$ *a crossderivative of constructed* SHK*-patches is governed by the "magic cubic"* mac_s*;*
(3) author (1999) shows how the classical pentagonal Sabin patch can be represented as a T*-patch of order* 3*; in this case it was checked a crossderivative is also governed by the* mac_s*;*
(4) the mentioned observations suggest to base a proof on the cubics mac_s *and it really does the trick.*

4.2 Reduction and scheme of a proof

Suppose we try to find the functions g_q, $q \in \mathcal{H}^n$, in a form, declared in Theorem 10, but with still unknown coefficients a, b_j, c_j, d_j, satisfying natural symmetry conditions $b_{n-j} = c_j$, $d_{n-j} = d_j$.

For $s = 0, 1, \ldots, m-1$ we set

$$
\begin{aligned}
&h_{0j}(t,u) = g_{s0j}(\mathrm{re}_s(t,u)),\ \ 0 \le j \le n, \\
&h_{1j}(t,u) = g_{s1j}(\mathrm{re}_s(t,u)),\ \ 1 \le j \le n-1, \\
&h_{10}(t,u) = g_{s-1,0,n-1}(\mathrm{re}_s(t,u)),\ \ h_{1n}(t,u) = g_{s+1,0,1}(\mathrm{re}_s(t,u)).
\end{aligned}
\tag{11}
$$

Our task is to show, that for the appropriate values of a,b_j, c_j, d_j there exist the functions $A(u), D(u)$, satisfying the assumptions of Lemma 9.
According to Proposition 3 only the functions

$$f_{[s,0,j]},\ \ 0 \le j \le n; \tag{12}$$

$$f_{[s,1,j]},\ \ 0 \le j \le n-1; \tag{13}$$

$$f_{[s-1,0,n-1]},\ \ f_{[s+1,0,1]}; \tag{14}$$

have nonzero crossderivatives along the edge $\overline{V_s V_{s+1}}$ and only the functions from (12) are nonzero on it. So finding the coefficients a, b_j, c_j, d_j we can set

$$
\begin{aligned}
&h_{00} = (f_{[s,0,0]} + a f_{[s,1,0]}) \circ \mathrm{re}_s,\ \ h_{0n} = (f_{[s,0,n]} + a f_{[s,1,n-1]}) \circ \mathrm{re}_s, \\
&h_{0j} = (f_{[s,0,j]} + b_j f_{[s,1,j-1]} + c_j f_{[s,1,j]}) \circ \mathrm{re}_s,\ \ 1 \le j \le n-1, \\
&h_{10} = (f_{[s-1,0,n-1]} + c_{n-1} f_{[s,1,0]}) \circ \mathrm{re}_s, \\
&h_{1n} = (f_{[s+1,0,1]} + b_1 f_{[s,1,n-1]}) \circ \mathrm{re}_s, \\
&h_{1j} = d_j (f_{[s,1,j-1]} + f_{[s,1,j]}) \circ \mathrm{re}_s,\ \ 1 \le j \le n-1.
\end{aligned}
\tag{15}
$$

For a simplicity in (15) the same letter h is used as in (11). Though the functions defined by the formulas (11) and (15) are, in general, different, the restrictions of these functions and of their crossderivatives are equal on $t = 0$ (just what exactly is needed).

In the next two steps we will find the coefficients a, b_j, c_j, d_j, insuring the assumptions of Lemma 9 are satisfied. Then Lemma 9, together with the just mentioned properties of the functions from (12), (13) and (14) will imply all conditions of Proposition 8 are satisfied. This will end a proof of Theorem 10.

4.3 Step 1: calculation of of the coefficients d_j, b_1, c_{n-1}

4.3.1 Canceling first unessential factor

Let $Y_s = \{0, 1, \ldots, m-1\} \setminus \{s-2, s-1, s, s+1, s+2\}$, $0 \le s \le m-1$. Using the formula (1) we get the functions f_q from (12), (13) and 14) can be written in a form $f_q = \check{f}_q M$, where $M = l_{s-2}^{n-1} l_{s+2}^{n-1} \prod_{r\in Y_s} l_r^n$,

$$\begin{aligned} &\check{f}_{[s,0,j]} = \binom{n}{j} l_{s-1}^j l_{s+1}^{n-j} C l_{s-2} l_{s+2},\ 0 \le j \le n, \\ &\check{f}_{[s-1,0,n-1]} = n l_s l_{s+1}^n C l_{s+2},\ \check{f}_{[s+1,0,1]} = n l_s l_{s-1}^n C l_{s-2}, \\ &\check{f}_{[s,1,j]} = l_s l_{s-1}^{j+1} l_{s+1}^{n-j} l_{s-2} l_{s+2},\ 0 \le j \le n-1. \end{aligned} \tag{16}$$

So we can write $h_{ij}(t,u) = \check{h}_{ij}(t,u) M(\mathrm{re}_s(t,u))$. The functions $\check{h}_{ij}$ are defined by the formulas (15), where h_{ij} is replaced by $\check{h}_{ij}$ and f_q is replaced by $\check{f}_q$. It follows from (16), that $\check{h}_{1j}(0,u) = 0$, $0 \le j \le n$. Suppose

$$\check{h}_{0j}(0,u) = B_j^n(u)\check{A}(u),\ \partial\check{h}_{1j}/\partial t\ |_{t=0} = nB_j^n(u)\check{A}(u).$$

Then we get

$$h_{0j}(0,u) = B_j^n(u)A(u),\ \partial h_{1j}/\partial t\ |_{t=0} = nB_j^n(u)A(u),$$

where $A(u) = \check{A}(u)M(\mathrm{re}_s(0,u))$. This means we can consider instead of h_{ij} more simple functions $\check{h}_{ij}$. In Section 4.3.2 we set for a simplicity $h_{ij} := \check{h}_{ij}$.

4.3.2 End of the step 1

- Using the formulas from (10) we get for $i = 0, 1, \ldots, m-1$

$$\begin{aligned} h_{0j}(0,u) &= \binom{n}{j} \bar{l}_{-1}^j \bar{l}_1^{n-j} \overline{C} \bar{l}_{-2} \bar{l}_2 = \binom{n}{j} 2^n \sin^n \alpha \sin^n \tfrac{\alpha}{2} \sin^2 2\alpha \overline{C}^2 u_1^j u_0^{n-j} \\ &= A(u)B_j^n(u),\ \text{where } A(u) = 2^n \sin^n \alpha \sin^n \tfrac{\alpha}{2} \sin^2 2\alpha \overline{C}^2. \end{aligned}$$

- Differentiating the function h_{10} respect to t and using the formulas from (10) we get

$$\begin{aligned} \partial h_{10}/\partial t\ |_{t=0} &= n\tilde{l}_0 \bar{l}_1^n \overline{C} \bar{l}_2 + c_{n-1} \tilde{l}_0 \bar{l}_{-1}^n \bar{l}_1^n \bar{l}_{-2} \bar{l}_2 = A(u)(\sin\tfrac{\alpha}{2}/\sin\tfrac{3\alpha}{2}) \\ &\quad (2n\cos\alpha + n + (c_{n-1}\sin^2 2\alpha - 2n\cos\alpha)u)u_0^n. \end{aligned}$$

  Let $c_{n-1} = 2n\cos\alpha/\sin^2 2\alpha$. Then a trigonometric identity $\sin(3\alpha/2) = (2\cos\alpha + 1)\sin(\alpha/2)$ implies $\partial h_{10}/\partial t\ |_{t=0} = nA(u)B_0^n(u)$. If $b_1 = c_{n-1} = 2n\cos\alpha/\sin^2 2\alpha$ it follows from a symmetric definition of the functions h_{ij}, that $\partial h_{1n}/\partial t\ |_{t=0} = nA(u)B_n^n(u)$.
- Differentiating the function h_{1j}, $1 \le j \le n-1$, respect to t and using the formulas from (10) we get

$$\begin{aligned} \partial h_{1j}/\partial t\ |_{t=0} &= d_j \tilde{l}_0 \bar{l}_{-1}^j \bar{l}_1^{n-j+1} \bar{l}_{-2} \bar{l}_2 + d_j \tilde{l}_0 \bar{l}_{-1}^{j+1} \bar{l}_1^{n-j} \bar{l}_{-2} \bar{l}_2 \\ &= d_j A(u)(\sin\tfrac{\alpha}{2}\sin^2 2\alpha/\sin\tfrac{3\alpha}{2}) u_0^{n-j} u_1^j. \end{aligned}$$

If $d_j = \binom{n}{j} n \sin\frac{3\alpha}{2}/(\sin^2 2\alpha \sin\frac{\alpha}{2}) = \binom{n}{j} n(2\cos\alpha + 1)/\sin^2 2\alpha$ then $\partial h_{1j}/\partial t\mid_{t=0} = nA(u)B_j^n(u)$.

4.4 *Step 2: calculation of the coefficients a, b_j, c_j*

4.4.1 *Canceling second unessential factor*

Let $R_s = \{0, 1, \dots, m-1\} \setminus \{s-1, s, s+1\}$, $0 \le s \le m-1$. Using the formula (1) we get the functions f_q from (12) and (13) can be written in a form $f_q = \breve{f}_q N$, where $N = \prod_{r \in R_s} l_r^n$,

$$\begin{aligned} \breve{f}_{[s,0,j]} &= \binom{n}{j} l_{s-1}^j l_{s+1}^{n-j} C, \ 0 \le j \le n, \\ \breve{f}_{[s,1,j]} &= l_s l_{s-1}^{j+1} l_{s+1}^{n-j}, \ 0 \le j \le n-1. \end{aligned} \tag{17}$$

The functions g_{s0j}, $0 \le j \le n$, are linear combinations only of the functions f_q from (12) and (13). So we can write $h_{0j}(t,u) = \breve{h}_{0j}(t,u)N(re_s(t,u))$. The functions $\breve{h}_{0j}$ are defined by the formulas (15), where h_{0j} is replaced by $\breve{h}_{0j}$ and f_q is replaced by $\breve{f}_q$. Suppose

$$\breve{h}_{0j}(0,u) = B_j^n(u)\breve{A}(u), \ \partial\breve{h}_{0j}/\partial t\mid_{t=0} = B_j^n(u)\breve{D}(u).$$

Then we get $\partial h_{0j}/\partial t\mid_{t=0} = B_j^n(u)D(u)$, where

$$D(u) = \breve{D}(u)N(re_s(0,u)) + \breve{A}(u)\partial N(re_s(t,u))/\partial t\mid_{t=0}.$$

This means we can consider instead of h_{0j} more simple functions $\breve{h}_{0j}$. In Section 4.4.2 we set for a simplicity $h_{0j} := \breve{h}_{0j}$.

4.4.2 *End of the step 2*

Here we similarly as in Section 4.3.2 differentiate the functions h_{0j}, $0 \le j \le n$, respect to t and use the formulas (10). Since the expressions for the derivatives of h_{0j} are more complicated we set for a simplicity

$$\begin{aligned} E_{-1} &= 2\cos\alpha + (2\cos 2\alpha + 5)u_1 - (2\cos 2\alpha + 4)u_1^2, \\ E_1 &= 2\cos\alpha + (2\cos 2\alpha + 5)u_0 - (2\cos 2\alpha + 4)u_0^2, \\ L &= 2^n \sin^{n-1}\alpha \sin^n\tfrac{\alpha}{2}/\sin\tfrac{3\alpha}{2}. \end{aligned}$$

- $\partial h_{00}/\partial t\mid_{t=0} = n\bar{l}_1^{n-1}\tilde{l}_1\overline{C} + \bar{l}_1^n\widetilde{C} + a\tilde{l}_0\bar{l}_{-1}\bar{l}_1^n = D_0 B_0^n(u)$, where

$$D_0 = L(n\sin 2\alpha \sin\frac{\alpha}{2}E_1\overline{C} + \sin\alpha\sin\frac{3\alpha}{2}\widetilde{C} + a\sin^2 2\alpha\sin\alpha\sin\frac{\alpha}{2}\overline{C}u_1).$$

- if $1 \le j \le n-1$ then

$$\partial h_{0j}/\partial t\ |_{t=0} = \binom{n}{j}(j\bar{l}_{-1}^{j-1}\tilde{l}_{-1}\bar{l}_1^{n-j}\overline{C} + (n-j)\bar{l}_{-1}^{j}\bar{l}_1^{n-j-1}\tilde{l}_1\overline{C} + \bar{l}_{-1}^{j}\bar{l}_1^{n-j}\tilde{C})$$
$$+b_j\tilde{l}_0\bar{l}_{-1}^{j}\bar{l}_1^{n-j+1} + c_j\tilde{l}_0\bar{l}_{-1}^{j+1}\bar{l}_1^{n-j} = D_jB_j^n(u),\ \text{where}$$

$$D_j = L(j\sin 2\alpha\sin\tfrac{\alpha}{2}\overline{C}E_{-1} + (n-j)\sin 2\alpha\sin\tfrac{\alpha}{2}\overline{C}E_1 + \sin\alpha\sin\tfrac{3\alpha}{2}\tilde{C}$$
$$+\sin^2 2\alpha\sin\alpha\sin\tfrac{\alpha}{2}\overline{C}(b_ju_0 + c_ju_1)/\tbinom{n}{j}).$$

- The expressions D_j must be independent of j ($D_j = D(u)$). So $D_j = D_0$ gives an equation

$$jE_{-1} + (n-j)E_1 + \sin\alpha\sin 2\alpha(b_ju_0 + c_ju_1)/\tbinom{n}{j} =$$
$$nE_1 + a\sin 2\alpha\sin\alpha u_1.$$

  This equation can be written in a form $Z_0 + Z_1u = 0$, where

$$Z_0 = -j + \sin\alpha\sin 2\alpha(b_j/\tbinom{n}{j}),$$
$$Z_1 = 2j - \sin\alpha\sin 2\alpha((b_j - c_j)/\tbinom{n}{j} + a).$$

  Solving the system $Z_0 = 0,\ Z_1 = 0$ we get

$$b_j = j\tbinom{n}{j}/(\sin\alpha\sin 2\alpha),\ \ c_j = (a\sin\alpha\sin 2\alpha - j)\tbinom{n}{j}/(\sin\alpha\sin 2\alpha).$$

  So b_1 is the same as b_1 derived in Section 4.3.2. Obviously c_{n-1} also must coincide with c_{n-1} from Section 4.3.2. This gives an equation

$$(a\sin\alpha\sin 2\alpha - n + 1)n/(\sin\alpha\sin 2\alpha) = 2n\cos\alpha/\sin^2 2\alpha.$$

  Solving it we get $a = n/(\sin\alpha\sin 2\alpha)$. The final expression for the coefficients c_j becomes $c_j = (n-j)\binom{n}{j}/(\sin\alpha\sin 2\alpha)$. It follows from a symmetric definition of the functions h_{ij} that $\partial h_{0n}/\partial t\ |_{t=0} = D(u)B_n^n(u)$.

5 Triangular patches

5.1 Triangular T-patches

In this section a domain D is a triangle with the vertices V_0, V_1, V_2. It is convenient to use barycentric coordinates for the patches defined over triangle domain. Therefore we change the notations for the functions l_s. Every point V in a plane, containing triangle D, can be written in a form $V = l_0V_0 + l_1V_1 + l_2V_2$

with $l_0+l_1+l_2=1$. The functions l_0, l_1, l_2 are called the barycentric coordinates of a point V. If D is a standard triangle in (x,y)-plane with the vertices $V_0=(0;0)$, $V_1=(1;0)$, $V_2=(0;1)$, then $l_0=1-x-y$, $l_1=x$, $l_2=y$.
For $s=0,1,2$ and any real number e we set

$$M_s = l_s^2 + l_{s+1}^2 + 2el_s l_{s+1},\ \ N_s = l_s^2 + l_{s+1}^2 + l_{s+2}^2 + 2el_s(l_{s+1}+l_{s+2}).$$

The functions f_q^n, $q=(s,i,j)\in\mathcal{T}^n$, are defined by the formulas

$$\begin{aligned} &f^n_{(s,0,0)} = N_s l_s^n,\ \ f^n_{(s,0,j)} = \binom{n}{j} M_s l_{s+1}^j l_s^{n-j},\ \ 1\le j\le n-1,\\ &f^n_{(s,i,j)} = l_{s+1}^{i+j} l_{s+2}^i l_s^{n-j},\ \ i\ge 1,\ \ 0\le j\le n-i. \end{aligned} \tag{18}$$

The index d is dropped from a notation of the functions f_q^n, since they are independent od d. Definition 1 from Section 2 is also valid for a triangle domain D if in the formula (2) the functions $f_q^{n,d}$ are replaced by the functions f_q^n. After this replacement the propositions 3, 5, 4 from Section 2.2 remains true (their proofs are similar). Since $N_s\mid_{l_{s+2}=0} = M_s\mid_{l_{s+2}=0}$ we get a map F_d^n, restricted to the edge $\overline{V_sV_{s+1}}$, defines a rational Bézier curve of degree n with the control points P_{s0j} and the weights w_{s0j}. It follows from the formulas (18) the maximum of the degrees of f_q^n, $q\in\mathcal{T}_d^n$, is $n+2d$. This implies a triangular T-patch of order n and depth d can be represented as a rational triangular Bézier patch of degree $n+2d$.

Remark 12 *Involving e in the definition of the functions f_q^n creates on the lines $l_s=0$ the basis points (maybe complex) of the map F_d^n. (If $m=3$ there are no points K_s). This simulation really does the trick in the construction of triangular SHK-patches.*

5.2 Triangular SHK-patches

In the triangular version of the main theorem we also use only the functions f_q^n, $q\in\mathcal{T}_1^n$. For the reasons, explained at the beginning of Section 3.2, we keep introduced there indexing notations. So if $q=(s,i,j)\in\mathcal{T}_1^n$ we set $f_{[s,i,j]} := f_q^n$.

Theorem 13 *Let $a=n(1-2e)+2e$, $b_j=\binom{n}{j}(j(1-2e)+2e)$,*
$c_j=\binom{n}{j}((n-j)(1-2e)+2e)$, $d_j=n\binom{n}{j}$, $1\le j\le n-1$, and for $s=0,1,2$
- $g_{s00} = f_{[s,0,0]} + af_{[s,1,0]}$;
- $g_{s0j} = f_{[s,0,j]} + b_j f_{[s,1,j-1]} + c_j f_{[s,1,j]}$, $1\le j\le n-1$;
- $g_{s11} = d_1(f_{[0,1,0]} + f_{[1,1,0]} + f_{[2,1,0]})$ *if* $n=2$;
  $g_{s11} = d_1(f_{[s-1,1,n-2]} + f_{[s,1,0]} + f_{[s,1,1]})$ *if* $n\ge 3$;
  $g_{s1j} = d_j(f_{[s,1,j-1]} + f_{[s,1,j]})$, $2\le j\le n-2$, *if* $n\ge 4$.

Then a patch defined by the formula (5) is a rational SHK-patch of order n.

Proof. This proof is similar to the proof of the main theorem: first we define the cubics mac_s and reparametrization maps re_s; the coefficients a, b_j, c_j, d_j, insuring the conditions of Lemma 9 are satisfied, are found step by step; Lemma 9 implies the conditions of Proposition 8 are satisfied and it will finish a proof.

For $s = 0, 1, 2$ we set

$$Q_0^s = V_{s+2},\ Q_1^s = (1 - \tfrac{4e}{3})V_s + (\tfrac{2e}{3} - \tfrac{1}{3})V_{s+1} + (\tfrac{1}{3} + \tfrac{2e}{3})V_{s+2},$$
$$Q_2^s = (\tfrac{2e}{3} - \tfrac{1}{3})V_s + (1 - \tfrac{4e}{3})V_{s+1} + (\tfrac{1}{3} + \tfrac{2e}{3})V_{s+2},\ Q_3^s = V_{s+2}.$$

The cubic curves mac_s and the reparametrization maps re_s are defined, as in Section 4.1, by the formula (9). In Figure 9 there are displayed the examples of the cubics mac_s together with the lines $\mathrm{re}_s(t, \mathrm{const})$. Some values of e are special: if $e = 1$ the cubic mac_s degenerates to a line; if $e = 0.5$ the lines $\mathrm{re}_s(t, \mathrm{const})$ go through the vertex V_{s+2}. Note $l_{s+2}(\mathrm{re}_s(0, u)) = 0$. We set:

$$\bar{l}_0 = l_s(\mathrm{re}_s(0, u)),\ \bar{l}_1 = l_{s+1}(\mathrm{re}_s(0, u)),\ \tilde{l}_0 = \partial l_s(\mathrm{re}_s(t, u))/\partial t\ |_{t=0},$$
$$\tilde{l}_1 = \partial l_{s+1}(\mathrm{re}_s(t, u))/\partial t\ |_{t=0},\ \tilde{l}_2 = \partial l_{s+2}(\mathrm{re}_s(t, u))/\partial t\ |_{t=0},$$
$$\overline{N}_0 = N_s(\mathrm{re}_s(0, u)),\ \overline{M}_0 = M_s(\mathrm{re}_s(0, u)),\ \overline{M}_2 = M_{s+2}(\mathrm{re}_s(0, u)),$$
$$\widetilde{N}_0 = \partial N_s(\mathrm{re}_s(t, u))/\partial t\ |_{t=0},\ \widetilde{M}_0 = \partial M_s(\mathrm{re}_s(t, u))/\partial t\ |_{t=0},$$
$$A(u) = 1 + 2(e - 1)u_0u_1.$$

Direct computation gives the following important identities:

$$\begin{aligned}
&\bar{l}_0 = u_0,\ \bar{l}_1 = u_1,\ \tilde{l}_0 = u_0E_0,\ \tilde{l}_1 = u_1E_1,\ \tilde{l}_2 = A(u),\\
&E_0 = (4 - 6e)u_0u_1 + (2e - 1)u_1 - 1,\\
&E_1 = (4 - 6e)u_0u_1 + (2e - 1)u_0 - 1;\\
&\widetilde{N}_0 = 2((e - 1)u_0^4 + (1 - 6e + 4e^2)u_0^3u_1 + (-4 + 7e - 6e^2)u_0^2u_1^2\\
&\qquad + (1 - 6e + 4e^2)u_0u_1^3 - u_1^4),\\
&\widetilde{M}_0 = 2(-u_0^4 + (1 - 7e + 2e^2)u_0^3u_1 + (-4 + 6e - 8e^2)u_0^2u_1^2\\
&\qquad + (1 - 7e + 2e^2)u_0u_1^3 - u_1^4),\\
&\overline{N}_0 = \overline{M}_0 = A(u),\ \overline{M}_2 = u_0^2,\ \widetilde{N}_0 - \widetilde{M}_0 = 2eu_0A(u).
\end{aligned} \tag{19}$$

Similar arguments as in Section 4.2 show the functions h_{ij} can be defined by the formulas (15). In the items 2–5 we simply differentiate the functions h_{ij} respect to t and use the identities from (19).

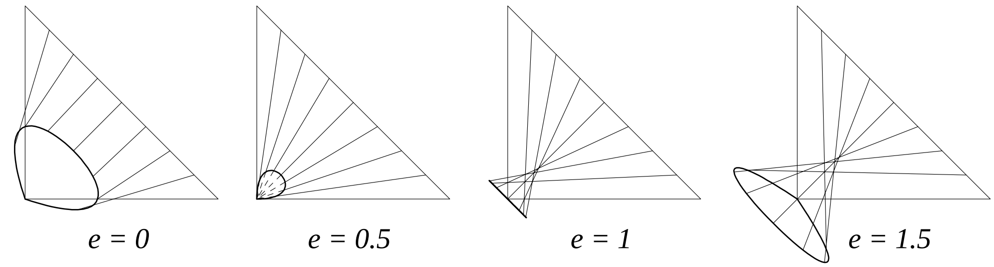

Fig. 9. Cubics mac$_s$ for a triangle domain

1. Since $\bar{l}_0 = u_0$, $\bar{l}_1 = u_1$, $\overline{N}_0 = \overline{M}_0 = A(u)$, we get for $j = 0, 1, \ldots, n$ $h_{0j}(0, u) = \binom{n}{j} A(u) u_0^{n-j} u_1^j = A(u) B_j^n(u)$.
2. $\partial h_{10}/\partial t\ |_{t=0} = n\tilde{l}_2 \bar{l}_0^{n-1} \overline{M}_2 + c_{n-1} \tilde{l}_2 \bar{l}_0^n \bar{l}_1 = A(u) n u_0^n (1 - u + u c_{n-1}/n)$. If $c_{n-1} = n$ then $\partial h_{10}/\partial t\ |_{t=0} = A(u) n B_0^n(u)$. If $b_1 = c_{n-1} = n$ it follows from a symmetric definition of the functions h_{ij} that $\partial h_{1n}/\partial t\ |_{t=0} = A(u) n B_n^n(u)$.
3. For $j = 1, \ldots, n-1$ we get $\partial h_{1j}/\partial t\ |_{t=0} = d_j(\tilde{l}_2 \bar{l}_1^j \bar{l}_0^{n-j+1} + \tilde{l}_2 \bar{l}_1^{j+1} \bar{l}_0^{n-j}) = d_j A(u) u_0^{n-j} u_1^j$. If $d_j = n\binom{n}{j}$ then $\partial h_{1j}/\partial t\ |_{t=0} = A(u) n B_j^n(u)$.
4. $\partial h_{00}/\partial t\ |_{t=0} = \widetilde{N}_0 \bar{l}_0^n + n \bar{l}_0^{n-1} \tilde{l}_0 \overline{N}_0 + a \tilde{l}_2 \bar{l}_0^n \bar{l}_1 = D_0 B_0^n(u)$, where $D_0 = \widetilde{N}_0 + nA(u)E_0 + aA(u)u_1$.
5. For $j = 1, \ldots, n-1$ we get

$$\begin{aligned}\partial h_{0j}/\partial t\ |_{t=0} = \binom{n}{j}(\widetilde{M}_0 \bar{l}_1^j \bar{l}_0^{n-j} + j \bar{l}_1^{j-1} \tilde{l}_1 \bar{l}_0^{n-j} \overline{M}_0 + (n-j) \bar{l}_0^{n-j-1} \tilde{l}_0 \bar{l}_1^j \overline{M}_0) \\ + b_j \tilde{l}_2 \bar{l}_1^j \bar{l}_0^{n-j+1} + c_j \tilde{l}_2 \bar{l}_1^{j+1} \bar{l}_0^{n-j} = D_j B_j^n(u), \text{ where}\end{aligned}$$

$D_j = \widetilde{M}_0 + jA(u)E_1 + (n-j)A(u)E_0 + A(u)(b_j u_0 + c_j u_1)/\binom{n}{j}$.
6. The expressions D_j must be independent of j ($D_j = D(u)$). The last identity from (19) $\widetilde{N}_0 - \widetilde{M}_0 = 2eu_0A(u)$ implies, that equation $D_j - D_0 = 0$ contains as a factor $A(u)$. Simple calculations show, that after cancellation of the factor $A(u)$ this equation can be written in a form $Z_0 + Z_1 u = 0$, where

$$\begin{aligned}Z_0 &= -2e + j(2e-1) + b_j/\binom{n}{j}, \\ Z_1 &= 2e - a - 2j(2e-1) - (b_j + c_j)/\binom{n}{j}.\end{aligned}$$

Solving the system $Z_0 = 0,\ Z_1 = 0$ we get

$$b_j = \binom{n}{j}(j(1-2e) + 2e), \quad c_j = \binom{n}{j}(a + j(2e-1)).$$

So b_1 is the same as b_1 from the item 2. Obviously c_{n-1} also must coincide with c_{n-1} from the item 2. This gives an equation $n(a+(n-1)(2e-1)) = n$. Solving it we get $a = n(1-2e)+2e$. The final expression for the coefficients c_j becomes $c_j = \binom{n}{j}((n-j)(1-2e) + 2e)$. It follows from a symmetric definition of the functions h_{ij} that $\partial h_{0n}/\partial t\ |_{t=0} = D(u) B_n^n(u)$. $\square$

A content of this section is similar to that of Section 3.3. We do not consider here extended structures, since everything from Section 3.3.5 is also valid for the triangular patches.

1. Convex hull property. The SHK-patches possess obviously convex hull property if all basis functions g_q are nonnegative on D. They are nonnegative for all orders n if $0 \le e \le 0.5$ (it follows from the expressions for a, b_j, c_j, d_j). Deeper analysis shows, that for the special values n we can take a wider range for e, insuring nonnegativity of basis functions g_q: for $n = 2$ we get $0 \le e \le 1.5$; for $n = 3$ – $0 \le e \le 1$.

2. Degree of SHK-patch. It follows from the formulas (18) the basis functions of SHK-patch of order n have degree $n+2$. This means triangular SHK-patch of order n can be represented as a rational triangular Bézier patch of degree $n+2$. Degrees of triangular SHK-patches of orders 2 and 3, derived by Loop & De Rose (1990) are respectively 5 and 6. Theorem 13 gives the patches of degrees 4 and 5 respectively. So let us compare the patches of orders 2 and 3 from Theorem 13 with the original triangular Sabin and Hosaka–Kimura patches.

3. Comparing with Sabin patch. Suppose in $\mathbb{R}^3$ with a coordinate system $(u; v; w)$ is fixed a domain triangle D with the vertices $V_0 = (1; 0; 0)$, $V_1 = (0; 1; 0)$; $V_2 = (0; 0; 1)$. Then $l_0 = u$, $l_1 = v$, $l_2 = w$. Let $n = 2$ and $e = 1$. Since $l_0 + l_1 + l_2 = 1$ on D it follows from Theorem 13 that for $s = 0, 1, 2$

$$g_{s00} = l_s^2(1 - 2l_{s+1}l_{s+2}), \;\; g_{s01} = 2l_s l_{s+1}(1 - l_{s+2}), \;\; g_{s11} = 4l_0 l_1 l_2.$$

We set $h_{2s} = g_{s00}$, $h_{2s+1} = g_{s01}$, $s = 0, 1, 2$, $h_6 = g_{011}$. Let us consider the functions $h_i = h_i(u, v, w)$, $0 \le i \le 6$, as the functions in $\mathbb{R}^3$. Then they coincide with the basis functions of the original Sabin patch, defined on a curved triangular domain $u, v, w \ge 0$, $u + v + w - 2uvw = 1$. For the control points P_i, $0 \le i \le 6$, and a real number r by $S(r)$ we denote a triangular patch, defined on the domain

$$u, v, w \ge 0, \;\; u + v + w - ruvw = 1 \tag{20}$$

via map $(u, v, w) \mapsto \sum_{i=0}^{6} h_i P_i / \sum_{i=0}^{6} h_i$. Direct calculations show they all are tangent to each other along a common boundary. But $S(2)$ is the original Sabin patch and $S(0)$ – patch from Theorem 13 with $e = 1$. So we get one parameter family of SHK-patches of order 2 (Sabin like patches). Theorem 13 gives another one parameter family of Sabin like patches. These two families consist, in general, of the different patches. They all are of degree 4 and they have one common patch with $e = 1$. The experiments show that for suitable values of e the energy of SHK-patch is less then the energy of the Sabin patch. It indicates e is an important shape parameter.

4. Comparing with Hosaka–Kimura patch. Similar results we get for Hosaka–Kimura like patches. If $n = 3$ and $e = 1$ then

$$g_{s00} = l_s^3(1 - 3l_{s+1}l_{s+2}),\ g_{s01} = 3l_s^2 l_{s+1}(1 - l_{s+2} - l_{s+2}l_{s+1}),$$
$$g_{s02} = 3l_s l_{s+1}^2(1 - l_{s+2} - l_{s+2}l_s),\ g_{s11} = 9l_s^2 l_{s+1}l_{s+2}.$$

For $s = 0, 1, 2$ we set $h_{3s} = g_{s00}$, $h_{3s+1} = g_{s02}$, $h_{3s+2} = g_{s02}$, $h_{9+s} = g_{s11}$. The functions $h_i = h_i(u, v, w)$, as the functions in $\mathbb{R}^3$, coincide with the basis functions of Hosaka–Kimura patches $\bar{h}_i$, except of "inner" functions h_9, h_{10}, h_{11}. Corresponding basis functions $\bar{h}_9$, $\bar{h}_{10}$, $\bar{h}_{11}$ of Hosaka–Kimura have a form $\bar{h}_i = h_i - 2(uvw)^2 + 4(uvw)^3/3$, $i = 9, 10, 11$. Similarly as in the previous case we define on the domain (20) triangular Hosaka–Kimura like patches via map $(u, v, w) \mapsto \sum_{i=0}^{11} \tilde{h}_i P_i / \sum_{i=0}^{11} \tilde{h}_i$, where $\tilde{h}_i = h_i$, $0 \leq i \leq 8$, $\tilde{h}_i = h_i + (r/2)(-2(uvw)^2 + 4(uvw)^3/3)$, $i = 9, 10, 11$. So we get a family of patches of degree 9, connecting original Hosaka–Kimura patch $(r = 2)$ with a patch of degree 5 $(r = 0)$ from Theorem 13.

5. Integral patches. Integral SHK-patches we get when a denominator in formula (5) sums to 1. This is impossible for $m \geq 5$, since all functions g_q are zero at the points K_s (base points of a map G^n). For triangle patches, if $e = 1$, base points of G^n are infinite points. So only in this case we could get integral patches. For example, it is true (denominator in formula (5) is equal to 1) if:

(case $n = 2$) inner weight $w_{011} = 2$ and other weights are equal to 1;

(case $n = 3$) inner weights $w_{011} = w_{111} = w_{211} = 5/3$ and other weights are equal to 1;

For $n > 3$ we get integral patches if additional basis functions are involved (extended SHK-patches). So triangular SHK-patch itself can be integral. But adjacent rectangular Bézier patches in a smooth spline surface must be rational. This means in applications the integral SHK-patches do not play an important role.

7. Twist incompatibility. Actually everything from Section 3.3.4 is true for the triangular patches. Since the notations of l_s are different for triangular patches the formula (6) becomes $P_{s11} = (l_{s-1}P^{s-1}_{s-1,1,n-1} + l_{s+1}P^s_{s,1,1})/(l_{s-1} + l_{s+1})$. The formula (7) remains true if we set $L_s = l_s l_{s+1}$.

6 Spline surfaces

Already a definition of SHK-patches suggests they will be used for a construction of spline surfaces, containing non rectangular patches. Such spline surfaces

can be constructed using already known generalized B-spline schemes (see Sabin (1983), Loop & De Rose (1990)). Generalized biquadratic scheme produces surfaces, build from the biquadratic Bézier patches and SHK-patches of order 2 (Sabin like patches). Generalized bicubic scheme produces surfaces, build from the bicubic Bézier patches and SHK-patches of order 3 (Hosaka–Kimura like patches). Bicubic scheme can be also applied for a construction of the interpolating spline surfaces. One of the standard methods of creating generalized biquadratic and bicubic schemes is a repeated application of the bicubic and biquadratic subdivision algorithms (see Cattmull & Clark (1978), Doo & Sabin (1978)).
The mentioned methods produce compatible data at the corners of the adjacent patches (no "twist incompatibility"). But if a surface is constructed by some other methods a "twist incompatibility" may arise at the corners of the patches. In this cases for a smooth filling of m-sided holes we use SHK-patches with variable inner control points (they can be treated as m-sided Gregory patches). Lodha (1993) applied this method for SHK-patches, developed by Loop & De Rose (1990).

7 Conclusion and future work

In this paper we have introduced new control point scheme for rational m-sided surface patches (T-patches). They are basis for a construction of rational m-sided Sabin–Hosaka–Kimura like (SHK) patches for arbitrary $m \geq 3$, $m \neq 4$, with boundary Bézier curves of arbitrary degree n. Derived SHK-patches can meet surrounding rectangular patches with C^1 continuity and possess convex hull property.
In preprint (1999) it is shown, that 5- and 6-sided patches, derived by author (1998), give even more efficient representation of 5- and 6-sided SHK-patches. Hence we do not consider here a relationship with the classical pentagonal Sabin and Hosaka-Kimura patches. Main interest of a current research of the author are rational m-sided patches, meeting surrounding rectangular patches with G^2 continuity. It would be also interesting to understand a geometric origin of "magic" cubics.

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