Fractals: a way to represent natural objects

In spatial information systems there are two kinds of entity to model:

- natural earth features like terrain and coastlines;
- human-made objects like buildings and roads.

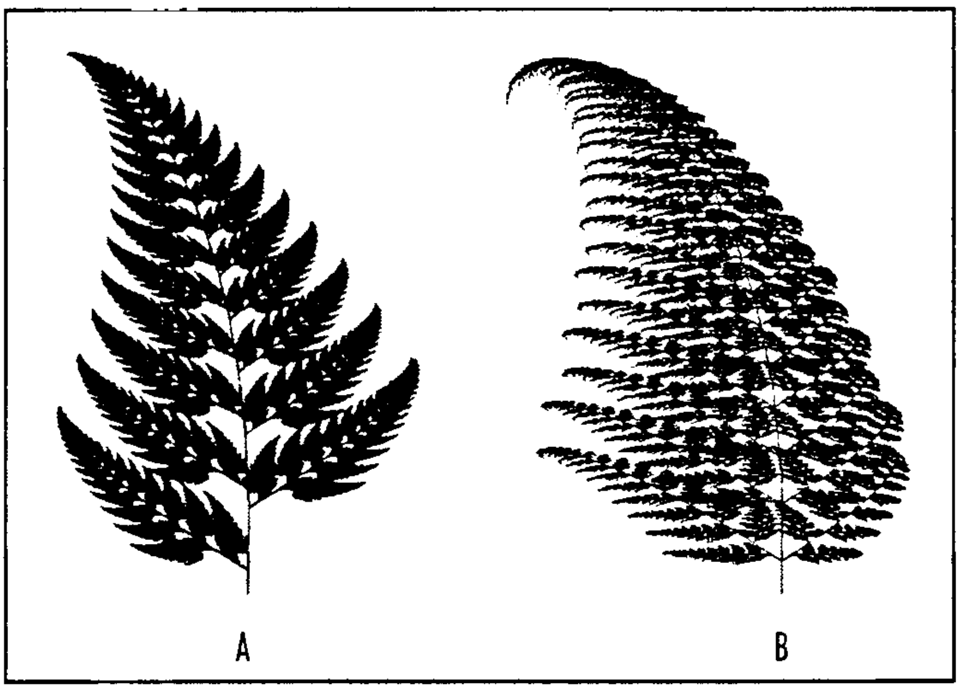
Start with a piece of blank paper with an *(x, y)*-coordinate system marked and pick an arbitrary point on the paper; then find its coordinates. ***Randomly*** select one of the four affine transformations listed below:

$$
\begin{aligned}
w_1 &: (x,y) \mapsto (0.85 \cdot x + 0.04 \cdot y, -0.04 \cdot x + 0.85 \cdot y + 40),\\
w_2 &: (x,y) \mapsto (0.20 \cdot x - 0.26 \cdot y, 0.23 \cdot x + 0.22 \cdot y + 40),\\
w_3 &: (x,y) \mapsto (-0.15 \cdot x + 0.28 \cdot y, 0.26 \cdot x + 0.24 \cdot y + 11),\\
w_4 &: (x,y) \mapsto (0, 0.16 \cdot y).
\end{aligned}
$$

Then apply the transformation to this point, and the coordinates of a new point are obtained. Notice the new point. Plot it in black on the paper.

Again select randomly one of the above four transformations and apply it to this point to obtain the next new point. Notice the new point. Plot it in black on the paper.

Again pick randomly one of the above four transformations and apply it to the point to obtain the next new point. Notice the new point. Plot it in black on the paper (you can repeat this process indefinitely). If you are patient and persistent enough, a spleentvort fern (*figure, a*), will appear on the paper like magic:



How does this magic happen?

The process above shows that with ***the right mathematical model*** a perfect picture can be described with infinitely fine and marvelously rich textures in only 24 numbers:

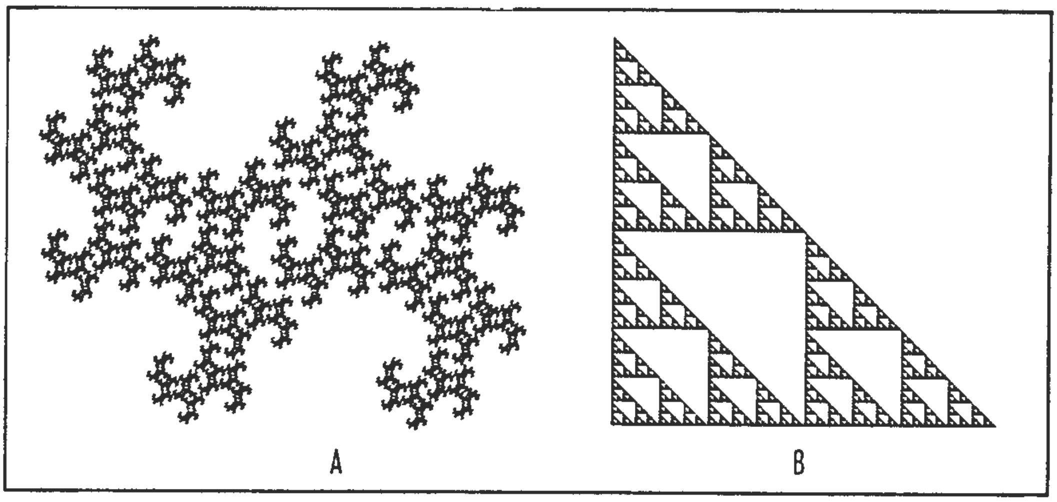
85,	4,	0,	-4,	85,	40;
20,	-26,	0,	23,	22,	40;
-15,	28,	0,	26,	24,	11;
0,	0,	0,	0,	16,	0;

Each 2-D affine transform will be characterized by 6 numbers. The ***dragon*** takes only 2 transforms, i.e. 12 numbers:

45,	-50,	0,	40,	55,	0;
45,	-50,	100,	40,	55,	0

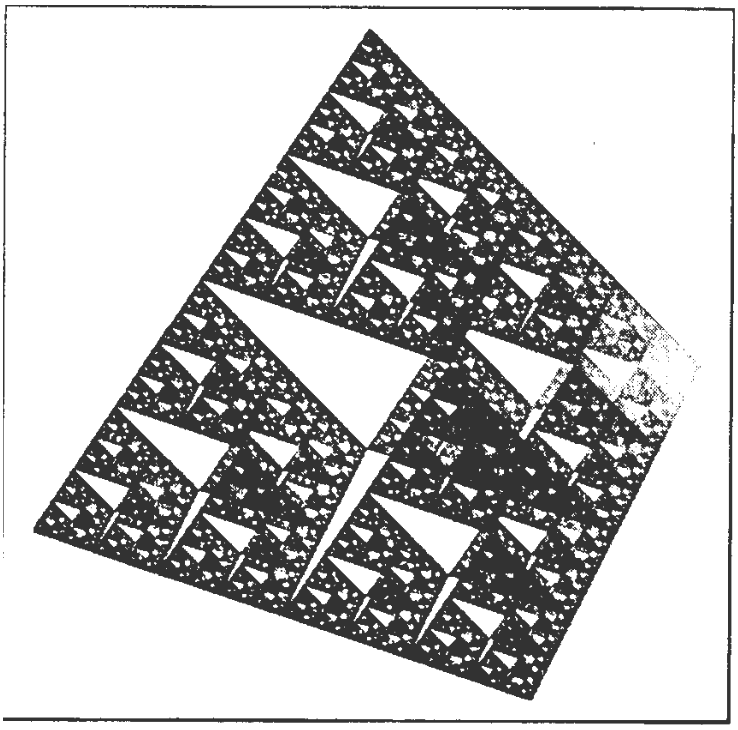
The ***Sierpinsky triangle*** is created using 18 numbers:

50,	0,	0,	0,	50,	0;
50,	0,	50,	0,	50,	0;
50,	0,	0,	0,	50,	50



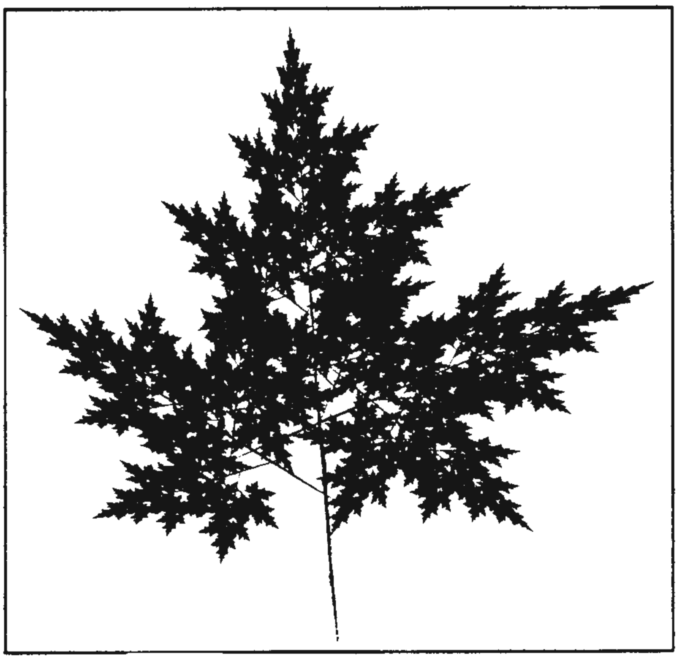
The ***3-D*** affine transform will be characterized by 12 coefficients. Then the ***3-D Sierpinsky gasket*** takes the following four affine transforms:

50,	0,	0,	0,	0,	50,	0,	0,	0,	0,	50,	0;
50,	0,	0,	50,	0,	50,	0,	0,	0,	0,	50,	0;
50,	0,	0,	25,	0,	50,	0,	50,	0,	0,	50,	0;
50,	o,	o,	25,	0,	50,	0,	25,	0,	0,	50,	50



The maple leaf:

49,	1,	25,	0,	62,	-2;
27,	52,	0,	-40,	36,	56;
18,	-73,	88,	50,	26,	8;
4,	-1,	52,	50,	0,	32

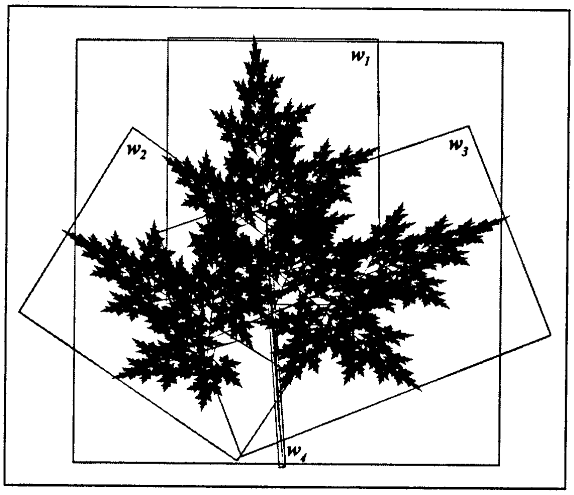


More details about the ***maple leaf***, there are four affine transforms:

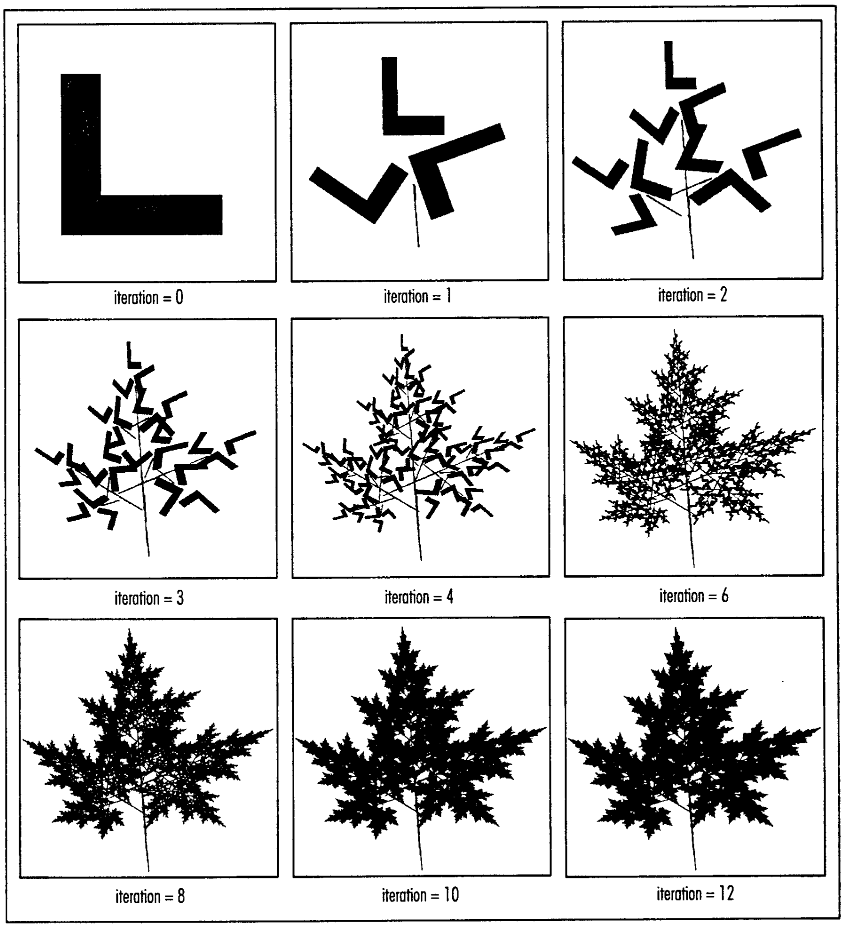
$$w_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0.49 & 0.01\\0 & 0.62\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}25\\-2\end{pmatrix},\quad w_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0.27 & 0.52\\-0.40 & 0.36\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\56\end{pmatrix},$$

$$w_3\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0.18 & -0.73\\0.50 & 0.26\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}88\\8\end{pmatrix},\quad w_4\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0.04 & -0.01\\0.50 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}52\\32\end{pmatrix}.$$

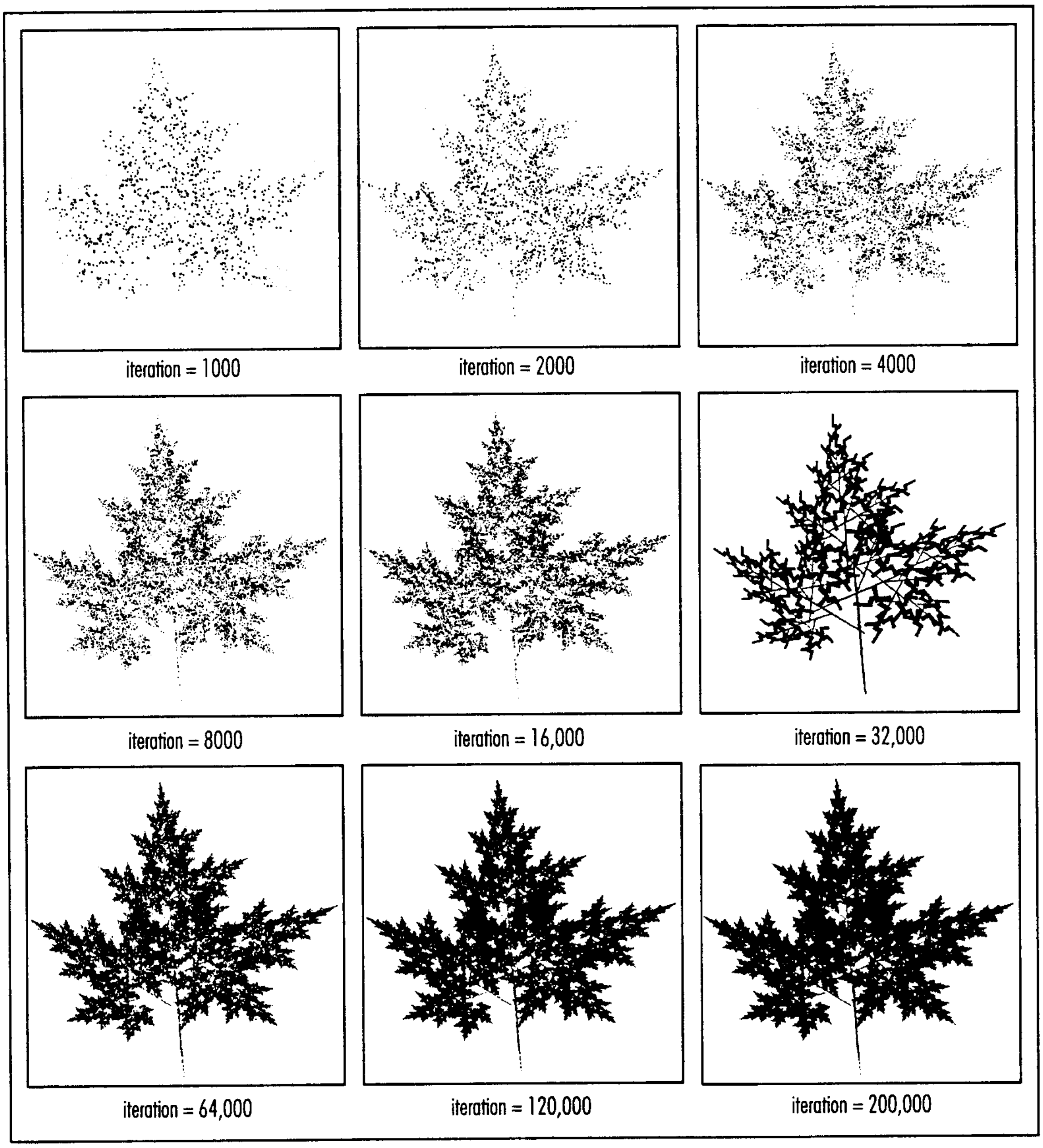
and visually their effects may be expressed:



in deterministic way:



or in random iteration way:



Def.: ***A fractal*** is an image or picture that obey self-similarity and can be completely described by a mathematical algorithm in its infinitely fine texture and detail. In essence, a fractal is a mathematical model of the self-similar nature of the real world.

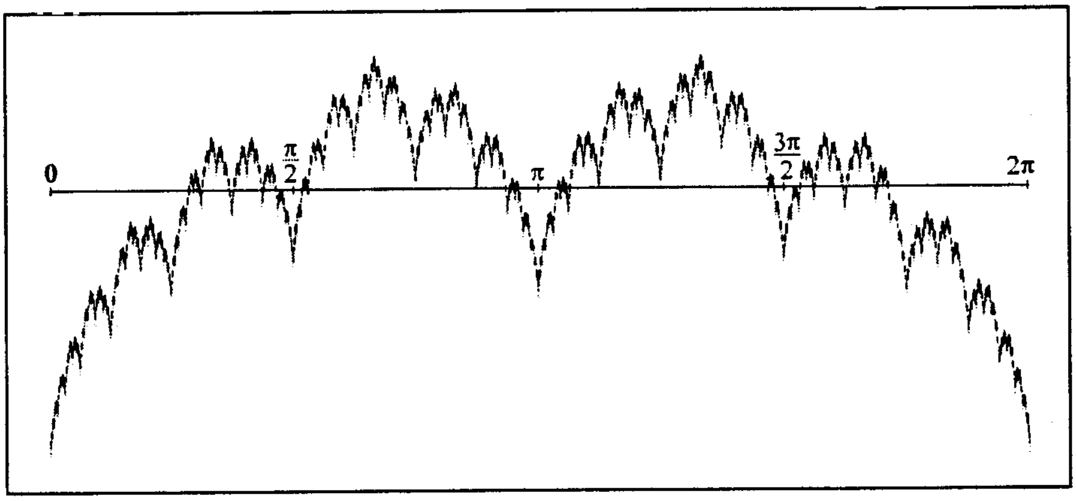
In the most interesting cases those textures and details cannot be predicted using classical geometry. Fractals bave fractional dimensions. Fractals can be generated by an algorithm, because of the self-similarity on every scale.

Some ***mathematical fractals***:

Weierstrass functions (introduced in 1875)

$$f(x) = \sum_{i=0}^{\infty} \lambda^{-si} \cos(\lambda^i x), \; 0 < s < 1 \; \text{and} \; \lambda > 1.$$

and for ë = 2 and s = 0.5



This is a fractal, because of for the function

$$f\left(\frac{x}{2}\right) = \sum_{i=0}^{\infty}\left(\sqrt{2}\right)^{-i}\cos\left(2^i\,\frac{x}{2}\right) = \frac{\sqrt{2}}{2}\left(f(x) + \cos\frac{x}{2}\right)$$

$$f\left(2\pi - \frac{x}{2}\right) = \sum_{i=1}^{\infty}\left(\sqrt{2}\right)^{-i}\cos\left(2^i\left(2\pi - \frac{x}{2}\right)\right) = f\left(\frac{x}{2}\right) = \frac{\sqrt{2}}{2}\left(f(x) + \cos\frac{x}{2}\right),$$

there are operators:

$$W(f)(x) = \begin{cases} \dfrac{\sqrt{2}}{2}\left(f(2x) + \cos x\right), & \text{if } x \in [0,\ \pi]; \\ \dfrac{\sqrt{2}}{2}\left(f(4\pi - 2x) + \cos x\right), & \text{if } x \in (\pi,\ 2\pi]. \end{cases}$$

and then the limit ***f*, *W(f)*, *W(W(f))*, *W(W(W(f)))*, ….** gives the fractal.

THE HILBERT CURVE

This function gives a bijection of a unit segment [0, 1] to [0, 1] x [0, 1], defined as follows:

$$\begin{array}{ccc} [0,\ 1] & \longleftrightarrow & [0,\ 1]\times[0,\ 1] \\ 0.d_1d_2d_3d_4d_5\cdots & \longrightarrow & (0.d_1d_3d_5\cdots, 0.d_2d_4d_6\cdots) \\ 0.a_1b_1a_2b_2a_3\cdots & \longleftarrow & (0.a_1a_2a_3\cdots, 0.b_1b_2b_3\cdots). \end{array}$$

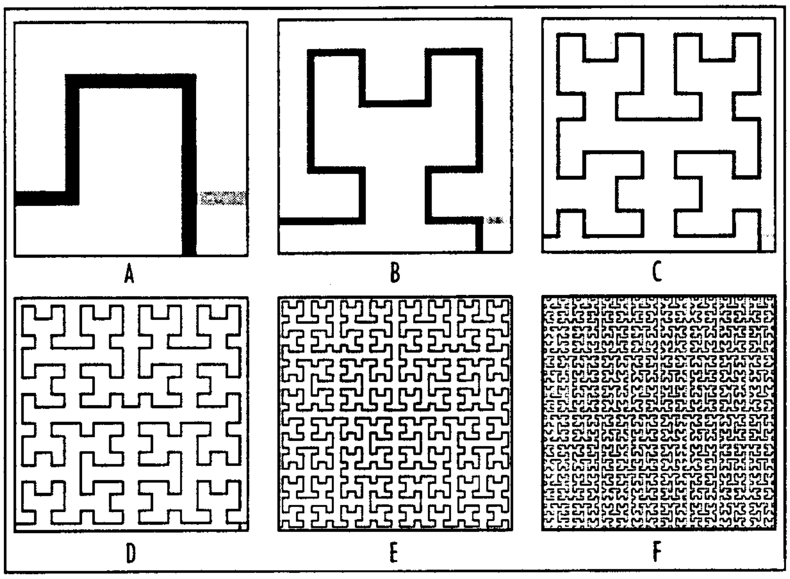
A geometric proof of this was given in 1890, by Peano. Later, Hilbert introduced an easier, iterative construction. Hilbert filled up the unit square with a segment, now is called the Hilbert space filling curve. The following four transformations are chosen:

$$w_0\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0 & 0.5\\-0.5 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\0.5\end{pmatrix},\quad w_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-0.5 & 0\\0 & 0.5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0.5\\0.5\end{pmatrix},$$

$$w_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0.5 & 0\\0 & 0.5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0.5\\0.5\end{pmatrix},\quad w_3\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0 & -0.5\\-0.5 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}1\\0.5\end{pmatrix}.$$

The transformations

- w_0 maps the whole square to the bottom left corner with a 90^0 rotation;
- w_1 maps the whole square to the top left corner with a vertical flip;
- w_2 shrinks the whole square to the top right corner;
- w_3 maps the whole square to the bottom right corner with a diagonal flip.

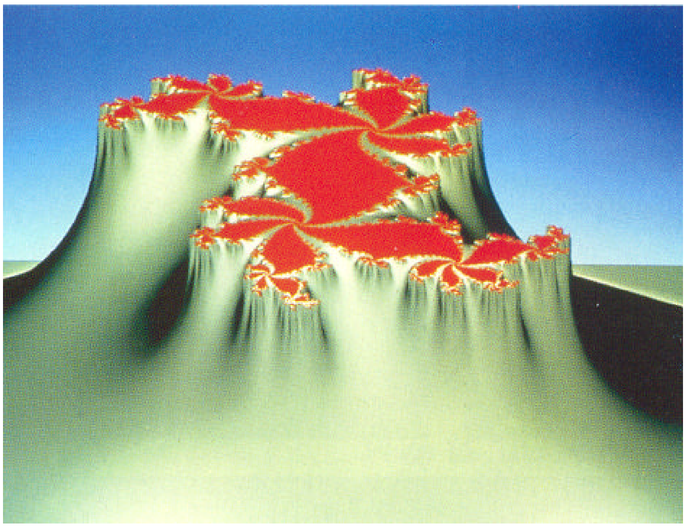


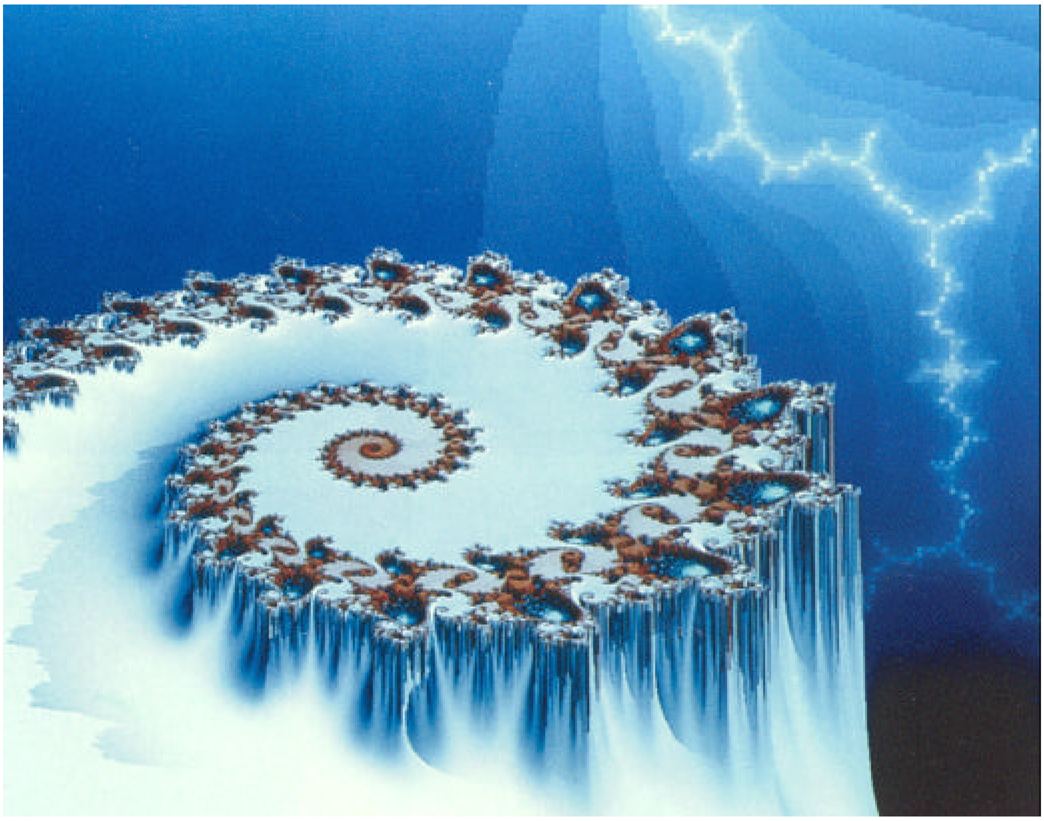
Julia sets and Mandelbrot set

In mathematics, conformal transformations play an essential role in the theory and applications of functions of complex variables. The conformal transforms

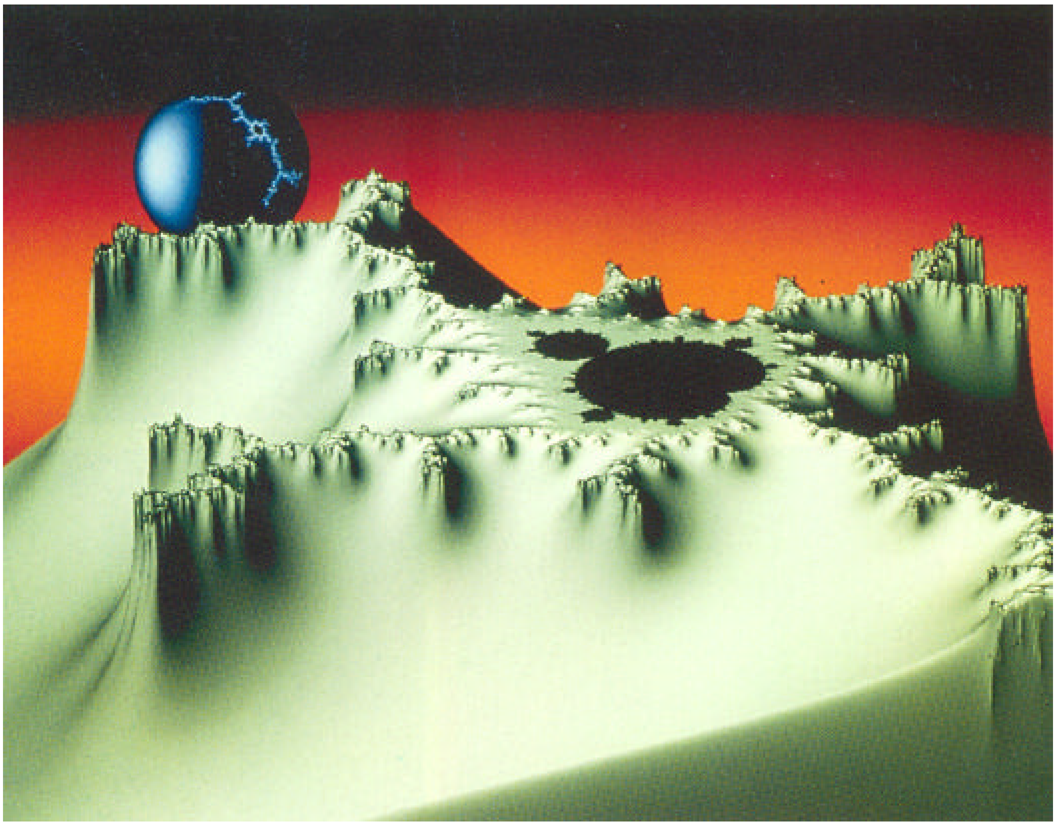
$$f(z) = z^2 + C$$

for complex number $\boldsymbol{C}$ and complex variable $\boldsymbol{z}$, expressed geometrically, correspond to ***Julia sets***:





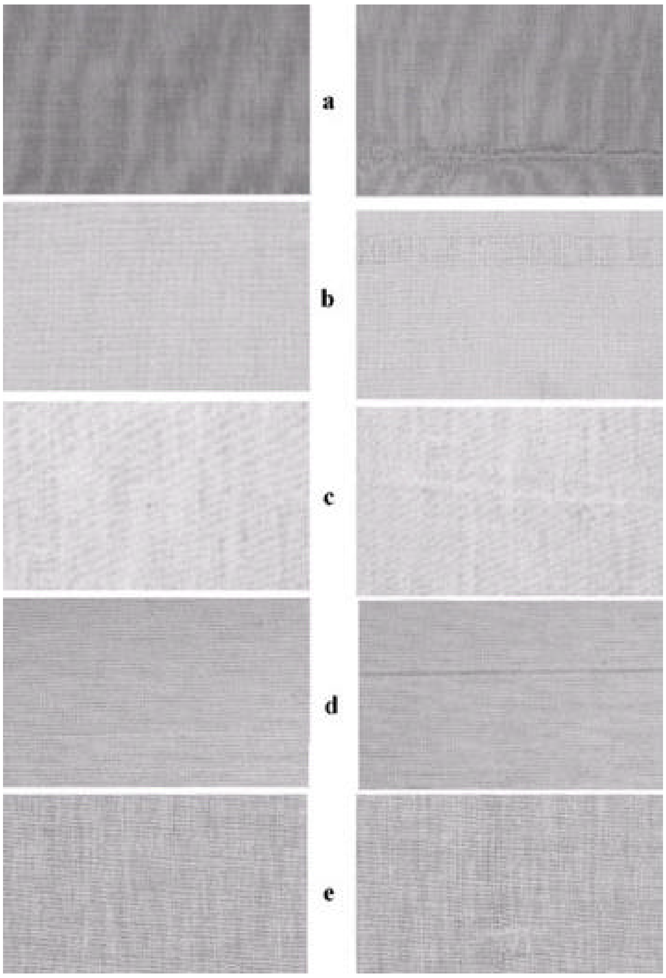
The ***Mandelbrot set*** is defined on complex plane as a set of complex numbers ***C***, for which the corresponding ***Julia set*** is connected:

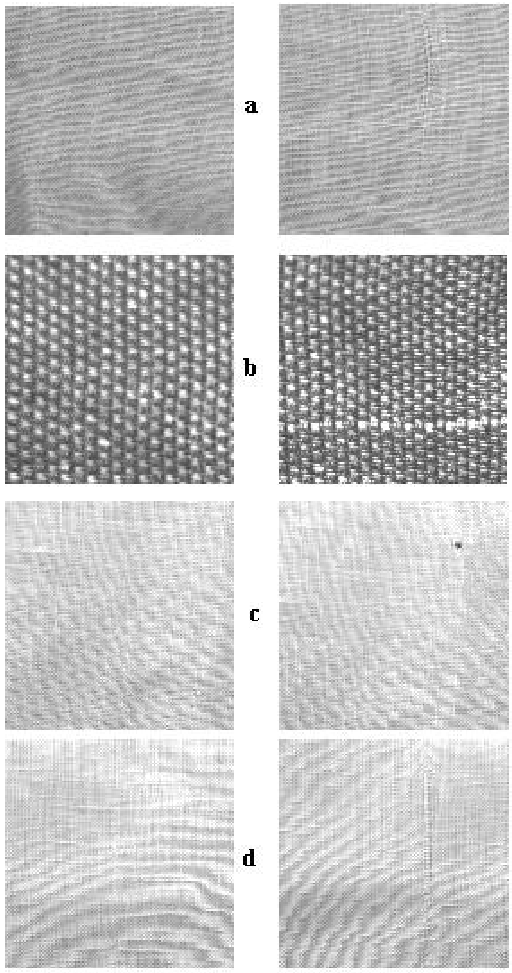


More Julia sets and Weierstrass functions?

A fractal image analysis system for fabric inspection:

The main difficulty is solving the problem of quantifying visual impressions in complex situation like those met in fabric manufacture. Defects which need to be found by the inspections are numerous and complex:





Stochastic fractals

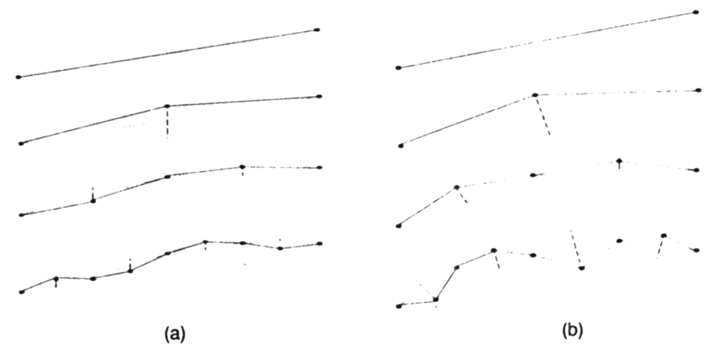
To create more "natural looking" shapes (involving variety in the sense that the leaves of tree may generally have the same form, but are individually different), we use randomization. The **fractional Brownian model,** a family of one-dimensional Gaussian stochastic processes of value in analyzing time series for natural phenomena, incorporates a curve which is selfsimilar.

To simulate fractional movement, an approximate solution may be given by the formula:

$$y_{new} = \frac{1}{2}(y_1 + y_2) + us_0 2^{-lh}$$

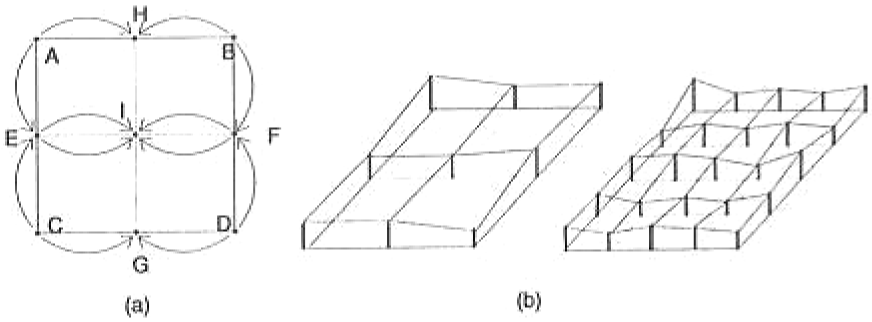
where *u* stands for a random number, $ó_0$ is the parameter for a Gaussian (normal curve) distribution, *l* is the level of recursivity, and *h* is a fractal parameter specifying the roughness of an object.

For example, for coastlines or terrain generation, this is often near 0.8. This formula means that instead of having the pure middle line segment one-half of the addition of the two y values, we add a small error term which has a Gaussian distribution, and fading with the level of recursivity.

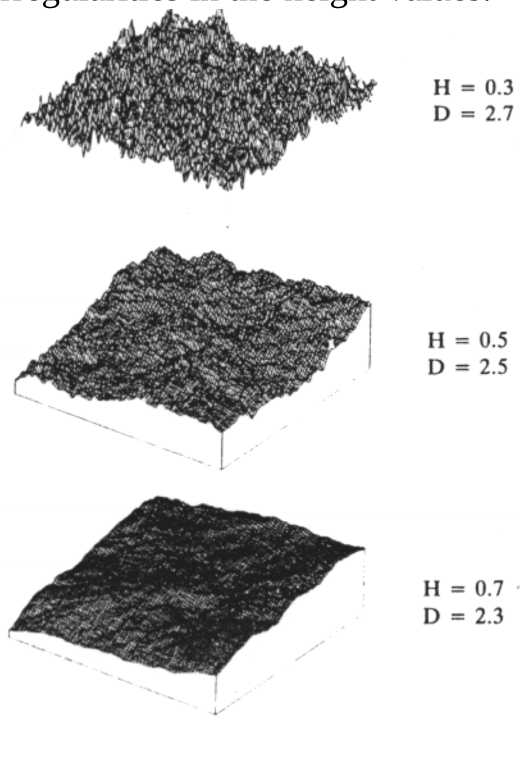


A slight variety of this process displaces the point along the segment mediator (b), the line perpendicular to the line joining the two points at the ends of the starting line segment. Some authors prefer to move the middle along the perpendicular bisector, and some prefer movement along a line parallel to an axis.

The process can also be used for two-dimensional contexts, to produce simulations of terrain:



Starting from points A, B, C and D we generate midpoints on each side of the original rectangle (or square), and then get I from the midpoints E, F, G and H. Gaussian displacement values are again generated, producing an effect as shown (figure b). Continuing this process to small line segment lengths will produce a terrain simulation from the non-systematic irregularities in the height values:



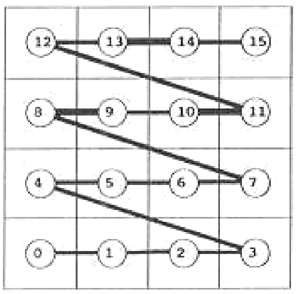
Space-Filling Curves And Dimensionality

Data processing and storage may be more economical if less information can be used to meet the same requirements. Thus area units may be represented by a centroid, a zero-dimensional object or by parametric curves. A data reduction can also occur if objects could be positioned in only one dimension rather than two or three.

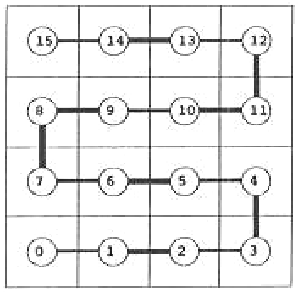
Paths through space

The matter of dimensionality is encountered in spatial information systems in different ways. In an earlier chapter we discussed the identification of material entities as having zero, one, two or three dimensions. It also arises in terms of addressing systems.

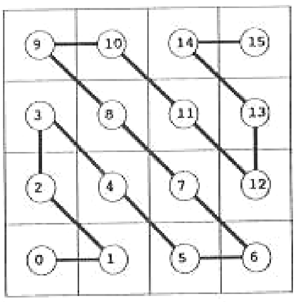
There are different orderings, that is **one-dimensional paths,** through the two-dimensional tiled space. Paths could zigzag, could go along a row in one direction and in a reverse direction in the next row, like a bidirectional computer printer (b), or could follow a path that reduces the total distance of travel through going to as many immediate neighbours as possible, and having a small number of longer connections, or could have diagonal or spiral forms (c and d):



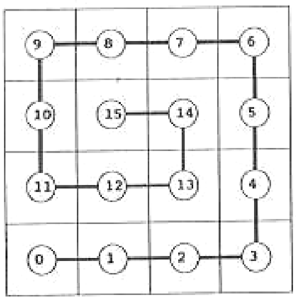
(a) Row Order



(b) Row–prime Order

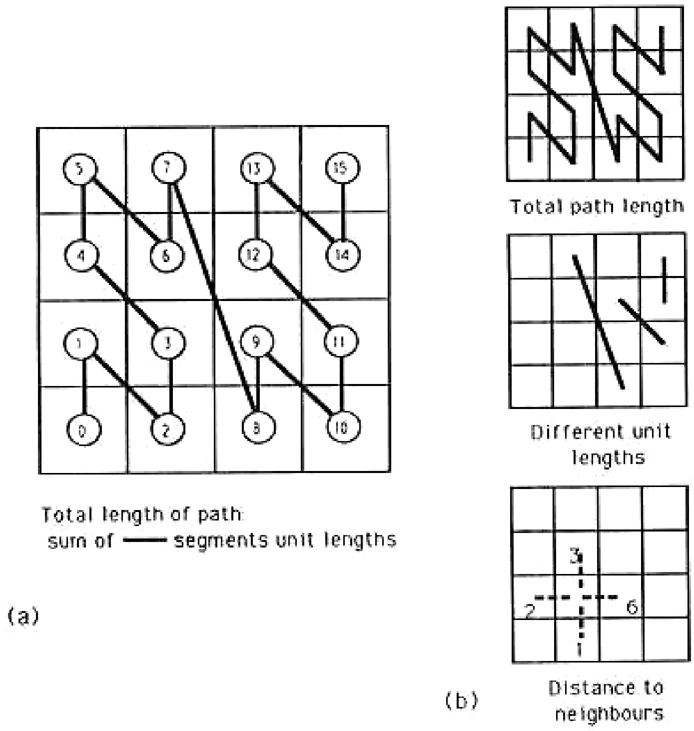


(c) Cantor–diagonal Order



(d) Spiral Order

A good sequential ordering should have certain properties that provide some conveniences in single dimension addressing for two- or three-dimensional sets of regularly shaped tiles. The path should pass only once to each tile in the two- or N-dimensional space, and neighbours in space should be adjacent on the path. The path should be useable even if there is a mixture of different sized spatial units, and should work equally well in two or three dimensions, and for connecting to adjacent blocks of space. In reality there is no ideal path; there are just orderings with some of these properties.



A comparison of different paths (for a given resolution) can use several measures:

1. Total length of the path.
2. Variability in unit lengths, where unit length is the distance from one point on the path to the next in sequence.
3. The average distance on the path from tiles to their four neighbours in space.

Comparative averages for the central block of four squares are shown in the table below along with other properties of a sixteen-tile mosaic.

Path type	Length (approximate)	Variability	Average distance
Row	22	2	10
Row-prime	*15*	1	10
Diagonal	18	2	18
Spiral	15	1	13
N	20	3	12

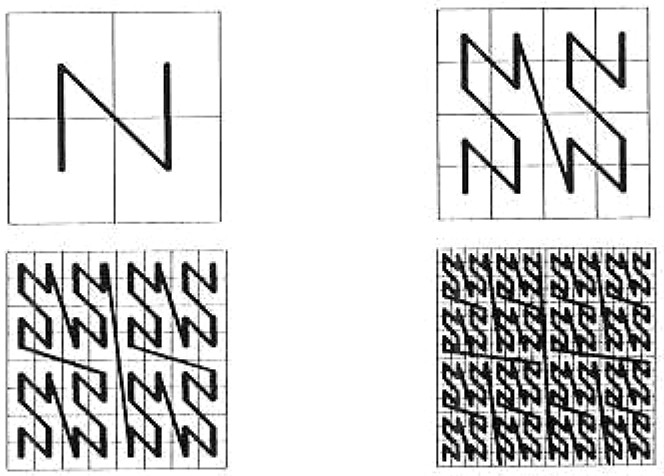
Space-filling curves

Often we talk about space-filling curves rather than paths through space. These curves are special fractal curves which have characteristics of completely covering an area or volume. While they have a topological dimension of two, their fractal dimension is two when filling an area, or three when completely occupying a volume space.

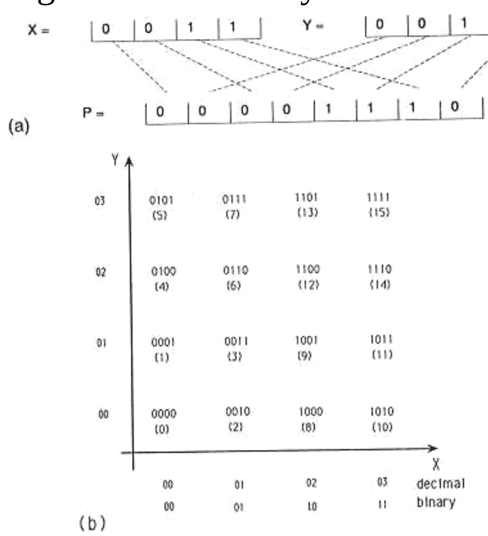
Consequently, thinking of paths in space now as space-filling curves, lines that pass to all possible points in space, they should have the following properties:

1. The curve must pass only once to every point in the multi-dimensional space.
2. Two points that are neighbours in space must be neighbours on the curve.
3. Two points that are neighbours on the curve must be neighbours in space.
4. It should be easy to retrieve the neighbours of any point.
5. The curve corresponds to a bijective mapping from a multi- to a onedimensional space.
6. The curve should be able to be used for variable spatial resolution, that is, a mixture of different-sized "points".
7. The curve should be stable, even when the space becomes very large or infinite.

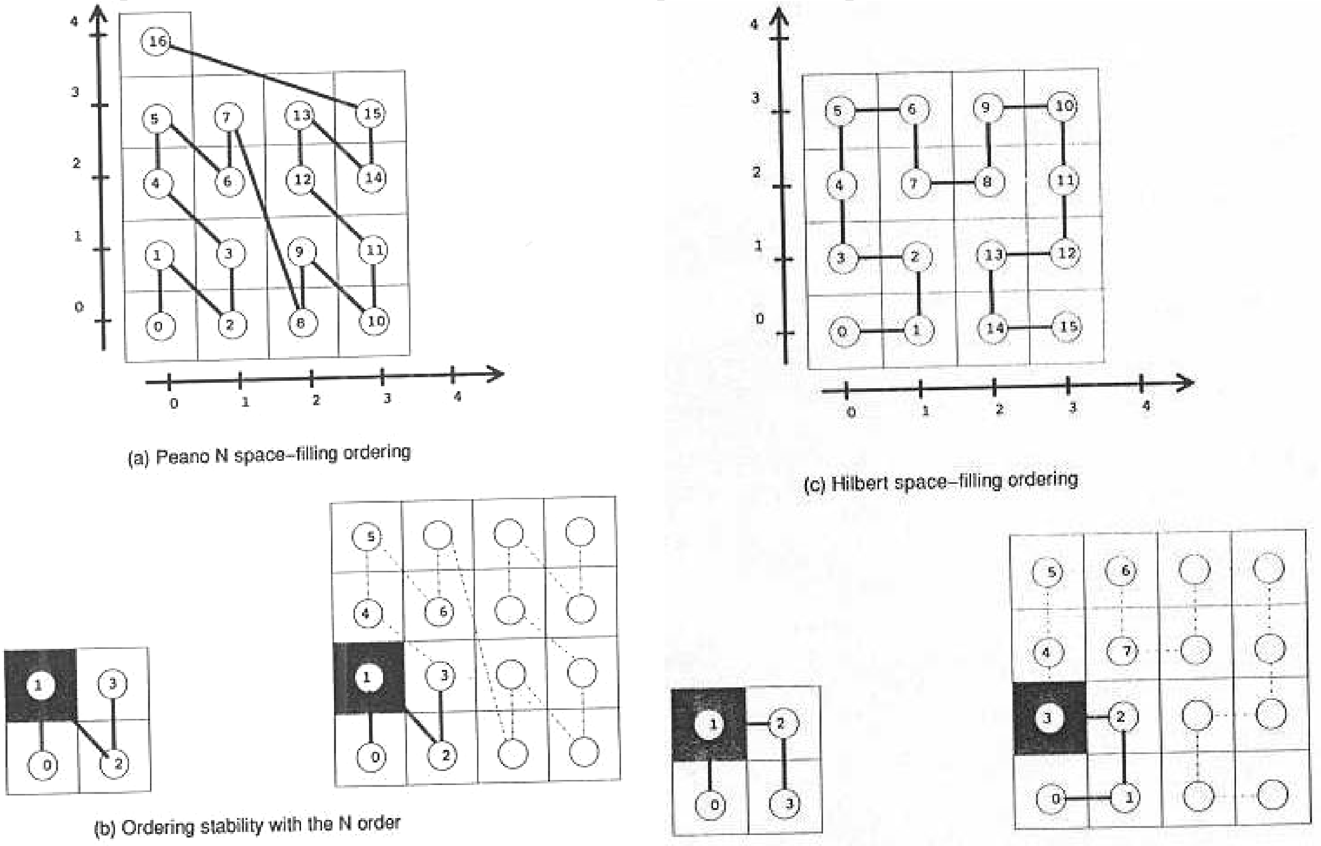
In reality, we do not possess such ideal curves, but there are some with valuable properties for our purposes. The original space-filling curve was exhibited in 1890 by the Italian mathematician Giuseppe Peano (Peano, 1890). A later variety, now known as the **Peano** or **N ordering** (a), facilitates retrieving **neighbours,** and although neighbouring points in space are not always neighbours on the curve, they generally are. It is also possible to deal with different resolutions as shown, and the curve is stable:



The **Hilbert** curve meets most of the conditions noted above, but does not provide an easy way to retrieve neighbours and is not stable. For the Peano curve the keys are easily obtained, the binary digits for the x and y values are interleaved:



Generally, the ordered paths have similar shapes at different scale levels, they are self-similar. However, the particular place of a point or tile in the sequence for a particular curve type may not be consistent across scales. While the N curve does have such **stability,** as revealed by the coded numbers, this is not so for the Hilbert curve. That is, for the Peano case, if the space is extended by doubling each side for the block of four quadrants, we see that the order of squares is not perturbed.



(a) Peano N space-filling ordering

(c) Hilbert space-filling ordering

(b) Ordering stability with the N order

Space-filling curves have two principal, practical uses in the domain of spatial information systems:

- firstly, they provide some efficiencies in scanning operations, either hardware devices or searches through datafiles;
- secondly, they are used as spatial indexes, simplifying two dimensional addressing as single dimension addressing.