the increase in the area of the cells proportional to k^2 must be considered:

$$\int_{-\infty}^{\infty} |\hat{g}(\boldsymbol{k})|^2 \, \mathrm{d}k_1 \, \mathrm{d}k_2 = \int_{-\infty}^{\infty} k^2 |\hat{g}(\boldsymbol{k})|^2 \, \mathrm{d} \ln k \, \mathrm{d}\varphi \tag{8.47}$$

Thus, the *power spectrum* $|\hat{g}(\boldsymbol{k})|^2$ in the log-polar representation is multiplied by k^2 and falls off much less steep than in the Cartesian representation. The representation in a log-polar coordinate system allows a much better evaluation of the directions of the spatial structures and of the smaller scales. Moreover, a change in scale or orientation just causes a shift of the signal in the log-polar representation. Therefore, it has gained importance in representation object for shape analysis ([CVA3, Chapter 8]).

8.6 Continuous Fourier transform (FT)

In this section, we give a brief survey of the continuous Fourier transform and we point out the properties that are most important for signal processing. Extensive and excellent reviews of the Fourier transform are given by Bracewell [8], Poularikas [7, Chapter 2], or Madisetti and Williams [9, Chapter 1]

8.6.1 One-dimensional FT

Definition 8.1 (1-D FT) *If $g(x) : \mathbb{R} \mapsto \mathbb{C}$ is a square integrable function, that is,*

$$\int_{-\infty}^{\infty} |g(x)| \, \mathrm{d}x < \infty \tag{8.48}$$

then the Fourier transform *of $g(x)$, $\hat{g}(k)$ is given by*

$$\hat{g}(k) = \int_{-\infty}^{\infty} g(x) \exp(-2\pi \mathrm{i} k x) \, \mathrm{d}x \tag{8.49}$$

The Fourier transform maps the vector space of absolutely integrable functions onto itself. The inverse Fourier transform *of $\hat{g}(k)$ results in the original function $g(x)$:*

$$g(x) = \int_{-\infty}^{\infty} \hat{g}(k) \exp(2\pi \mathrm{i} k x) \, \mathrm{d}k \tag{8.50}$$

It is convenient to use an operator notation for the Fourier transform. With this notation, the Fourier transform and its inverse are simply written as

$$\hat{g}(k) = \mathcal{F}g(x) \quad \text{and} \quad g(x) = \mathcal{F}^{-1}\hat{g}(k) \tag{8.51}$$

A function and its transform, a *Fourier transform pair* is simply denoted by $g(x) \Longleftrightarrow \hat{g}(k)$.

In Eqs. (8.49) and (8.50) a definition of the wave number *without* the factor 2π is used, that is $k = 1/\lambda$, in contrast to the notation often used in physics with $k' = 2\pi/\lambda$. For signal processing, the first notion is more useful, because k directly gives the number of periods per unit length.

With the notation that includes the factor 2π in the wave number, two forms of the Fourier transform are common: the asymmetric form

$$\begin{aligned} \hat{g}(k') &= \int\limits_{-\infty}^{\infty} g(x)\exp(-\mathrm{i}k'x)\,\mathrm{d}x \\ g(x) &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \hat{g}(k)\exp(\mathrm{i}k'x)\,\mathrm{d}k \end{aligned} \tag{8.52}$$

and the symmetric form

$$\begin{aligned} \hat{g}(k') &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} g(x)\exp(-\mathrm{i}k'x)\,\mathrm{d}x \\ g(x) &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} \hat{g}(k')\exp(\mathrm{i}k'x)\,\mathrm{d}k' \end{aligned} \tag{8.53}$$

As the definition of the Fourier transform takes the simplest form in Eqs. (8.49) and (8.50), most other relations and equations also become simpler than with the definitions in Eqs. (8.52) and (8.53). In addition, the relation of the continuous Fourier transform with the discrete Fourier transform (Section 8.7) and the Fourier series (Table 8.3) becomes more straightforward.

Because all three versions of the Fourier transform are in common use, it is likely that wrong factors in Fourier transform pairs will be obtained. The rules for conversion of Fourier transform pairs between the three versions can directly be inferred from the definitions and are summarized here:

$$\begin{array}{lrcl} k \text{ without } 2\pi, \text{ Eq. (8.49)} & g(x) & \Longleftrightarrow & \hat{g}(k) \\ k' \text{ with } 2\pi, \text{ Eq. (8.52)} & g(x) & \Longleftrightarrow & \hat{g}(k'/2\pi) \\ k' \text{ with } 2\pi, \text{ Eq. (8.53)} & g(x/\sqrt(2\pi)) & \Longleftrightarrow & \hat{g}(k'/\sqrt(2\pi)) \end{array} \tag{8.54}$$

Table 8.3: *Comparison of the continuous Fourier transform (FT), the Fourier series (FS), the infinite discrete Fourier transform (IDFT), and the discrete Fourier transform (DFT) in one dimension*

Type	Forward transform	Backward transform
FT: $\mathbb{R} \Longleftrightarrow \mathbb{R}$	$\int\limits_{-\infty}^{\infty} g(x) \exp(-2\pi \mathrm{i} kx)\, \mathrm{d}x$	$\int\limits_{-\infty}^{\infty} \hat{g}(k) \exp(2\pi \mathrm{i} kx)\, \mathrm{d}k$
FS: $[0, \Delta x] \Longleftrightarrow \mathbb{Z}$	$\frac{1}{\Delta x} \int\limits_{0}^{\Delta x} g(x) \exp\left(-2\pi \mathrm{i} \frac{vx}{\Delta x}\right) \mathrm{d}x$	$\sum\limits_{v=-\infty}^{\infty} \hat{g}_v \exp\left(2\pi \mathrm{i} \frac{vx}{\Delta x}\right)$
IDFT: $\mathbb{Z} \Longleftrightarrow [0, 1/\Delta x]$	$\sum\limits_{n=-\infty}^{\infty} g_n \exp(-2\pi \mathrm{i} n \Delta x k)$	$\Delta x \int\limits_{0}^{1/\Delta x} \hat{g}(k) \exp(2\pi \mathrm{i} n \Delta x k)\, \mathrm{d}k$
DFT: $\mathbb{N}_N \Longleftrightarrow \mathbb{N}_N$	$\frac{1}{N} \sum\limits_{n=0}^{N-1} g_n \exp\left(-2\pi \mathrm{i} \frac{vn}{N}\right)$	$\sum\limits_{v=0}^{N-1} \hat{g}_v \exp\left(2\pi \mathrm{i} \frac{vn}{N}\right)$

8.6.2 Multidimensional FT

The Fourier transform can easily be extended to multidimensional signals.

Definition 8.2 (Multidimensional FT) *If $g(\boldsymbol{x}) : \mathbb{R}^D \mapsto \mathbb{C}$ is a square integrable function, that is,*

$$\int\limits_{-\infty}^{\infty} |g(\boldsymbol{x})|\, \mathrm{d}^D x < \infty \tag{8.55}$$

then the Fourier transform *of $g(\boldsymbol{x})$, $\hat{g}(\boldsymbol{k})$ is given by*

$$\hat{g}(\boldsymbol{k}) = \int\limits_{-\infty}^{\infty} g(\boldsymbol{x}) \exp\left(-2\pi \mathrm{i} \boldsymbol{k}^T \boldsymbol{x}\right) \mathrm{d}^D x \tag{8.56}$$

and the inverse Fourier transform *by*

$$g(\boldsymbol{x}) = \int\limits_{-\infty}^{\infty} \hat{g}(\boldsymbol{k}) \exp\left(2\pi \mathrm{i} \boldsymbol{k}^T \boldsymbol{x}\right) \mathrm{d}^D k \tag{8.57}$$

The scalar product in the exponent of the kernel $\boldsymbol{x}^T \boldsymbol{k}$ makes the kernel of the Fourier transform separable, that is, it can be written as

$$\exp\left(-2\pi \mathrm{i} \boldsymbol{k}^T \boldsymbol{x}\right) = \prod_{d=1}^{D} \exp(-\mathrm{i} k_d x_d) \tag{8.58}$$

***Table 8.4:** Summary of the properties of the continuous D-dimensional Fourier transform; $g(\boldsymbol{x})$ and $h(\boldsymbol{x})$ are complex-valued functions, the Fourier transforms of which, $\hat{g}(\boldsymbol{k})$ and $\hat{h}(\boldsymbol{k})$, do exist; s is a real and a and b are complex constants; $\boldsymbol{A}$ and $\boldsymbol{U}$ are $D \times D$ matrices, $\boldsymbol{U}$ is unitary ($\boldsymbol{U}^{-1} = \boldsymbol{U}^T$, see Section 8.5.2)*

Property	Spatial domain	Fourier domain
Linearity	$ag(\boldsymbol{x}) + bh(\boldsymbol{x})$	$a\hat{g}(\boldsymbol{k}) + b\hat{h}(\boldsymbol{k})$
Similarity	$g(s\boldsymbol{x})$	$\hat{g}(\boldsymbol{k}/s)/\lvert s\rvert$
Similarity	$g(\boldsymbol{A}\boldsymbol{x})$	$\hat{g}\left((\boldsymbol{A}^{-1})^T\boldsymbol{k}\right)/\lvert\boldsymbol{A}\rvert$
Rotation	$g(\boldsymbol{U}\boldsymbol{x})$	$\hat{g}(\boldsymbol{U}\boldsymbol{k})$
Separability	$\prod_{d=1}^{D} g(x_d)$	$\prod_{d=1}^{D} \hat{g}(k_d)$
Shift in x space	$g(\boldsymbol{x} - \boldsymbol{x}_0)$	$\exp(-2\pi \mathrm{i}\boldsymbol{k}\boldsymbol{x}_0)\hat{g}(\boldsymbol{k})$
Shift in k space	$\exp(2\pi \mathrm{i}\boldsymbol{k}_0\boldsymbol{x})g(\boldsymbol{x})$	$\hat{g}(\boldsymbol{k} - \boldsymbol{k}_0)$
Differentiation in x space	$\dfrac{\partial g(\boldsymbol{x})}{\partial x_p}$	$2\pi \mathrm{i} k_p \hat{g}(\boldsymbol{k})$
Differentiation in k space	$-2\pi \mathrm{i} x_p g(\boldsymbol{x})$	$\dfrac{\partial \hat{g}(\boldsymbol{k})}{\partial k_p}$
Definite integral	$\int_{-\infty}^{\infty} g(\boldsymbol{x}')\,\mathrm{d}^D x'$	$\hat{g}(\boldsymbol{0})$
Moments	$\int_{-\infty}^{\infty} x_p^m x_q^n g(\boldsymbol{x})\,\mathrm{d}^D x$	$\left(\dfrac{1}{-2\pi \mathrm{i}}\right)^{m+n}\left(\dfrac{\partial^m \hat{g}(\boldsymbol{k})}{\partial k_p^m}\dfrac{\partial^n \hat{g}(\boldsymbol{k})}{\partial k_q^n}\right)\Bigg\rvert_{\boldsymbol{0}}$
Convolution	$\int_{-\infty}^{\infty} h(\boldsymbol{x}')g(\boldsymbol{x} - \boldsymbol{x}')\,\mathrm{d}^D x'$	$\hat{h}(\boldsymbol{k})\hat{g}(\boldsymbol{k})$
Multiplication	$h(\boldsymbol{x})g(\boldsymbol{x})$	$\int_{-\infty}^{\infty} \hat{h}(\boldsymbol{k}')\hat{g}(\boldsymbol{k} - \boldsymbol{k}')\,\mathrm{d}^D k'$
Finite difference	$g(\boldsymbol{x} + V x_0) - g(\boldsymbol{x} - V x_0)$	$2\mathrm{i}\sin(2\pi \boldsymbol{x}_0\boldsymbol{k})$
Modulation	$\cos(2\pi \boldsymbol{k}_0\boldsymbol{x})g(\boldsymbol{x})$	$(\hat{g}(\boldsymbol{k} - \boldsymbol{k}_0) + \hat{g}(\boldsymbol{k} + \boldsymbol{k}_0))/2$
Spatial correlation	$\int_{-\infty}^{\infty} g(\boldsymbol{x}')h(\boldsymbol{x}' + \boldsymbol{x})\,\mathrm{d}^D x'$	$\hat{g}(\boldsymbol{k})\hat{h}^*(\boldsymbol{k})$
Inner product	$\int_{-\infty}^{\infty} g(\boldsymbol{x})h^*(\boldsymbol{x})\,\mathrm{d}^D x$	$\int_{-\infty}^{\infty} \hat{g}(\boldsymbol{k})\hat{h}^*(\boldsymbol{k})\,\mathrm{d}^D k$

8.6.3 Basic properties

For reference, the basic properties of the Fourier transform are summarized in Table 8.4. An excellent review of the Fourier transform and its applications are given by [8]. Here we will point out some of the properties of the FT that are most significant for multidimensional signal processing.

Symmetries. Four types of symmetries are important for the Fourier transform:

$$\begin{array}{ll} \text{even} & g(-\boldsymbol{x}) = g(\boldsymbol{x}), \\ \text{odd} & g(-\boldsymbol{x}) = -g(\boldsymbol{x}), \\ \text{Hermitian} & g(-\boldsymbol{x}) = g^*(\boldsymbol{x}), \\ \text{anti-Hermitian} & g(-\boldsymbol{x}) = -g^*(\boldsymbol{x}) \end{array} \tag{8.59}$$

Any function $g(\boldsymbol{x})$ can be split into its even and odd parts by

$${}^e g(\boldsymbol{x}) = \frac{g(\boldsymbol{x}) + g(-\boldsymbol{x})}{2} \quad \text{and} \quad {}^o g(\boldsymbol{x}) = \frac{g(\boldsymbol{x}) - g(-\boldsymbol{x})}{2} \tag{8.60}$$

With this partition, the Fourier transform can be parted into a cosine and a sine transform:

$$\hat{g}(\boldsymbol{k}) = 2\int\limits_0^\infty {}^e g(\boldsymbol{x}) \cos(2\pi \boldsymbol{k}^T \boldsymbol{x})\, \mathrm{d}^D x + 2\mathrm{i}\int\limits_0^\infty {}^o g(\boldsymbol{x}) \sin(2\pi \boldsymbol{k}^T \boldsymbol{x})\, \mathrm{d}^D x \tag{8.61}$$

It follows that if a function is even or odd, its transform is also even or odd. The full symmetry results are:

$$\begin{array}{lcl} \text{real} & \Longleftrightarrow & \text{Hermitian} \\ \text{real and even} & \Longleftrightarrow & \text{real and even} \\ \text{real and odd} & \Longleftrightarrow & \text{imaginary and odd} \\ \text{imaginary} & \Longleftrightarrow & \text{anti-Hermitian} \\ \text{imaginary and even} & \Longleftrightarrow & \text{imaginary and even} \\ \text{imaginary and odd} & \Longleftrightarrow & \text{real and odd} \\ \text{Hermitian} & \Longleftrightarrow & \text{real} \\ \text{anti-Hermitian} & \Longleftrightarrow & \text{imaginary} \\ \text{even} & \Longleftrightarrow & \text{even} \\ \text{odd} & \Longleftrightarrow & \text{odd} \end{array} \tag{8.62}$$

Separability. As the kernel of the Fourier transform (Eq. (8.58)) is separable, the transform of a separable function is also separable:

$$\prod_{d=1}^{D} g(x_d) \Longleftrightarrow \prod_{d=1}^{D} \hat{g}(k_d) \tag{8.63}$$

This property is essential to compute transforms of multidimensional functions efficiently from 1-D transforms because many of them are separable.

Convolution. Convolution is one of the most important operations for signal processing. It is defined by

$$(h * g)(\boldsymbol{x}) = \int_{-\infty}^{\infty} g(\boldsymbol{x}')h(\boldsymbol{x} - \boldsymbol{x}')\,\mathrm{d}^D x' \tag{8.64}$$

In signal processing, the function $h(\boldsymbol{x})$ is normally zero except for a small area around zero and is often denoted as the *convolution mask*. Thus, the convolution with $h(\boldsymbol{x})$ results in a new function $g'(\boldsymbol{x})$ whose values are a kind of weighted average of $g(\boldsymbol{x})$ in a small neighborhood around $\boldsymbol{x}$. It changes the signal in a defined way, that is, makes it smoother, etc. Therefore it is also called a *filter operation*. The *convolution theorem* states:

Theorem 8.3 (Convolution) *If $g(\boldsymbol{x})$ has the Fourier transform $\hat{g}(\boldsymbol{k})$ and $h(\boldsymbol{x})$ has the Fourier transform $\hat{h}(\boldsymbol{k})$ and if the convolution integral (Eq. (8.64)) exists, then it has the Fourier transform $\hat{h}(\boldsymbol{k})\hat{g}(\boldsymbol{k})$.*

Thus, convolution of two functions means multiplication of their transforms. Likewise, convolution of two functions in the Fourier domain means multiplication in the space domain. The simplicity of convolution in the Fourier space stems from the fact that the base functions of the Fourier domain, the complex exponentials $\exp\left(2\pi \mathrm{i}\boldsymbol{k}^T\boldsymbol{x}\right)$, are joint eigenfunctions of all convolution operators. This means that these functions are not changed by a convolution operator except for the multiplication by a factor.

From the convolution theorem, the following properties are immediately evident. Convolution is

$$\begin{array}{ll} \text{commutative} & h * g = g * h, \\ \text{associative} & h_1 * (h_2 * g) = (h_1 * h_2) * g, \\ \text{distributive over addition} & (h_1 + h_2) * g = h_1 * g + h_2 * g \end{array} \tag{8.65}$$

In order to grasp the importance of these properties of convolution, we note that two operations that do not look so at first glance, are also convolution operations: the shift operation and all derivative operators. This can immediately be seen from the shift and derivative theorems (Table 8.4; [8, Chapters 5 and 6]).

In both cases the Fourier transform is just multiplied by a complex factor. The convolution mask for a shift operation S is a shifted δ distribution:

$$S(\boldsymbol{s})g(\boldsymbol{x}) = \delta(\boldsymbol{x} - \boldsymbol{s}) * g(\boldsymbol{x}) \tag{8.66}$$

The transform of the first derivative operator in x_1 direction is $2\pi\mathrm{i}k_1$. The corresponding inverse Fourier transform of $2\pi\mathrm{i}k_1$, that is, the convolution mask, is no longer an ordinary function ($2\pi\mathrm{i}k_1$ is not absolutely integrable) but the derivative of the δ distribution:

$$2\pi\mathrm{i}k_1 \quad \Longleftrightarrow \quad \delta'(x) = \frac{\mathrm{d}\delta(x)}{\mathrm{d}x} = \lim_{a\to 0}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\exp(-\pi x^2/a^2)}{a}\right) \tag{8.67}$$

Of course, the derivation of the δ distribution exists—as all properties of distributions—only in the sense as a limit of a sequence of functions as shown in the preceding equation.

With the knowledge of derivative and shift operators being convolution operators, we can use the properties summarized in Eq. (8.65) to draw some important conclusions. As any convolution operator commutes with the shift operator, convolution is a shift-invariant operation. Furthermore, we can first differentiate a signal and then perform a convolution operation or vice versa and obtain the same result.

The properties in Eq. (8.65) are essential for an effective computation of convolution operations [CVA2, Section 5.6]. As we already discussed qualitatively in Section 8.5.3, the convolution operation is a linear shift-invariant operator. As the base functions of the Fourier domain are the common eigenvectors of all linear and shift-invariant operators, the convolution simplifies to a complex multiplication of the transforms.

Central-limit theorem. The central-limit theorem is mostly known for its importance in the theory of probability [2]. It also plays, however, an important role for signal processing as it is a rigorous statement of the tendency that cascaded convolution tends to approach Gaussian form ($\propto \exp(-ax^2)$). Because the Fourier transform of the Gaussian is also a Gaussian (Table 8.5), this means that *both* the Fourier transform (the transfer function) and the mask of a convolution approach Gaussian shape. Thus the central-limit theorem is central to the unique role of the Gaussian function for signal processing. The sufficient conditions under which the central-limit theorem is valid can be formulated in different ways. We use here the conditions from [2] and express the theorem with respect to convolution.

Theorem 8.4 (Central-limit theorem) *Given N functions $h_n(x)$ with zero mean $\int_{-\infty}^{\infty} h_n(x)\,\mathrm{d}x$ and the variance $\sigma_n^2 = \int_{-\infty}^{\infty} x^2 h_n(x)\,\mathrm{d}x$ with $z = x/\sigma$, $\sigma^2 = \sum_{n=1}^{N}\sigma_n^2$ then*

$$h = \lim_{N\to\infty} h_1 * h_2 * \ldots * h_N \propto \exp(-z^2/2) \tag{8.68}$$

provided that

$$\lim_{N\to\infty} \sum_{n=1}^{N} \sigma_n^2 \to \infty \tag{8.69}$$

and there exists a number $\alpha > 2$ *and a finite constant* c *such that*

$$\int_{-\infty}^{\infty} x^{\alpha} h_n(x)\,\mathrm{d}x < c < \infty \quad \forall n \tag{8.70}$$

The theorem is of much practical importance because—especially if h is smooth—the Gaussian shape is approximated sufficiently accurate already for values of n as low as 5.

Smoothness and compactness. The smoother a function is, the more compact is its Fourier transform. This general rule can be formulated more quantitatively if we express the smoothness by the number of derivatives that are continuous and the compactness by the asymptotic behavior for large values of k. Then we can state: If a function $g(x)$ and its first $n-1$ derivatives are continuous, its Fourier transform decreases at least as rapidly as $|k|^{-(n+1)}$ for large k, that is, $\lim_{|k|\to\infty} |k|^n g(k) = 0$.

As simple examples we can take the box and triangle functions (see next section). The box function is discontinuous ($n = 0$), its Fourier transform, the sinc function, decays with $|k|^{-1}$. In contrast, the triangle function is continuous, but its first derivative is discontinuous. Therefore, its Fourier transform, the sinc^2 function, decays steeper with $|k|^{-2}$. In order to include also impulsive functions (δ distributions) in this relation, we note that the derivative of a discontinous function becomes impulsive. Therefore, we can state: If the nth derivative of a function becomes impulsive, the function's Fourier transform decays with $|k|^{-n}$.

The relation between smoothness and compactness is an extension of reciprocity between the spatial and Fourier domain. What is strongly localized in one domain is widely extended in the other and vice versa.

Uncertainty relation. This general law of reciprocity finds another quantitative expression in the classical *uncertainty relation* or the *bandwidth-duration product*. This theorem relates the mean square width of a function and its Fourier transform. The mean square width $(\Delta x)^2$ is defined as

$$(\Delta x)^2 = \frac{\int\limits_{-\infty}^{\infty} x^2 \left|g(x)\right|^2}{\int\limits_{-\infty}^{\infty} \left|g(x)\right|^2} - \left(\frac{\int\limits_{-\infty}^{\infty} x \left|g(x)\right|^2}{\int\limits_{-\infty}^{\infty} \left|g(x)\right|^2}\right)^2 \tag{8.71}$$

Table 8.5: *Functions and distributions that are invariant under the Fourier transform; the table contains 1-D and multidimensional functions with the dimension D*

Space domain	Fourier domain
Gauss, $\exp\left(-\pi \boldsymbol{x}^T \boldsymbol{x}\right)$	Gauss, $\exp\left(-\pi \boldsymbol{k}^T \boldsymbol{k}\right)$
$\operatorname{sech}(\pi x) = \dfrac{1}{\exp(\pi x) + \exp(-\pi x)}$	$\operatorname{sech}(\pi k) = \dfrac{1}{\exp(\pi k) + \exp(-\pi k)}$
Pole, $\lvert\boldsymbol{x}\rvert^{-D/2}$	Pole, $\lvert\boldsymbol{k}\rvert^{-D/2}$
δ comb, $Ш(x/\Delta x) = \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x)$	δ comb, $Ш(k\Delta x) = \sum_{v=-\infty}^{\infty} \delta(k - v/\Delta x)$

It is essentially the variance of $|g(x)|^2$, a measure of the width of the distribution of the "energy" of the signal. The uncertainty relation states:

Theorem 8.5 (Uncertainty relation) *The product of the variance of* $|g(x)|^2$, $(\Delta x)^2$, *and of the variance of* $|\hat{g}(k)|^2$, $(\Delta k)^2$, *cannot be smaller than* $1/4\pi$*:*

$$\Delta x \Delta k \geq \frac{1}{4\pi} \tag{8.72}$$

The relations between compactness and smoothness and the uncertainty relation give some basic guidance for the design of linear filter (convolution) operators [CVA2, Chapter 6].

Invariant functions. It is well known that the Fourier transform of a Gaussian function is again a Gaussian function with reciprocal variance:

$$\exp\left(\frac{-\pi x^2}{a^2}\right) \quad \Longleftrightarrow \quad \exp\left(\frac{-\pi k^2}{a^{-2}}\right) \tag{8.73}$$

But it is less well known that there are other functions that are invariant under the Fourier transform (Table 8.5). Each of these functions has a special meaning for the Fourier transform. The δ-comb function Ш is the basis for the sampling theorem and establishes the relation between the lattice in the spatial domain and the *reciprocal* lattice in the Fourier domain. The functions with a pole at the origin, $|x|^{D/2}$ in a D-dimensional space, are the limiting signal form for which the integral over the square of the function diverges (physically speaking, the total energy of a signal just becomes infinite). Tables with Fourier transform pairs can be found in Bracewell [8].

8.7 The discrete Fourier transform (DFT)

8.7.1 One-dimensional DFT

Definition 8.3 (1-D DFT) *If $\boldsymbol{g}$ is an N-dimensional complex-valued vector,*

$$\boldsymbol{g} = [g_0, g_1, \ldots, g_{N-1}]^T \tag{8.74}$$

then the discrete Fourier transform of $\boldsymbol{g}$, $\hat{\boldsymbol{g}}$ is defined as

$$\hat{g}_v = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} g_n \exp\left(-\frac{2\pi \mathrm{i} n v}{N}\right), \quad 0 \le v < N \tag{8.75}$$

The DFT maps the vector space of N-dimensional complex-valued vectors onto itself. The index v denotes how often the wavelength of the corresponding discrete exponential $\exp(-2\pi \mathrm{i} n v/N)$ with the amplitude $\hat{g}_v$ fits into the interval $[0, N]$.

The back transformation is given by

$$g_n = \frac{1}{\sqrt{N}} \sum_{v=0}^{N-1} \hat{g}_v \exp\left(\frac{2\pi \mathrm{i} n v}{N}\right), \quad 0 \le n < N \tag{8.76}$$

We can consider the DFT as the inner product of the vector $\boldsymbol{g}$ with a set of M orthonormal basis vectors, the *kernel* of the DFT:

$$\boldsymbol{b}_v = \frac{1}{\sqrt{N}} \left[1, W_N^v, W_N^{2v}, \ldots, W_N^{(N-1)v}\right]^T \text{ with } W_N = \exp\left(\frac{2\pi \mathrm{i}}{N}\right) \tag{8.77}$$

Using the base vectors $\boldsymbol{b}_v$, the DFT reduces to

$$\hat{g}_v = \boldsymbol{b}^{*T} \boldsymbol{g} \quad \text{or} \quad \hat{\boldsymbol{g}} = \boldsymbol{F}\boldsymbol{g} \quad \text{with} \quad \boldsymbol{F} = \begin{bmatrix} \boldsymbol{b}_0^{*T} \\ \boldsymbol{b}_1^{*T} \\ \ldots \\ \boldsymbol{b}_{N-1}^{*T} \end{bmatrix} \tag{8.78}$$

This means that the coefficient $\hat{g}_v$ in the Fourier space is obtained by projecting the vector $\boldsymbol{g}$ onto the basis vector $\boldsymbol{b}_v$. The N basis vectors $\boldsymbol{b}_v$ form an *orthonormal base* of the vector space:

$$\boldsymbol{b}_v^{*T} \boldsymbol{b}_v' = \delta_{v-v'} = \begin{cases} 1 & \text{if} \quad v = v' \\ 0 & \text{otherwise} \end{cases} \tag{8.79}$$

The real and imaginary parts of the basis vectors are sampled sine and cosine functions of different wavelengths with a characteristic periodicity:

$$\exp\left(\frac{2\pi \mathrm{i} n + pN}{N}\right) = \exp\left(\frac{2\pi \mathrm{i} n}{N}\right), \quad \forall p \in \mathbb{Z} \tag{8.80}$$

The basis vector $\boldsymbol{b}_0$ is a constant real vector.

With this relation and Eqs. (8.75) and (8.76) the DFT and the inverse DFT extend the vectors $\hat{\boldsymbol{g}}$ and $\boldsymbol{g}$, respectively, periodically over the whole space:

$$\begin{array}{lll} \text{Fourier domain} & \hat{g}_{v+pN} = \hat{g}_v, & \forall p \in \mathbb{Z} \\ \text{space domain} & g_{n+pN} = g_n & \forall p \in \mathbb{Z} \end{array} \tag{8.81}$$

This periodicity of the DFT gives rise to an interesting geometric interpretation. According to Eq. (8.81) the border points g_{M-1} and $g_M = g_0$ are neighboring points. Thus it is natural to draw the points of the vector not on a finite line but on a unit circle, or *Fourier ring*.

With the double periodicity of the DFT, it does not matter which range of N indices we chose. The most natural choice of wave numbers is $v \in [-N/2, N/2-1]$, N even. With this index range the 1-D DFT and its inverse are defined as

$$\hat{g}_v = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} g_n W_N^{-nv} \Longleftrightarrow g_n = \frac{1}{\sqrt{N}} \sum_{v=-N/2}^{N/2-1} \hat{g}_v W_N^{nv} \tag{8.82}$$

Then the wave numbers are restricted to values that meet the sampling theorem (Section 8.4.2), that is, are sampled at least two times per period. Note that the exponentials $\boldsymbol{b}_{N-v} = \boldsymbol{b}_{-v} = \boldsymbol{b}_v^*$ according to Eqs. (8.77) and (8.80).

As in the continuous case further variants for the definition of the DFT exist that differ by the factors applied to the forward and backward transform. Here again a symmetric definition was chosen that has the benefit that the base vectors become unit vectors. Other variants use the factor $1/N$ either with the forward or backward transform and not, as we did $1/\sqrt{N}$ with both transforms. The definition with the factor $1/N$ has the advantage that the zero coefficient of the DFT, $\hat{g}_0 = (1/N)\sum_{n=0}^{N-1} g_n$, directly gives the mean value of the sequence. The various definitions in use are problematic because they cause considerable confusion with factors in DFT pairs and DFT theorems.

8.7.2 Multidimensional DFT

As with the continuous FT (Section 8.6.2), it is easy to extend the DFT to higher dimensions. In order to simplify the equations, we use the abbreviation for the complex exponentials already used in Eq. (8.77)

$$W_N = \exp\left(\frac{2\pi \mathrm{i}}{N}\right) \quad \text{with} \quad W_N^{n+pN} = W_N^n, \; W_N^{-n} = W_N^{*n} \tag{8.83}$$

In two dimensions the DFT operates on $M \times N$ matrices.

Definition 8.4 (2-D DFT) *The 2-D DFT: $\mathbb{C}^{M\times N} \mapsto \mathbb{C}^{M\times N}$ is defined as*

$$\hat{G}_{u,v} = \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \left(\sum_{n=0}^{N-1} G_{m,n} W_N^{-nv} \right) W_M^{-mu} \tag{8.84}$$

and the inverse DFT as

$$G_{mn} = \frac{1}{\sqrt{MN}} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \hat{G}_{u,v} W_M^{mu} W_N^{nv} \tag{8.85}$$

As in the 1-D case, the DFT expands a matrix into a set of NM orthonormal basis matrices $\boldsymbol{B}_{u,v}$, which span the $N \times M$-dimensional vector space over the field of complex numbers:

$$\boldsymbol{B}_{u,v} = \frac{1}{\sqrt{MN}} W_N^{-nv} W_M^{-mu} = \frac{1}{\sqrt{MN}} \boldsymbol{b}_u \boldsymbol{b}_v^T \tag{8.86}$$

In this equation, the basis matrices are expressed as an *outer product* of the column and the row vector that form the basis vectors of the 1-D DFT. Thus as in the continuous case, the kernel of the multidimensional DFTs are *separable*.

As in the 1-D case (Section 8.7.1), the definition of the 2-D DFT implies a periodic extension in both domains beyond the original matrices into the whole 2-D space.

8.7.3 Basic properties

The theorems of the 2-D DFT are summarized in Table 8.6. They are very similar to the corresponding theorems of the continuous Fourier transform, which are listed in Table 8.4 for a D-dimensional FT. As in Section 8.6.3, we discuss some properties that are of importance for signal processing in more detail.

Symmetry. The DFT shows the same symmetries as the FT (Eq. (8.59)). In the definition for even and odd functions $g(-\boldsymbol{x}) = \pm g(\boldsymbol{x})$ only the continuous functions must be replaced by the corresponding vectors $g_{-n} = \pm g_n$ or matrices $G_{-m,-n} = \pm G_{m,n}$. Note that because of the periodicity of the DFT, these symmetry relations can also be written as

$$G_{-m,-n} = \pm G_{m,n} \equiv G_{M-m,N-n} = \pm G_{m,n} \tag{8.87}$$

for even (+ sign) and odd (− sign) functions. This is equivalent to shifting the symmetry center from the origin to the point $[M/2, N/2]^T$.

The study of symmetries is important for practical purposes. Careful consideration of symmetry allows storage space to be saved and algorithms to speed up. Such a case is real-valued images. Real-valued

Table 8.6: *Summary of the properties of the 2-D DFT;* $\mathbf{G}$ *and* $\mathbf{H}$ *are complex-valued* $M \times N$ *matrices,* $\hat{\mathbf{G}}$ *and* $\hat{\mathbf{H}}$ *their Fourier transforms, and* a *and* b *complex-valued constants; for proofs see Poularikas [7], Cooley and Tukey [10]*

Property	Space domain	Wave-number domain
Mean	$\dfrac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} G_{mn}$	$\hat{G}_{0,0}/\sqrt{MN}$
Linearity	$a\mathbf{G} + b\mathbf{H}$	$a\hat{\mathbf{G}} + b\hat{\mathbf{H}}$
Shifting	$G_{m-m',n-n'}$	$W_M^{-m'u} W_N^{-n'v} \hat{G}_{uv}$
Modulation	$W_M^{u'm} W_N^{v'n} G_{m,n}$	$\hat{G}_{u-u',v-v'}$
Finite differences	$(G_{m+1,n} - G_{m-1,n})/2$ $(G_{m,n+1} - G_{m,n-1})/2$	$\mathrm{i}\sin(2\pi u/M)\hat{G}_{uv}$ $\mathrm{i}\sin(2\pi v/N)\hat{G}_{uv}$
Spatial stretching	$G_{Pm,Qn}$	$\hat{G}_{uv}/(\sqrt{PQ})$
Frequency stretching	$G_{m,n}/(\sqrt{PQ})$	$\hat{G}_{Pu,Qv}$
Spatial sampling	$G_{m/P,n/Q}$	$\dfrac{1}{\sqrt{PQ}}\sum_{p=0}^{P-1}\sum_{q=0}^{Q-1} \hat{G}_{u+pM/P,v+qN/Q}$
Frequency sampling	$\dfrac{1}{\sqrt{PQ}}\sum_{p=0}^{P-1}\sum_{q=0}^{Q-1} G_{m+pM/P,n+qN/Q}$	$\hat{G}_{pu,qv}$
Convolution	$\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1} H_{m'n'} G_{m-m',n-n'}$	$\sqrt{MN}\hat{H}_{uv}\hat{G}_{uv}$
Multiplication	$\sqrt{MN} G_{mn} H_{mn}$	$\sum_{u'=0}^{M-1}\sum_{v'=0}^{N-1} H_{u'v'} G_{u-u',v-v'}$
Spatial correlation	$\sum_{m'=0}^{M-1}\sum_{n'=0}^{N-1} H_{m'n'} G_{m+m',n+n'}$	$\sqrt{N}\hat{H}_{uv}\hat{G}^*_{uv}$
Inner product	$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} G_{mn} H^*_{mn}$	$\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} \hat{G}_{uv}\hat{H}^*_{uv}$
Norm	$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \lvert G_{mn}\rvert^2$	$\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} \lvert\hat{G}_{uv}\rvert^2$

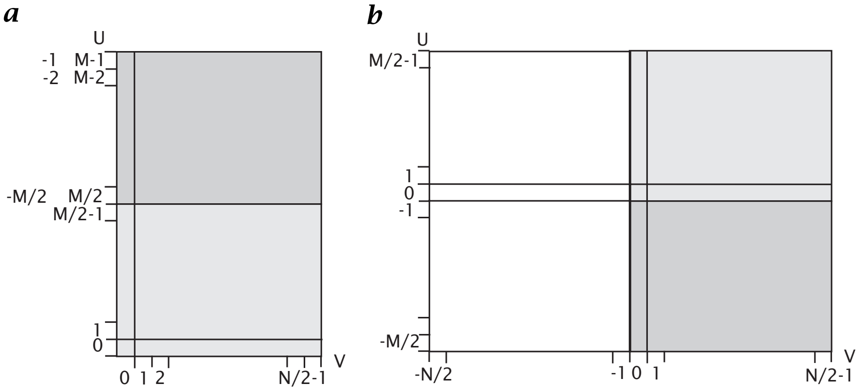

Figure 8.11: ***a*** *Half-space as computed by an in-place Fourier transform algorithm; the wave number zero is in the upper left corner;* ***b*** *FT with the missing half appended and remapped so that the wave number zero is in the center.*

images can be stored in half of the space as complex-valued images. From the symmetry relations Eq. (8.62) we can conclude that real-valued functions exhibit a Hermitian DFT:

$$G_{mn} = G^*_{mn} \quad \Longleftrightarrow \quad \hat{G}_{M-u,N-v} = \hat{G}^*_{uv} \tag{8.88}$$

The complex-valued DFT of real-valued matrices is, therefore, completely determined by the values in one half-space. The other half-space is obtained by mirroring at the symmetry center $(M/2, N/2)$. Consequently, we need the same amount of storage space for the DFT of a real image as for the image itself, as only half of the complex spectrum needs to be stored.

In two and higher dimensions, matters are slightly more complex. The spectrum of a real-valued image is determined completely by the values in one half-space, but there are many ways to select the half-space. This means that all except for one component of the wave number can be negative, but that we cannot distinguish between $\boldsymbol{k}$ and $-\boldsymbol{k}$, that is, between wave numbers that differ only in sign. Therefore, we can again represent the Fourier transform of real-valued images in a half-space where only one component of the wave number includes negative values. For proper representation of the spectra with zero values of this component in the middle of the image, it is necessary to interchange the upper (positive) and lower (negative) parts of the image as illustrated in Fig. 8.11.

For real-valued image sequences, again we need only a half-space to represent the spectrum. Physically, it makes the most sense to choose the half-space that contains positive frequencies. In contrast to a single image, we obtain the full wave-number space. Now we can identify the spatially identical wave numbers $\boldsymbol{k}$ and $-\boldsymbol{k}$ as structures propagating in opposite directions.

Convolution. One- and two-dimensional discrete convolution are defined by

$$g'_n = \sum_{n'=0}^{N-1} h_{n'} g_{n-n'}, \quad G'_{m,n} = \sum_{m'=0}^{M-1} \sum_{n'=0}^{N-1} H_{m'n'} G_{m-m',n-n'} \tag{8.89}$$

The *convolution theorem* states:

Theorem 8.6 (Discrete convolution) *If $\boldsymbol{g}$ ($\boldsymbol{G}$) has the Fourier transform $\hat{\boldsymbol{g}}$ ($\hat{\boldsymbol{G}}$) and $\boldsymbol{h}$ ($\boldsymbol{H}$) has the Fourier transform $\hat{\boldsymbol{h}}$ ($\hat{\boldsymbol{H}}$), then $\boldsymbol{h} * \boldsymbol{g}$ ($\boldsymbol{H} * \boldsymbol{G}$) has the Fourier transform $\sqrt{N}\hat{\boldsymbol{h}}\hat{\boldsymbol{g}}$ ($\sqrt{MN}\hat{\boldsymbol{H}}\hat{\boldsymbol{G}}$).*

Thus, also in the discrete case convolution of two functions means multiplication of their transforms. This is true because the *shift theorem* is still valid, which ensures that the eigenfunctions of all convolution operators are the basis functions $\boldsymbol{b}_v$ of the Fourier transform.

Convolution for arbitrary dimensional signals is also

$$\begin{array}{ll} \text{commutative} & \boldsymbol{h} * \boldsymbol{g} = \boldsymbol{g} * \boldsymbol{h}, \\ \text{associative} & \boldsymbol{h}_1 * (\boldsymbol{h}_2 * \boldsymbol{g}) = (\boldsymbol{h}_1 * \boldsymbol{h}_2) * \boldsymbol{g}, \\ \text{distributive over addition} & (\boldsymbol{h}_1 + \boldsymbol{h}_2) * \boldsymbol{g} = \boldsymbol{h}_1 * \boldsymbol{g} + \boldsymbol{h}_2 * \boldsymbol{g} \end{array} \tag{8.90}$$

These equations show only the 1-D case.

8.7.4 Fast Fourier transform algorithms (FFT)

Without an effective algorithm to calculate the discrete Fourier transform, it would not be possible to apply the FT to images and other higher-dimensional signals. Computed directly after Eq. (8.84), the FT is prohibitively expensive. Not counting the calculations of the cosine and sine functions in the kernel, which can be precalculated and stored in a lookup table, the FT of an $N \times N$ image needs in total N^4 complex multiplications and $N^2(N^2-1)$ complex additions. Thus it is an operation of $O(N^4)$ and the urgent need arises to minimize the number of computations by finding a suitable fast algorithm. Indeed, the fast Fourier transform (FFT) algorithm first published by Cooley and Tukey [10] is the classical example of a fast algorithm. A detailed discussion on FFT-algorithms can be found in Bracewell [8], Blahut [11], and Besslich and Lu [6].

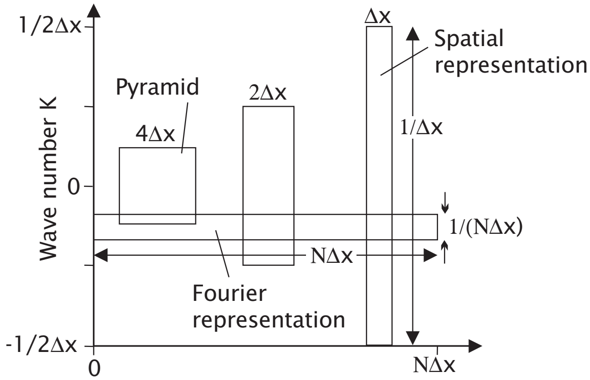

Figure 8.12: *Illustration of the interdependence of resolution in the spatial and wave-number domain in one dimension. Representations in the space domain, the wave-number domain, and the space/wave-number domain (2 planes of pyramid with half and quarter resolution) are shown.*

8.8 Scale of signals

8.8.1 Basics

In Sections 8.5 and 8.7 the representation of images in the spatial and wave-number domain were discussed. If an image is represented in the *spatial domain*, we do not have any information at all about the wave numbers contained at a point in the image. We know the position with an accuracy of the lattice constant Δx, but the local wave number at this position may be anywhere in the range of the possible wave numbers from $-1/(2\Delta x)$ to $1/(2\Delta x)$ (Fig. 8.12).

In the *wave-number domain* we have the reverse case. Each pixel in this domain represents one wave number with the highest wave-number resolution possible for the given image size, which is $-1/(N\Delta x)$ for an image with N pixels in each coordinate. But any positional information is lost, as one point in the wave-number space represents a periodic structure that is spread over the whole image (Fig. 8.12). Thus, the position uncertainty is the linear dimension of the image $N\Delta x$. In this section we will revisit both representations under the perspective of how to generate a multiscale representation of an image.

The foregoing discussion shows that the representations of an image in either the spatial or wave-number domain constitute two opposite extremes. Although the understanding of both domains is essential for any type of signal processing, the representation in either of these domains is inadequate to analyze objects in images.

In the wave-number representation the spatial structures from various independent objects are mixed up because the extracted periodic