We examine the ratio of the output of two different radial center frequencies k_1 and k_2 and obtain:

$$\begin{aligned}
\frac{\hat{r}_2}{\hat{r}_1} &= \exp\left[-\frac{(\ln k - \ln k_2)^2 - (\ln k - \ln k_1)^2}{2\sigma^2 \ln 2}\right] \\
&= \exp\left[\frac{2(\ln k_2 - \ln k_1)\ln k + \ln^2 k_2 - \ln^2 k_1}{2\sigma^2 \ln 2}\right] \\
&= \exp\left[\frac{(\ln k_2 - \ln k_1)[\ln k - 1/2(\ln k_2 + \ln k_1)]}{\sigma^2 \ln 2}\right] \\
&= \exp\left[\frac{\ln(k/\sqrt{k_2 k_1})\ln(k_2/k_1)}{\sigma^2 \ln 2}\right] \\
&= \left(\frac{k}{\sqrt{k_1 k_2}}\right)^{\ln(k_2/k_1)/(\sigma^2 \ln 2)}
\end{aligned}$$

Generally, the ratio of two different radial filters is directly related to the local wave number. The relation becomes particularly simple if the exponent in the last expression is one. This is the case, for example, if the wave-number ratio of the two filters is two ($k_2/k_1 = 2$ and $\sigma = 1$). Then

$$\frac{\hat{r}_2}{\hat{r}_1} = \frac{k}{\sqrt{k_1 k_2}} \tag{8.109}$$

8.9 Scale space and diffusion

As we have seen with the example of the windowed Fourier transform in the previous section, the introduction of a characteristic *scale* adds a new coordinate to the representation of image data. Besides the spatial resolution, we have a new parameter that characterizes the current resolution level of the image data. The scale parameter is denoted by ξ. A data structure that consists of a sequence of images with different resolutions is known as a *scale space*; we write $g(\boldsymbol{x}, \xi)$ to indicate the scale space of the image $g(\boldsymbol{x})$. Such a sequence of images can be generated by repeated convolution with an appropriate smoothing filter kernel.

This section is considered a brief introduction into scale spaces. For an authoritative monograph on scale spaces, see Lindeberg [16].

8.9.1 General properties of a scale space

In this section, we discuss some general conditions that must be met by a filter kernel generating a scale space. We will discuss two basic requirements. First, new details must not be added with increasing scale parameter. From the perspective of information theory, we may say that the information content in the signal should continuously decrease with the scale parameter.

The second property is related to the general principle of *scale invariance*. This basically means that we can start smoothing the signal at any scale parameter in the scale space and still obtain the same scale space.

Minimum-maximum principle. The information-decreasing property of the scale space with ξ can be formulated mathematically in different ways. We express it here with the *minimum-maximum principle*, which states that local extrema must not be enhanced. This means that the gray value at a local maximum or minimum must not increase or decrease, respectively. For the physical process of diffusion this is an intuitive property. For example, in a heat transfer problem, a hot spot must not become hotter or a cool spot cooler.

Semigroup property. The second important property of the scale space is related to the *scale invariance* principle. We want to start the generating process at any scale parameter and still obtain the same scale space. More quantitatively, we can formulate this property as

$$\mathcal{B}(\xi_2)\mathcal{B}(\xi_1) = \mathcal{B}(\xi_1 + \xi_2) \tag{8.110}$$

This means that the smoothing of the scale space at the scale ξ_1 by an operator with the scale ξ_2 is equivalent to the application of the scale space operator with the scale $\xi_1 + \xi_2$ to the original image. Alternatively, we can state that the representation at the coarser level ξ_2 can be computed from the representation at the finer level ξ_1 by applying

$$\mathcal{B}(\xi_2) = \mathcal{B}(\xi_2 - \xi_1)\mathcal{B}(\xi_1) \quad \text{with} \quad \xi_2 > \xi_1 \tag{8.111}$$

In mathematics the properties Eqs. (8.110) and (8.111) are referred to as the *semigroup property*.

Conversely, we can ask what scale space generating kernels exist that meet both the minimum-maximum principle and the semigroup property. The answer to this question may be surprising. As shown by Lindeberg [16, Chapter 2], the Gaussian kernel is the *only* convolution kernel that meets both criteria and is in addition isotropic and homogeneous. From yet another perspective this feature puts the Gaussian convolution kernel into a unique position for signal processing. With respect to the Fourier transform we have already discussed that the Gaussian function is one of the few functions with a shape that is invariant under the Fourier transform (Table 8.5) and optimal in the sense of the uncertainty relation (Section 8.6.3). In Section 9.5.4 we will see in addition that the Gaussian function is the only function that is separable and isotropic.

8.9.2 Linear scale spaces

Generation by a diffusion process. The generation of a scale space requires a process that can blur images to a controllable degree. Diffusion is a transport process that tends to level out concentration differences. In physics, diffusion processes govern the transport of heat, matter, and momentum [20] leading to an ever increasing equalization of spatial concentration differences. If we identify the time with the scale parameter ξ, the diffusion process thus establishes a scale space.

To apply a diffusion process to an image, we regard the gray value g as the concentration of a scalar property. The elementary law of diffusion states that the flux density $\boldsymbol{j}$ is directed against the concentration gradient $\boldsymbol{\nabla} g$ and is proportional to it:

$$\boldsymbol{j} = -D\boldsymbol{\nabla} g \tag{8.112}$$

where the constant D is known as the *diffusion coefficient*. Using the continuity equation

$$\frac{\partial g}{\partial t} + \boldsymbol{\nabla}\boldsymbol{j} = 0 \tag{8.113}$$

the diffusion equation is

$$\frac{\partial g}{\partial t} = \boldsymbol{\nabla}(D\boldsymbol{\nabla} g) \tag{8.114}$$

For the case of a homogeneous diffusion process (D does not depend on the position), the equation reduces to

$$\frac{\partial g}{\partial t} = D\Delta g \quad \text{where} \quad \Delta = \sum_{d=1}^{D} \frac{\partial^2}{\partial x_d^2} \tag{8.115}$$

It is easy to show that the general solution to this equation is equivalent to a convolution with a smoothing mask. To this end, we perform a spatial Fourier transform that results in

$$\frac{\partial \hat{g}(\boldsymbol{k})}{\partial t} = -4\pi^2 D|\boldsymbol{k}|^2 \hat{g}(\boldsymbol{k}) \tag{8.116}$$

reducing the equation to a linear first-order differential equation with the general solution

$$\hat{g}(\boldsymbol{k}, t) = \exp(-4\pi^2 Dt|\boldsymbol{k}|^2)\hat{g}(\boldsymbol{k}, 0) \tag{8.117}$$

where $\hat{g}(\boldsymbol{k}, 0)$ is the Fourier-transformed image at time zero.

Multiplication of the image in Fourier space with the Gaussian function in Eq. (8.117) is equivalent to a convolution with the same function but of reciprocal width. Using

$$\exp\left(-\pi a\,|\boldsymbol{k}|^2\right) \Longleftrightarrow \frac{1}{a^{d/2}} \exp\left(-\frac{|\boldsymbol{x}|^2}{a/\pi}\right) \tag{8.118}$$

we obtain with $a = 4\pi Dt$ for a d-dimensional space

$$g(\boldsymbol{x}, t) = \frac{1}{(2\pi)^{d/2}\sigma^d(t)} \exp\left(-\frac{|\boldsymbol{x}|^2}{2\sigma^2(t)}\right) * g(\boldsymbol{x}, 0) \qquad (8.119)$$

with

$$\sigma(t) = \sqrt{2Dt} \qquad (8.120)$$

Now we can replace the physical time coordinate by the scale parameter ξ with

$$\xi = 2Dt = \sigma^2 \qquad (8.121)$$

and finally obtain

$$g(\boldsymbol{x}, \xi) = \frac{1}{(2\pi\xi)^{d/2}} \exp\left(-\frac{|\boldsymbol{x}|^2}{2\xi}\right) * g(\boldsymbol{x}, 0) \qquad (8.122)$$

We have written all equations in such a way that they can be used for signals of any dimension. Thus, Eqs. (8.117) and (8.119) can also be applied to scale spaces of image sequences. The scale parameter is *not* identical to the time although we used a physical diffusion process that proceeds with time to derive it. If we compute a scale-space representation of an image sequence, it is useful to scale the time coordinate with a characteristic velocity u_0 so that it has the same dimension as the spatial coordinates: $t' = u_0 t$. For digital signals (Section 8.3), of course, no such scaling is required. It is automatically fixed by the spatial and temporal sampling intervals: $u_0 = \Delta x / \Delta t$.

As an illustration, Fig. 8.16 shows some individual images of the scale space of a 2-D image at values of ξ as indicated. This example nicely demonstrates a general property of scale spaces. With increasing scale parameter ξ, the signals become increasingly blurred, more and more details are lost. This feature can be most easily seen by the transfer function of the scale-space representation in Eq. (8.117). The transfer function is always positive and monotonically decreasing with the increasing scale parameter ξ for all wave numbers. This means that no structure is amplified. All structures are attenuated with increasing ξ, and smaller structures always faster than coarser structures. In the limit of $\xi \to \infty$ the scale space converges to a constant image with the mean gray value. A certain feature exists only over a certain scale range. We can observe that edges and lines disappear and two objects merge into one.

Accelerated scale spaces. Despite the mathematical beauty of scale-space generation with a Gaussian convolution kernel, this approach has

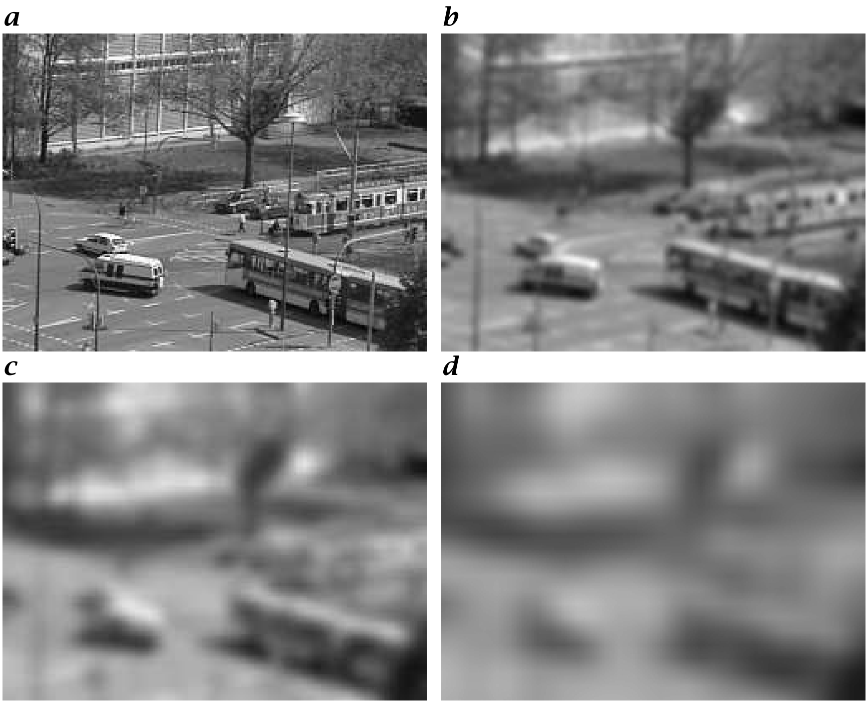

Figure 8.16: *Scale space of a 2-D image:* ***a*** *original image;* ***b***, ***c***, *and* ***d*** *at scale parameters* σ *1, 2, and 4, respectively.*

one significant disadvantage. The standard deviation of the smoothing increases only with the square root of the scale parameter ξ (see Eq. (8.121)). While smoothing goes fast for fine scales, it becomes increasingly slower for larger scales.

There is a simple cure for this problem. We need a diffusion process where the diffusion constant increases with time. We first discuss a diffusion coefficient that increases linearly with time. This approach results in the differential equation

$$\frac{\partial g}{\partial t} = D_0 t \Delta g \qquad (8.123)$$

A spatial Fourier transform results in

$$\frac{\partial \hat{g}(\boldsymbol{k})}{\partial t} = -4\pi^2 D_0 t |\boldsymbol{k}|^2 \hat{g}(\boldsymbol{k}) \qquad (8.124)$$

This equation has the general solution

$$\hat{g}(\boldsymbol{k}, t) = \exp(-2\pi^2 D_0 t^2 |\boldsymbol{k}|^2) \hat{g}(\boldsymbol{k}, 0) \qquad (8.125)$$

which is equivalent to a convolution in the spatial domain as in Eq. (8.121) with $\xi = \sigma^2 = D_0 t^2$. Now the standard deviation for the smoothing is proportional to time for a diffusion process with a diffusion coefficient that increases linearly in time. As the scale parameter ξ is proportional to the time squared, we denote this scale space as the *quadratic scale space*. This modified scale space still meets the minimum-maximum principle and the semigroup property.

For even more accelerated smoothing, we can construct a *logarithmic scale space*, that is, a scale space where the scale parameter increases logarithmically with time. We use a diffusion coefficient that increases exponentially in time:

$$\frac{\partial g}{\partial t} = D_0 \exp(t/\tau) \Delta g \tag{8.126}$$

A spatial Fourier transform results in

$$\frac{\partial \hat{g}(\boldsymbol{k})}{\partial t} = -4\pi^2 D_0 \exp(t/\tau) |\boldsymbol{k}|^2 \hat{g}(\boldsymbol{k}) \tag{8.127}$$

The general solution of this equation in the Fourier domain is

$$\hat{g}(\boldsymbol{k}, t) = \exp(-4\pi^2 D_0 (\exp(t/\tau)/\tau) |\boldsymbol{k}|^2) \hat{g}(\boldsymbol{k}, 0) \tag{8.128}$$

Again,the transfer function and thus the convolution kernel have the same form as in Eqs. (8.117) and (8.125), now with the scale parameter

$$\xi_l = \sigma^2 = \frac{2D_0}{\tau} \exp(t/\tau) \tag{8.129}$$

This means that the logarithm of the scale parameter ξ is now proportional to the limiting scales still contained in the scale space. Essentially, we can think of the quadratic and logarithmic scale spaces as a coordinate transform of the scale parameter that efficiently compresses the scale space coordinate:

$$\xi_q \propto \sqrt{\xi}, \qquad \xi_l \propto \ln(\xi) \tag{8.130}$$

8.9.3 Differential scale spaces

The interest in a *differential scale space* stems from the fact that we want to select optimum scales for processing of features in images. In a differential scale space, the change of the image with scale is emphasized. We use the transfer function of the scale-space kernel Equation (8.117), which is also valid for quadratic and logarithmic scale spaces. The general solution for the scale space can be written in the Fourier space as

$$\hat{g}(\boldsymbol{k}, \xi) = \exp(-2\pi^2 |\boldsymbol{k}|^2 \xi) \hat{g}(\boldsymbol{k}, 0) \tag{8.131}$$

Differentiating this signal with respect to the scale parameter ξ yields

$$\begin{aligned}\frac{\partial \hat{g}(\boldsymbol{k},\xi)}{\partial \xi} &= -2\pi^2\,|\boldsymbol{k}|^2 \exp(-2\pi^2\,|\boldsymbol{k}|^2\,\xi)\hat{g}(\boldsymbol{k},0)\\ &= -2\pi^2\,|\boldsymbol{k}|^2\,\hat{g}(\boldsymbol{k},\xi)\end{aligned} \tag{8.132}$$

The multiplication with $-4\pi^2|\boldsymbol{k}|^2$ is equivalent to a second-order spatial derivative (Table 8.4), the *Laplacian operator*. Thus we can write in the spatial domain

$$\frac{\partial g(\boldsymbol{x},\xi)}{\partial \xi} = \frac{1}{2}\Delta g(\boldsymbol{x},\xi) \tag{8.133}$$

Equations (8.132) and (8.133) constitute a basic property of the differential scale space. The differential scale space is equivalent to a second-order spatial derivation with the Laplacian operator and thus leads to an isotropic *bandpass decomposition* of the image. This is, of course, not surprising as the diffusion equation in Eq. (8.115) relates just the first-order temporal derivative with the second-order spatial derivative. The transfer function at the scale ξ is

$$-2\pi^2\,|\boldsymbol{k}|^2 \exp(-2\pi^2\xi\,|\boldsymbol{k}|^2) \tag{8.134}$$

For small wave numbers, the transfer function is proportional to $-|\boldsymbol{k}|^2$. It reaches a maximum at

$$k_{\max} = \frac{1}{\sqrt{2\pi^2\xi}} \tag{8.135}$$

and then decays exponentially.

8.9.4 Discrete scale spaces

The construction of a *discrete scale space* requires a discretization of the diffusion equation and *not* of the convolution kernel [16]. We start with a discretization of the 1-D diffusion equation

$$\frac{\partial g(x,\xi)}{\partial \xi} = \frac{\partial^2 g(x,\xi)}{\partial x^2} \tag{8.136}$$

The derivatives are replaced by discrete differences in the following way:

$$\begin{aligned}\frac{\partial g(x,\xi)}{\partial \xi} &\approx \frac{g(x,\xi+\Delta\xi) - g(x,\xi)}{\Delta \xi}\\ \frac{\partial^2 g(x,\xi)}{\partial x^2} &\approx \frac{g(x+\Delta x,\xi) - 2g(x,\xi) + g(x-\Delta x,\xi)}{\Delta x^2}\end{aligned} \tag{8.137}$$

This leads to the following iteration scheme for computing a discrete scale space:

$$g(x, \xi + \Delta\xi) = \Delta\xi g(x + \Delta x, \xi) + (1 - 2\Delta\xi) g(x, \xi) + \Delta\xi g(x - \Delta x, \xi) \tag{8.138}$$

or written with discrete coordinates

$$g_{n,\xi+1} = \Delta\xi g_{n+1,\xi} + (1 - 2\Delta\xi) g_{n,\xi} + \Delta\xi g_{n-1,\xi} \tag{8.139}$$

Lindeberg [16] shows that this iteration results in a discrete scale space that meets the minimum-maximum principle and the semi-group property if and only if

$$\Delta\xi \le \frac{1}{4} \tag{8.140}$$

The limit case of $\Delta\xi = 1/4$ leads to the especially simple iteration

$$g_{n,\xi+1} = 1/4 g_{n+1,\xi} + 1/2 g_{n,\xi} + 1/4 g_{n-1,\xi} \tag{8.141}$$

Each step of the scale-space computation is given by a smoothing of the signal with the binomial mask $\boldsymbol{B}^2 = [1/4\ 1/2\ 1/4]$ (Section 9.5.4). We can also formulate the general scale-space generating operator in Eq. (8.139) using the convolution operator $\mathcal{B}$. Written in the operator notation introduced in Section 9.1.3, the operator for one iteration step to generate the discrete scale space is

$$(1 - \epsilon)\mathcal{I} + \epsilon\mathcal{B}^2 \quad \text{with} \quad \epsilon \le 1 \tag{8.142}$$

where $\mathcal{I}$ denotes the identy operator.

This expression is significant, as it can be extended directly to higher dimensions by replacing $\mathcal{B}^2$ with a correspondingly higher-dimensional smoothing operator. The convolution mask $\boldsymbol{B}^2$ is the simplest mask in the class of smoothing binomial filters. These filters will be discussed in Section 9.5.4. A detailed discussion of discrete linear scale spaces is given by Lindeberg [16, Chapters 3 and 4].

8.10 Multigrid representations

8.10.1 Basics

The scale space discussed in Section 8.9 has one significant disadvantage. The use of the additional scale parameter adds a new dimension to the images and thus leads to an explosion of the data storage requirements and, in turn, the computational overhead for generating the scale space and for analyzing it. Thus, it is not surprising that before the evolution of the scale space more efficient multiscale storage