Duomenų struktūros ir algoritmai

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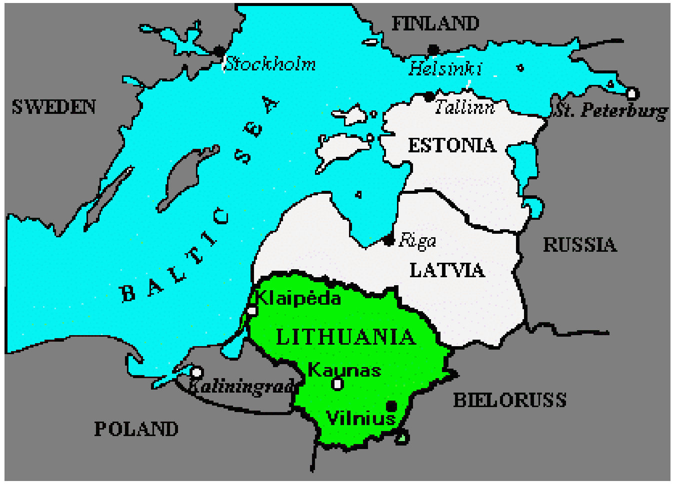


The goal of the course:

provide a comprehensive introduction to the modern study of computer algorithms. The algorithms and data structures selected are basic ones, constituting the background of computer science. The exposition of the theory, the design, and implementation of data structures and algorithms, as well as their issues are especially oriented to the applications typical for databases, information systems, geographic information systems, computer graphics, and multimedia. The content of the course:

Algorithms, abstract data types, computers, memory

- outline of topic, Euclid algorithm and it's implication to objects like very big numbers and polynomials, basic data types
- pseudocode (and other conventions)
- abstract data types (stack ADT, queue ADT, list ADT, matrix ADT, the dynamic set ADT)
- models of memory, data in physical memory, data in multimedia systems

Trees and priority queues

- trees, binary trees, AVL trees, 2-3-4 trees, red-black trees
- heaps, heapsort, priority queues
- B-trees
- data compression and Huffman codes
- trees in computer memory

Hashing and indexing

- hashing and hashing functions
- linear chaining, open addressing
- extendable hashing

Sorting algorithms

- elementary sorting methods
- quicksort
- radix sorting
- mergesort
- external sorting

Radix searching

- digital search trees
- radix search trees
- multiway radix searching
- Patricia algorithm

External searching

- indexed sequential access
- virtual memory

String processing

- brute-force algorithm
- Knuth-Morris-Pratt algorithm
- Boyer-Moore algorithm
- Rabin-Karp algorithm

Pattern matching

- finite automata
- longest subsequence problem
- constrained patterns

Range searching

- elementary methods
- grid method
- two-dimensional trees
- multidimensional range searching

Hierarchical multidimensional data

- quadtrees and octrees
- R-trees

Algorithms and abstract data types

Informally, ***algorithm*** means is a well-defined computational procedure that takes some value, or set of values, as ***input*** and produces some other value, or set of values, as ***output***. An algorithm is thus a sequence of computational steps that transform the input into the output.

Algorithm is also viewed as a ***tool*** for solving a well-specified problem, involving computers. The statement of the problem specifies in general terms the desired input/output relationship. The algorithm describes a specific computational procedure for achieving that input/output relationship.

There exist many points of view to algorithms. One if these points is a computational one. A good example of this is a famous Euclid’s algorithm:

for two integers x, y calculate the greatest common divisor gcd (x, y)

Direct implementation of the algorithm (in Pascal) looks like:

```
program euclid (input, output);
var x,y: integer;
function gcd (u,v: integer): integer;
    var t: integer;
    begin
    repeat
          if u<v then
          begin t := u; u := v; v := t end;
          u := u-v;
    until u = 0;
    gcd := v
    end;
begin
    while not eof do
    begin
    readln (x, y);
    if (x>0) and (y>0) then writeln (x, y, gcd (x, y))
    end;
end.
```

The same algorithm in C++:

```
#include
int gcd(int u, int v)
  {
    int t;
    while (u > 0)
      {
        if (u < v) { t = u; u = v; v = t; }
        u = u - v;
      }
    return v;
   }
main()
  {
    int x, y;
    while (cin >> x && cin >> y)
      if (x>0 && y>0) cout << x << ' ' << y << ' '
                          << gcd(x,y) << '\n';
  }
```

This algorithm has some exceptional features, nevertheless specific to the computational procedures:

- it is computationally intensive (uses a lot of processor mathematical instructions);
- it is applicable only to numbers;
- it has to be changed every time when the environment changes, say if numbers are to be very long and consequently do not fit to size of variable (numbers like 1000!).

(d1)

```
402387260077093773543702433923003985719374864210↵
714632543799910429938512398629020592044208486969↵
404800479988610197196058631666872994808558901323↵
829669944590997424504087073759918823627727188732↵
519779505950995276120874975462497043601418278094↵
646496291056393887437886487337119181045825783647↵
849977012476632889835955735432513185323958463075↵
557409114262417474349347553428646576611667797396↵
668820291207379143853719588249808126867838374559↵
731746136085379534524221586593201928090878297308↵
431392844403281231558611036976801357304216168747↵
609675871348312025478589320767169132448426236131↵
412508780208000261683151027341827977704784635868↵
170164365024153691398281264810213092761244896359↵
928705114964975419909342221566832572080821333186↵
116811553615836546984046708975602900950537616475↵
847728421889679646244945160765353408198901385442↵
487984959953319101723355556602139450399736280750↵
137837615307127761926849034352625200015888535147↵
331611702103968175921510907788019393178114194545↵
257223865541461062892187960223838971476088506276↵
862967146674697562911234082439208160153780889893↵
964518263243671616762179168909779911903754031274↵
622289988005195444414282012187361745992642956581↵
746628302955570299024324153181617210465832036786↵
906117260158783520751516284225540265170483304226↵
143974286933061690897968482590125458327168226458↵
066526769958652682272807075781391858178889652208↵
164348344825993266043367660176999612831860788386↵
150279465955131156552036093988180612138558600301↵
435694527224206344631797460594682573103790084024↵
432438465657245014402821885252470935190620929023↵
136493273497565513958720559654228749774011413346↵
962715422845862377387538230483865688976461927383↵
814900140767310446640259899490222221765904339901↵
886018566526485061799702356193897017860040811889↵
729918311021171229845901641921068884387121855646↵
124960798722908519296819372388642614839657382291↵
123125024186649353143970137428531926649875337218↵
940694281434118520158014123344828015051399694290↵
153483077644569099073152433278288269864602789864↵
321139083506217095002597389863554277196742822248↵
757586765752344220207573630569498825087968928162↵
753848863396909959826280956121450994871701244516↵
461260379029309120889086942028510640182154399457↵
156805941872748998094254742173582401063677404595↵
741785160829230135358081840096996372524230560855↵
903700624271243416909004153690105933983835777939↵
410970027753472000000000000000000000000000000000↵
000000000000000000000000000000000000000000000000↵
000000000000000000000000000000000000000000000000↵
000000000000000000000000000000000000000000000000↵
000000000000000000000000000000000000000000000000↵
000000000000000000000000
```

For algorithms of applications, like databases, information systems, they are usually understood in a slightly different way, more like ***tools*** to achieve input/output relationship. The example of an algorithm in such sense would be a sorting procedure. This procedure is frequently appearing in a transaction *(a prime operation in information systems or databases)*, and even while the same transaction, the sorting:

- is repeated many times;
- in a various circumstancies;
- with different types of data.

Pseudocode

The notation language to describe algorithms is needed. This language called ***pseudocode***, often ***is a lingua franca***.

Pseudocode *is* used to express algorithms in a manner that is independent of a particular programming language. The prefix ***pseudo*** is used to emphasize that this code is not meant to be compiled and executed on a computer. The reason for using pseudocode is that it allows one to convey basic ideas about an algorithm in general terms. This, in essence, is the difference between pseudocode and a computer program. A pseudocode program simply states the steps necessary to perform some computation, while the corresponding computer program is the translation of these steps into the syntax of a particular programming language.

The ability to ignore implementation details when using pseudocode will facilitate analysis by allowing us to focus solely on the computational or behavioral aspects of an algorithm. Constructs of pseudocode:

- assignments;
- for … to … [step] … do [in steps of];
- while … do;
- do … while;
- begin … end;
- if … then … else;
- pointer, *and* null pointer;
- arrays;
- composite data types;
- procedure *and* its name;
- formal parameters *versus* actual parameters.

Abstract Data Types

A theoretical description of an algorithm, if implemented in application is affected very much by:

- computer resources,
- implementation,
- data.

To avoid unnecessary troubles, limitations, specificity, etc. in the design of algorithm, some additional theory has to be used.

Such a theory include fundamental concepts (guide lining the content of the course):

- concepts of Abstract Data Type (ADT) or data type, or data structures;
- tools to express operations of algorithms;
- computational resources to implement the algorithm and test its functionality;
- evaluation of the complexity of algorithms.

Level of Abstraction

The level of abstraction is one of the most crucial issues in the design of algorithms. The term abstraction refers to the ***intellectual capability*** of considering an entity apart from any specific instance of that entity. This involves an abstract or logical description of components:

- the data required by the software system,
- the operations that can be performed on this data.

The use of data abstraction during software development allows the software designer to concentrate on how the data in the system is used to solve the problem at hand, without having to be concerned with how the data is represented and manipulated in computer memory.

Abstract Data Types

The development of computer programs is simplified by using abstract representations of data types (i.e., representations that are devoid of any implementation considerations), especially during the design phase.

Alternatively, utilizing concrete representations of data types (i.e., representations that specify the physical storage of the data in computer memory) during design introduces:

- unnecessary complications in programming (to deal with all of the issues involved in implementing a data type in the software development process),
- a yield a program that is dependent upon a particular data type implementation.

An ***abstract data type (ADT)*** is defined as ***a mathematical model*** of the data objects that make up a data type, as well as the functions that operate on these objects (and sometime impose logical or other relations between objects).

So ADT consist of two parts: data objects and operations with data objects. The operations that manipulate data objects are included in the specification of an ADT.

At this point it is useful to distinguish between ADTs, data types, and data structures.

The term ***data type*** refers to the *implementation* of the mathematical model specified by an ADT. That is, a data type is a computer representation of an ADT.

The term ***data structure*** refers to a collection of computer variables that are connected in some specific manner. This course is concerned also with using data structures to implement various data types in the most efficient manner possible.

The notion of ***data type*** include ***built-in data types***. Built-in data types are related to a programming language.

A programming language typically provides a number of ***built-in data types.*** For example, the ***integer*** data type in ***Pascal,*** or ***int*** data type available in the ***C*** programming language provide an implementation of the mathematical concept of an integer number.

Consider ***INTEGER*** ADT, which:

- defines the set of objects as numbers (*–infinity, … -2, -1, 0, 1, 2, ..., +infinity*);
- specifies the set of operations: integer addition, integer subtraction, integer multiplication, div (divisor), mod (remainder of the divisor), logical operations like <, >, =, etc.

The specification of the ***INTEGER*** ADT does not include any indication of how the data type should be implemented. For example, it is impossible to represent the full range of integer numbers in computer memory; however, the range of numbers that will be represented must be determined in the data type implementation.

Built-in data type ***INT*** in **C**, or ***INTEGER*** in **Pascal** are dealing with the set of objects as numbers in the range (*minint, … -2, -1, 0, 1, 2, …, maxint*), and format of these numbers in computer memory can vary between one's complement, two's complement, sign-magnitude, binary coded decimal (BCD), or some other format.

The implementation of an ***INTEGER*** ADT involves a translation of the ADT's specifications into the syntax of a particular programming language. This translation consists of the appropriate variable declarations necessary to define the data elements, and a procedure or ***accessing routine*** that implements each of the operations required by the ADT.

The ***INTEGER*** ADT when implemented according to specification gives a freedom to programmers. Generally they do not have to concern themselves with these implementation considerations when they use the data type in a program.

In many cases the design of a computer program will ***call*** for data types that ***are not available*** in the programming language used to implement the program. In these cases, programmers must be able to construct the necessary data types by using built-in data types. This will often involve the construction of quite complicated data structures. The data types constructed in this manner are called ***user-defined data types***.

The ***design and implementation*** of data types as well as ADTs are often focused on ***user-defined*** data types. Then a new data type has to be considered from two different viewpoints:

- a logical view;
- an implementation view.

The ***logical view*** of a data type should be used during program design. This is simply the model provided by the ADT specification.

The ***implementation view*** of a data type considers the manner in which the data elements are represented in memory, and how the accessing functions are implemented. Mainly it is to be concerned with how alternative data structures and accessing routine implementations affect the efficiency of the operations performed by the data type.

There should be ***only one*** logical view of a data type, however, there may be ***many*** different approaches to implementing it.

Taking the whole process of ADT modeling and implementation into account, many different features have to be considered. One of these characteristics is an interaction between various ADTs, data types, etc. In this course a so-called ***covering ADT*** will be used to model the behavior of a specific ADT and to implement it.

The *STACK* ADT

This ADT covers a set of objects as well as operations performed on these objects:

- *initialize* (*S*) – creates a necessary structured space in computer memory to locate objects in *S*;
- *push(S, x)* **–** inserts *x* into *S*;
- *pop(S)* **–** deletes object from the stack that was most recently inserted into;
- *top(S)* – returns an object from the stack that was most recently inserted into;
- *kill*(*S*) - releases an amount of memory occupied by *S*.

The operations with stack objects obey ***LIFO*** property: *Last-In-First-Out*. This is a logical constrain or logical condition.

The operations *Initialize* and *Kill* are more oriented to an implementation of this ADT, but they are important in some algorithms and applications too.

The stack is a dynamic data set with a limited access to objects.

The ***model of application*** to illustrate usage of a stack is:

calculate the value of an algebraic expression.

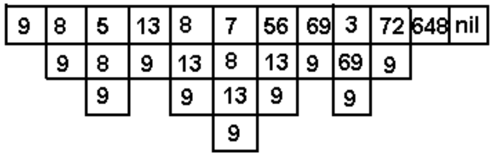
If the algebraic expression is like:

9*(((5+8)+(8*7))+3)

then the sequence of operations of stack to calculate the value would be:

push(9);
push(5);
push(8);
push(pop+pop);
push(8);
push(7);
*push(pop*pop);*
push(pop+pop);
push(3);
push(pop+pop);
*push(pop*pop);*
writeln(pop).

The content of stack after each operation will be:



9	8	5	13	8	7	56	69	3	72	648	nil
	9	8	9	13	8	13	9	69	9		
		9		9	13	9		9			
					9						

The stack is useful also to verify the ***correctness of parentheses*** in an algebraic expression.

If ***objects of stack*** fit to ordinary variables the straightforward implementation will look like expressed below.

```
type link=^node;
  node=record key:integer; next:link; end;
  var head,z:link;
procedure stackinit;
    begin
        new(head); new(z);
        head^.next:=z; z^.next:=z
    end;
procedure push(v:integer);
  var t:link;
    begin
        new(t);
        t^.key:=v; t^.next:=head^.next;
        head^.next:=t
    end;
function pop:integer;
  var t:link;
    begin
        t:=head^.next;
        pop:=t^.key;
        head^.next:=t^.next;
        dispose(t)
```

```
     end;
  function stackempty:boolean;
  begin stackempty:=(head^.next=z) end.
```

Implementation by the built-in data type of **array**:

```
  const maxP=100;
     var stack: array[0..maxP]of integer; p:integer;
  procedure push(v:integer);
       begin stack[p]:=v; p:=p+1 end;
  function pop:integer;
       begin p:=p-1; pop:=stack[p] end;
  procedure stackinit;
       begin p:=0 end;
  function stackempty:boolean;
       begin stackempty:=(p=<0) end.
```

The algebraic expression is implemented by using stack:

```
stackinit;
   repeat
     repeat read(c) until c<>'';
     if c=')' then write(chr(pop));
     if c='+' then push(ord(c));
     if c='*' then push(ord(c));
     while (c=>'0') and (c=<'9') do
          begin write(c); read(c) end;
     if c<>'(' then write('');
   until eoln;
```

The *QUEUE* ADT

This ADT covers a set of objects as well as operations performed on objects:

- *queueinit* (*Q*) – creates a necessary structured space in computer memory to locate objects in *Q*;
- *put (Q, x)* **–** inserts *x* into *Q*;
- *get (Q)* **–** deletes object from the queue that has been residing in *Q* the longest;
- *head (Q)* – returns an object from the queue that has been residing in *Q* the longest;
- *kill* (*Q*) – releases an amount of memory occupied by *Q*.

The operations with queue obey ***FIFO*** property: *First-In-First-Out*. This is a logical constrain or logical condition. The queue is a dynamic data set with a limited access to objects. The application to illustrate usage of a queue is:

queuing system simulation (system with waiting lines)

(implemented by using the built-in type of ***pointer***)

```
type link=^node;
  node=record key:integer; next:link; end;
var head,tail,z:link;

procedure queueinit;
    begin
        new(head); new(z);
        head^.next:=z; tail^.next=z; z^.next:=z
    end;

procedure put(v:integer);
  var t:link;
    begin
        new(t);
        t^.key:=v; t^.next:=tail^.next;
        tail^.next:=t
    end;

function get:integer;
  var t:link;
    begin
        t:=head^.next;
        get:=t^.key;
        head^.next:=t^.next;
        dispose(t)
    end;

function queueempty:boolean;
    begin queueempty:=(head^.next=z;tail^.next=z)
    end.
```

The queue operations by *array*:

```
const max=100;
var queue:array[0..max] of integer;
  head, tail:integer;
procedure put(v:integer);
    begin
        queue[tail]:=v; tail:=tail+1;
        if tail>max then tail:=0
    end;
function get: integer;
    begin  get:=queue[head]; head:=head+1;
        if head>max then head:=0
    end;
procedure queueinitialize;
    begin head:=0; tail:=0
    end;
function queueempty:boolean;
    begin queueempty:=(head=tail)
    end.
```

The Queue Implementation

A queue is used in computing in much the same way as it is used in everyday life:

- to allow a sequence of items to be processed on a ***first-come-first-served*** basis.

In most computer installations, for example, one printer is connected to several different machines, so that more than one user can submit printing jobs to the same printer. Since printing a job takes much longer than the process of actually transmitting the data from the computer to the printer, a queue of jobs is formed so that the jobs print out in the same order in which they were received by the printer. This has the irritating consequence that if your job consists of printing only a single page while the job in front of you is printing an entire 200-page thesis, you must still wait for the large job to finish before you can get your page.

The actions allowed on a queue are:

- ***creating an empty queue.***
- ***testing if a queue is empty.***
- ***adding data to the tail of the queue.***
- ***removing data from the head of the queue.***

Just as with stacks, queues can be implemented using arrays or lists. For the first of all, let's consider the implementation using arrays.

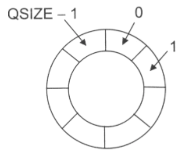
Define an array for storing the queue elements, and two markers:

- one pointing to the location of the *Head* of the queue,
- the other to the first empty space following the *Tail*.

When an item is to be added to the queue, a test to see if the *Tail* marker points to a valid location is made, then the item is added to the queue and the *Tail* marker is incremented by 1. When an item is to be removed from the queue, a test is made to see if the queue is empty and, if not, the item at the location pointed to by the *Head* marker is retrieved and the *Head* marker is incremented by 1.

This procedure works well until the first time when the *Tail marker reaches the end of the array*. If some removals have occurred during this time, there will be empty space at the beginning of the array. However, because the *Tail* marker points to the end of the array, the queue is thought to be 'full' and no more data can be added.

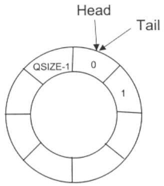
We could shift the data so that the *Head* of the queue returns to the beginning of the array each time this happens, but shifting data is costly in terms of computer time, especially if the data being stored in the array consist of large data objects.



A more efficient way of storing a queue in an array is to "***wrap around***" the end of the array so that it joins the front of the array. Such a circular array allows the entire array (well, almost, as we'll see in a bit) to be used for storing queue elements without ever requiring any data to be shifted. A circular array with QSIZE elements (numbered from 0 to QSIZE-1) may be visualized:

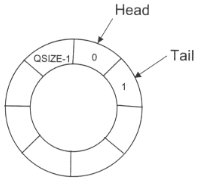
The array is, of course, stored in the normal way in memory, as a linear block of QSIZE elements. The circular diagram is just a convenient way of representing the data structure.

We will need *Head* and *Tail* markers to indicate the location of the head and the location just after the tail where the next item should be added to the queue,

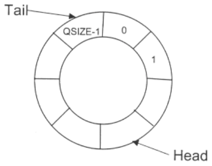


respectively. An empty queue is denoted by the condition *Head* = *Tail*:

At this point, the first item of data would be added at the location indicated by the *Tail* marker, that is, at array index 0. Adding this element gives us the situation:



Let us use the queue until the *Tail* marker reaches QSIZE-1. We will assume that some items have been removed from the queue, so that *Head* has moved along as well:



Now we add another element to the queue at the location marked by *Tail*, that is, at array index QSIZE-1. The *Tail* marker now advances one step, which positions it at array index 0. The *Tail* marker has wrapped around the array and come back to its starting point. Since the *Head* marker has moved along, those elements at the beginning of the array from index 0 up to index *Head*-1 are available for storage. Using a circular array means that we can make use of these elements without having to shift any data.

In a similar way, if we keep removing items from the queue, eventually *Head* will point to array index QSIZE-1. If we remove another element, *Head* will advance another step and wrap around the array, returning to index 0.

We have seen that the condition for an empty queue is that *Head* == *Tail*. What is the condition for a full queue? If we try to make use of all the array elements, then in a full queue, the tail of the queue must be the element immediately prior to the head. Since we are using the *Tail* marker to point to the array element immediately following the tail element in the queue, *Tail* would have to point to the same location as *Head* for a full queue. But we have just seen that the condition *Head* == *Tail* is the condition for an empty queue. Therefore, if we try to make use of all the array elements, the conditions for full and empty queues become identical. We therefore impose the rule that we must always keep at least ***one free space in the array***, and that a queue becomes full when the *Tail* marker points to the location immediately prior to *Head*.

We may now formalize the algorithms for dealing with queues in a circular array.

- Creating an empty queue: Set Head = Tail = 0.
- Testing if a queue is empty: is Head == Tail?
- Testing if a queue is full: is (Tail + 1) **mod** QSIZE == Head?
- Adding an item to a queue: if queue is not full, add item at location Tail and set Tail = (Tail + 1) **mod** QSIZE.
- Removing an item from a queue: if queue is not empty, remove item from location *Head* and set *Head* = (*Head* + 1) **mod** QSIZE.

The **mod** operator ensures that *Head* and *Tail* wrap around the end of the array properly. For example, suppose that *Tail* is QSIZE-1 and we wish to add an item to the queue. We add the item at location *Tail* (assuming that the queue is not full) and then set *Tail*

$$((\text{QSIZE} - 1) + 1)\ \textbf{mod}\ \text{QSIZE} = \text{QSIZE}\ \textbf{mod}\ \text{QSIZE} = 0$$

The following program codes present implementation of elementary abstract data types (stacks and queues) in C++.

```
const int N = 1000;
main()
  {
   int i, j, a[N+1];
   for (a[1] = 0, i = 2; i <= N; i++) a[i] = 1;
   for (i = 2; i <= N/2; i++)
     for (j = 2; j <= N/i; j++)
```

```
    a[i*j] = 0;
  for (i = 1; i <= N; i++)
   if (a[i]) cout << i << ' ';
  cout << '\n';
 }

  struct node
   { int key; struct node *next; };

  struct node *head, *z;
  head = new node; z = new node;
  head->next = z; z->next = z;

struct node
 { int key; struct node *next; };
main()
 {
  int i, N, M;
  struct node *t, *x;
  cin >> N >> M;
  t = new node; t->key = 1; x = t;
  for (i = 2; i <= N; i++)
   {
    t->next = new node;
    t = t->next; t->key = i;
   }
  t->next = x;
  while (t != t->next)
   {
    for (i = 1; i < M; i++) t = t->next;
    cout << t->next->key << ' ';
    x = t->next; t->next = x->next;
    delete x;
   }
  cout << t->key << '\n';
 }

  key[x] = v; next[x] = next[t]; next[t] = x++;

class Stack
```

```
{
 private:
  itemType *stack;
  int p;
 public:
  Stack(int max=100)
   { stack = new itemType[max]; p = 0; }
 ~Stack()
   { delete stack; }
  inline void push(itemType v)
   { stack[p++] = v; }
  inline itemType pop()
   { return stack[--p]; }
  inline int empty()
   { return !p; }
};

 char c; Stack acc(50); int x;
 while (cin.get(c))
  {
   x = 0;
   while (c == ' ') cin.get(c);
   if (c == '+') x = acc.pop() + acc.pop();
   if (c == '*') x = acc.pop() * acc.pop();
   while (c>='0' && c<='9')
    { x = 10*x + (c-'0');  cin.get(c); }
   acc.push(x);
  }
 cout << acc.pop() << '\n';

 char c; Stack save(50);
 while (cin.get(c))
  {
   if (c == ')') cout.put(save.pop());
   if (c == '+') save.push(c);
   if (c == '*') save.push(c);
   while (c>='0' && c<='9')
    { cout.put(c); cin.get(c); }
   if (c != '(') cout << ' ';
  }
```

```
    cout << '\n';

class Stack
  {
    public:
      Stack(int max);
     ~Stack();
      void push(itemType v);
      itemType pop();
      int empty();
    private:
      struct node
       { itemType key; struct node *next; };
      struct node *head, *z;
  };

Stack::Stack(int max)
  {
    head = new node; z = new node;
    head->next = z;  z->next = z;
  }
Stack::~Stack()
  {
    struct node *t = head;
    while (t != z)
      { head = t; t = t->next; delete head; }
  }

void Stack::push(itemType v)
  {
    struct node *t = new node;
    t->key = v; t->next = head->next;
    head->next = t;
  }
itemType Stack::pop()
  {
    itemType x;
    struct node *t = head->next;
    head->next = t->next; x = t->key;
    delete t; return x;
```

```
  }
int Stack::empty()
  { return head->next == z; }

void Queue::put(itemType v)
  {
    queue[tail++] = v;
    if (tail > size) tail = 0;
  }
itemType Queue::get()
  {
    itemType t = queue[head++];
    if (head > size) head = 0;
    return t;
  }
int Queue::empty()
  { return head == tail; }
```

Generalized Queues

Specifically, pushdown stacks and FIFO queues are special instances of a more general ADT: the *generalized queue*. Instances *generalized queues* differ in only the rule used when items are removed:

- for stacks, the rule is *"remove the item that was most recently inserted"*;
- for FIFO queues, the rule is *"remove the item that was least recently inserted"*;
- there are many other possibilities to consider.

A powerful alternative is the ***random queue***, which uses the rule:

- *"remove a random item"*

The algorithm can expect to get any of the items on the queue with equal probability. The operations of a random queue can be implemented:

- in constant time using an array representation (it requires to reserve space ahead of time)
- using linked-list alternative (which is less attractive however, because implementing both insertion and deletion efficiently is a challenging task).

Random queues can be used as the basis for randomized algorithms, to avoid, with high probability, worst-case performance scenarios.

Stacks and FIFO queues are identifying items according to ***the time*** that they were inserted into the queue. Alternatively, the abstract concepts may be identified in terms of a sequential listing of the ***items in order***, and refer to the basic operations of inserting and deleting items from the beginning and the end of the list:

- if we insert at the end and delete at the end, we get a stack (precisely as in array implementation);
- if we insert at the beginning and delete at the beginning, we also get a stack (precisely as in linked-list implementation);
- if we insert at the end and delete at the beginning, we get a FIFO queue (precisely as in linked-list implementation);
- if we insert at the beginning and *delete* at the end, we also get a FIFO queue (this option does not correspond to any of implementations given).

Building on this point of view, the ***deque*** ADT may be defined, where either insertion or deletion at either end are allowed. The implementation of *deque* is a good exercise to program.

The ***priority queue*** ADT is another example of *generalized queue*. The items in a priority queue have keys and the rule for deletion is:

"remove the item with the smallest key"

The priority queue ADT is useful in a variety of applications, and the problem of finding efficient implementations for this ADT has been a research goal in computer science for many years. Identifying and using the ADT in applications has been an important factor in this research:

- an immediate indication can be given for whether or not a new algorithm is correct by substituting its implementation for an old implementation in a huge, complex application and checking if there has been got the same result;
- an immediate indication can be given for whether a new algorithm is more efficient than an old one by noting the extent to which substituting the new implementation improves the overall running time (the data structures and algorithms for solving this problem will be considered later, they are interesting, ingenious, and effective).

The ***symbol tables*** ADT is one more example of *generalized queues,* where the items have keys and the rule for deletion is:

"remove an item whose key is equal to a given key, if there is one"

This ADT is perhaps the most important one to consider, and dozens of implementations will be examined.

Each of these ADTs also give rise to a number of related, but different, ADTs that suggest themselves as an outgrowth of careful examination of application programs and the performance of implementations.

Duplicate and Index Items

For many applications, the abstract items to be processed are *unique,* a quality that lead to modification of idea how stacks, FIFO queues, and other generalized ADTs should operate. Specifically, in this section, the effect of changing the specifications of stacks, FIFO queues, and generalized queues to disallow duplicate items in the data structure will be considered.

For example, a company that maintains a mailing list of customers might want to try to grow the list by performing *insert op*erations from other lists gathered from *many* sources, but would not want the list to grow for an *insert* operation that refers to a customer already on the list. The same principle applies in a variety of other applications. For another example, consider the problem of routing a message through a complex communications network. It might be trials to go through several paths simultaneously in the network, but there is only one message. So any particular node in the network would want to have only one copy in its internal data structures.

One approach to handle this situation is *to leave up* to the programs the task of ensuring that duplicate items are not presented to the ADT. But since the purpose of an ADT is to provide clients with clean solution to application problems, the detection and resolution of duplicates have to be a part of ADT.

Disallowing duplicate items is a change in the *abstraction*:

- the interface, names of operations, and so forth for such an ADT are the same as those for the corresponding original ADT, but the *behavior* of the implementation changes in a fundamental way.

In general, modification of the specification of a structure gives a *completely new* ADT – one that has completely different properties. This situation also demonstrates the precarious nature of ADT specification:

- being sure that clients and implementations adhere to the specifications in an interface is *difficult enough,* but enforcing a high-level statement such as this one is *another matter entirely*.

In general, a generic decision has to be made when a client makes an *insert* request for an item that is already in the data structure:

- should it be proceeded as though the request never happened?

- or should it be proceeded as though the client had performed a *delete* followed by *an insert?*

This decision affects the order in which items are ultimately processed for ADTs such as stacks and FIFO queues, and the distinction is significant for programs. For example, the company using such an ADT for a mailing list might prefer to use the new item (perhaps assuming that it has more up-to-date information about the customer), and the switching mechanism using such an ADT might prefer to ignore the new item (perhaps it has already taken steps to send along the message).

Furthermore, such choice affects the implementations:

- the *forget-the-old-item* statement is generally more difficult to implement than the *ignore-the-new-item* statement, because it requires to modify the data structure.

To implement generalized queues with no duplicate items:

- an abstract operation for testing item equality has to be presented;
- the determination whether a new item to be inserted is already in the data structure has to be available.

There is an important special case with a straightforward solution:

- if the items are integers in the range [0, …, N-1], then a second array of size N, indexed by the item itself, to determine whether that item is in the stack, may be used.

Inserting the item, the i^{th} entry in the second array may be set to 1; and deleting item i, the i^{th} entry in the array may be set to 0. The same code as before may be used to insert and delete items, with one additional test:

the test to see whether the item is already in the structure.

If it is, the insert or delete operation have to be ignored. This solution does not depend on whether an array or linked-list (or some other) representation for the ADT. Implementing an *ignore-the-old-item* case involves more work.

In summary, one way to implement a generalized queue ADT with no duplicates using an *ignore-the-new-item* case is to maintain *two* data structures: the first contains the items in the structure, as before, to keep track of the order in which the items in the queue were inserted; the second is an array that allows us to keep track of which items are in the queue, by using the item as an index. Using an array in this way is a special case of a symbol-table implementation, which is discussed later.

This special case arises frequently. The most important example is when the items in the data structure are array indices themselves, so such items are referred as *index items*. Typically, a set of N objects, kept in yet another array, has to be passed through a generalized queue structure as a part of a more complex algorithm. Objects are put on the queue by index and processed when they are removed, and each object is to be processed precisely once. Using array indices in a queue with no duplicates accomplishes this goal directly.

Each of these choices (disallow duplicates, or do not; and use the new item, or do not) leads to a new ADT The differences may seem minor, but they obviously affect the *dynamic behavior* of the ADT as seen by programs, and affect the choice of algorithm and data structure to implement the various operations, so there is no alternative but to treat all the ADTs as different.

Furthermore, in many cases additional options have to be considered. For example, there might be the wish to modify the interface to inform the client program when it attempts to insert a duplicate item, or to give the client the option whether to ignore the new item or to forget the old one.

To conclude, informally using a term such as *pushdown stack, FIFO queue, deque, priority queue, or symbol table,* it potentially referees to a *family* of ADTs, each with different sets of defined operations and different sets of conventions about the meanings of the operations, each requiring different and, in some cases, more sophisticated implementations to be able to support those operations efficiently.

First class ADT

The objects of built-in data types and in ADTs considered above are disarmingly simple, there is only one object in a given program and no possibility to declare variables of different types in client programs for the same ADT.

A **first-class data type** *is one for which there is potentially many different instances, and which can be assigned to variables declared to hold the instances.*

For example, it could be used first-class data types as arguments and return values to functions.

The implemention of first-class data types have to provide us with the capability to write programs that manipulate stacks and FIFO queues in much the same way as with types of data in programming language like *C*. This capability is important in the study of algorithms because it gives a natural way to express high level operations involving such ADTs. For example, two queues can be joined into one.

Some modern languages provide specific mechanisms for building first-class ADTs. Being able to manipulate instances of ADTs in much the same way that of built-in data types *int* or *float,* it allows any application program to be written such that the program manipulates the objects of central concern to the application; it allows many programmers to work simultaneously on large systems, all using a precisely defined set of abstract operations, and it provides for those abstract operations to be implemented in many different ways without any changes to the applications code - for example for new machines and programming environments. Some languages even allow *operator overloading,* to use basic symbols such as **+** or ***** to define operators.

First-class ADTs play a central role in many of implementations because they provide the necessary support for the abstract mechanisms for generic objects and collections of objects.

Nevertheless, the ability to have multiple instances of a given ADT in a single program can lead to complicated situations:

- Do we want to be able to have stacks or queues with different types of objects on them?
- How about different types of objects on the same queue?
- Do we want to use different implementations for queues of the same type in a single client because we know of performance differences?
- Should information about the efficiency of implementations be included in the interface?
- What form should that information take?

Such questions underscore the importance of understanding the basic characteristics of algorithms and data structures and how client programs may use them effectively.

The List ADT

A list is one of the most fundamental data structures used to store a collection of data items.

The importance of the List ADT is that it can be used to implement a wide variety of other ADTs. That is, the LIST ADT often serves as a basic building block in the construction of more complicated ADTs. A list may be defined as a dynamic ordered n-tuple:

$$L == (l_1, 1_2, \ldots.., l_n)$$

The use of the term ***dynamic*** *in* this definition is meant to emphasize that the elements in this n-tuple may change over time.

Notice that these elements have a linear order based upon their position in list.

- the first element in the list, 1_1, is called the ***head*** of the list.
- the last element, l_n, is referred to as the ***tail*** of the list.
- the number of elements in a list L is referred to as the length of the list.
- the empty list, represented by (), has length 0.
- list can be *homogeneous or heterogeneous.*

In many applications it is also useful to work with lists of lists. In this case, each element of the list is itself a list. For example, consider the list

((3), (4, 2, 5), (12, (8, 4)), ())

The operations we will define for accessing list elements are given below. For each of these operations, *L* represents a specific list. It is also assumed that a list has a current position variable that refers to some element in the list. This variable can be used to iterate through the elements of a list.

1. ***Initialize (L).*** This operation is needed to allocate the amount of memory and to give a structure to this amount.
2. ***Insert (L, x, i).*** If this operation is successful, the boolean value *true* is returned; otherwise, the boolean value *false* is returned.
3. ***Append (L, x).*** Adds element ***x*** to the tail of ***L,*** causing the length of the list to become ***n+1***. If this operation is successful, the boolean value ***true*** is returned; otherwise, the boolean value ***false*** is returned.
4. ***Retrieve (L, i).*** Returns the element stored at position ***i*** of ***L,*** or the null value if position ***i*** does not exist.
5. ***Delete (L, i).*** Deletes the element stored at position ***i*** of ***L,*** causing elements to move in their positions.
6. ***Length (L).*** Returns the length of ***L***.
7. ***Reset (L).*** Resets the current position in ***L*** to the head (i.e., to position ***1***) and returns the value ***1***. If the list is empty, the value ***0*** is returned.
8. ***Current (L).*** Returns the current position in ***L***.
9. ***Next (L).*** Increments and returns the current position in ***L***.

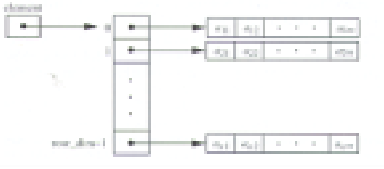
Note that only the ***Insert, Delete, Reset,*** and ***Next*** operations modify the lists to which they are applied. The remaining operations simply query lists in order to obtain information about them.

Sequential Mapping

If all of the elements that comprise a given data structure are stored one after the other in consecutive memory locations, we say that the data structure ***is sequentially mapped*** into computer memory.

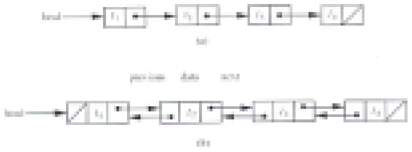
Sequential mapping makes it possible to access any element in the data structure in constant time. Given the starting address of the data structure in memory, we can find the address of any element in the data structure by simply calculating its offset from the starting address.

An array is an example of a sequentially mapped data structure:

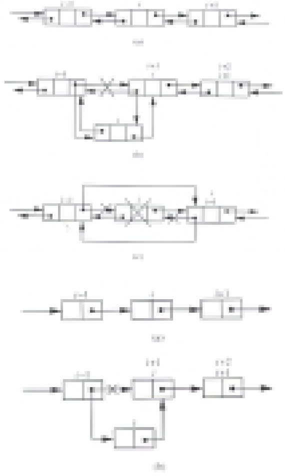


Because it takes the same amount of time to access any element, a sequentially-mapped data structure is also called a ***random access data structure***. That is, the accessing time is independent of the size of data structure, and requires ***O(l)*** time (constant time).

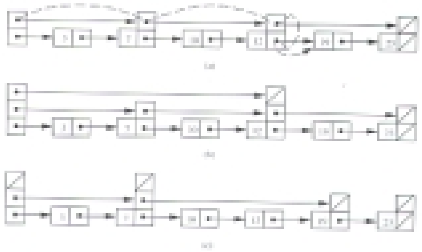
Schematic representations of (a) a singly-linked list and (b) a doubly-linked list:



List operations ***Insert and Delete*** with double-linked list and single-linked list:



The skip lists:



The ***MATRIX*** ADT

The abstract data type ***MATRIX*** is used to represent matrices, as well as the operations defined on matrices. A matrix is defined as a rectangular array of elements arranged by rows and columns. A matrix with *n* rows and *m* columns is said to have row dimension *n*, column dimension *m*, and order *n x m*. An element of a matrix *M* is denoted by $a_{i,j}$. representing the element at row *i* and column *j*.

The example of matrix:

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

Numerous operations are defined on matrices. A few of these are:

1. *InitializeMatrix* (M) – creates a necessary structured space in computer memory to locate matrix.
2. *RetrieveElement* (i, j, M) – returns the element at row i and column j of matrix M.
3. *AssignElement* (i, j, x, M) – assigns the value x to the element at row i and column j of matrix M.
4. *Assignment* (M_1, M_2) – assigns the elements of matrix M_1 to those of matrix M_2. Logical condition: matrices M_1 and M_2 must have the same order.
5. *Addition* (M_1, M_2) – returns the matrix that results when matrix M_1 *is* added to matrix M_2. Logical condition: matrices M_1 and M_2 must have the same order.
6. *Negation* (M) – returns the matrix that results when matrix M is negated.
7. *Subtraction* (M_1, M_2) – returns the matrix that results when matrix M_1 *is* subtracted from matrix M_2. Logical condition: matrices M_1 and M_2 must have the same order.
8. *Scalar Multiplication* (s, M) – returns the matrix that results when matrix M is multiplied by scalar s.
9. *Multiplication* (M_1, M_2) – returns the matrix that results when matrix M_1 *is* multiplied by matrix M_2. The column dimension of M_1 must be the same as the row dimension of M_2. The resultant matrix has the same row dimension as M_1, the same column dimension as M_2.
10. *Transpose(M)* – returns the transpose of matrix M.
11. *Determinant(M)* – returns the determinant of matrix M.
12. *Inverse(M)* – returns the inverse of matrix M.
13. *Kill* (M) – releases an amount of memory occupied by M.

Many programming languages have this data type implemented as ***built-in one***, but usually in some ***restricted*** way. Nevertheless an implementation of this ADT must provide a means for representing matrix elements, and for implementing the operations described. It is highly desirable to treat elements of the matrix in a ***uniform*** way, paying no attention whether elements are ***numbers, long numbers, polynomials***, or other types of data.

The *DYNAMIC SET* ADT

The ***set*** is a fundamental structure in mathematics, and the computer science view to ***a set*** is as follows:

- it groups objects together;
- the objects in a set are called the elements or members of the set;
- these elements are taken from the universal set ***U***, which contains all possible set elements;
- all the members of- a given set are unique.

The number of elements contained in a set ***S*** is referred to as the cardinality of ***S***, denoted by **|*S*|**. It is often referred to a set with cardinality ***n*** as an ***n-set***. The elements of a set are not ordered. Thus, {1, 2, 3} and {3, 2, 1} represent the same set. Mathematical operations with sets:

- an element ***x*** is (or is not) a member of the set ***S***;
- the empty set;
- two sets ***A*** and ***B*** are equal (or not);
- an ***A*** is said to be a subset of ***B*** (the empty set is a subset of every set);
- the union of ***A*** and **B**;
- the intersection of ***A*** and ***B***;
- the difference of ***A*** and ***B***;
- the Cartesian product of two sets.

In computer science it is often useful to consider set-like structures in which the ordering of elements is important; such sets are referred as an ordered n-tuple, like (a_1, a_2, … a_n).

The concept of a set serves as the basis for a wide variety of useful abstract data types. A large number of computer applications involve the manipulation of sets of data elements. Thus, it makes sense to investigate data structures and algorithms that support efficient implementation of various operations on sets.

Another important difference between the mathematical concept of a set and the sets considered in computer science:

- a set in mathematics is unchanging, while the sets in computer science are considered to change over time as data elements are added or deleted.

Thus, sets are referred here as ***dynamic sets***. In addition, it is assumed each element in a dynamic set contains an identifying field called a key, and that ***a total ordering relationship*** exists on these keys. It is often assumed *no two elements of a dynamic set contain the same key*.

If the *dynamic set* ADT is implemented properly, application programmers will be able to use dynamic sets without having to understand their implementation details. The use of ADTs in this manner simplifies design and development, and promotes reusability of software components.

A list of general operations for the ***dynamic set*** ADT (***S*** represents a specific dynamic set):

1. ***Search(S, k)***. Returns the element with key ***k*** in ***S***, or the ***null*** value if an element with key ***k*** is not in ***S***.
2. ***Insert(S, x)***. Adds element ***x*** to ***S***. If this operation is successful, the boolean value ***true*** is returned; otherwise, the boolean value ***false*** is returned.
3. ***Delete(S, k)***. Removes the element with key ***k*** in ***S***. If this operation is successful, the boolean value ***true*** is returned; otherwise, the boolean value ***false*** is returned.
4. ***Minimum(S)***. Returns the element in dynamic set ***S*** that has the smallest key value, or the ***null*** value if ***S*** is empty.
5. ***Maximum(S)***. Returns the element in ***S*** that has the largest key value, or the ***null*** value if ***S*** is empty.
6. ***Predecessor(S, k)***. Returns the element in ***S*** that has the largest key value less than ***k***, or the ***null*** value if no such element exists.
7. ***Successor(S, k)***. Returns the element in S that has the smallest key value greater than ***k***, or the ***null*** value if no such element exists.

In addition, when considering the ***dynamic set*** ADT (or any modifications of this ADT) the following operations are available:

8. ***Empty(S)***. Returns a boolean value, with ***true*** indicating that ***S*** is an empty dynamic set, and ***false*** indicating that ***S*** is not.
9. ***MakeEmpty(S)***. Clears ***S*** of all elements, causing ***S*** to become an empty dynamic set.

Since these last two operations are often trivial to implement, they generally are omitted. In many instances an application will only require the use of ***a few dynamic set*** operations. Some groups of these operations are used so frequently that they are given special names:

- the ADT that supports ***Search, Insert,*** and ***Delete*** operations is called the ***dictionary*** ADT;
- the ***stack***, ***queue***, and ***priority queue*** ADTs are all special types of dynamic sets.

A variety of data structures will be described in forthcoming considerations that they can be used to implement either the ***dynamic set*** ADT, or ADTs that support specific subsets of the ***dynamic set*** ADT operations.

Each of the data structures described will be analyzed in order to determine ***how efficiently*** they support the implementation of these operations. In each case, the analysis is performed in dependence of ***n***, the number of data elements stored in the dynamic set. This analysis demonstrates - there is ***no optimal data structure*** for implementing dynamic sets. Rather, the best implementation choice will depend upon:

- which operations need to be supported,
- the frequency with which specific operations are used,
- and possibly many other factors.

As always, ***the more we know*** about how a specific application will use data, the better we can ***fine tune*** the associated data structures so that this data can be accessed efficiently.

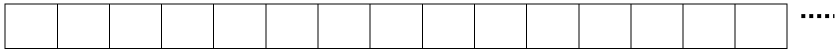
Models of Memory

Organizing data in computer memory, three types of memory can be considered:

- primary (RAM)
- secondary (HDD)
- ternary (CD-ROM, Video-tape, magnetic tape like streamer, etc.)

Primary memory

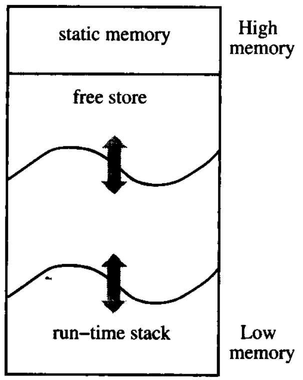
Primary memory is ***a homogeneous*** one, which means the ***accessible amount*** of memory is a smallest one (byte) and ***the access time*** to any element of memory is a constant, denoted by ***O(1)*** and measured ***in nanoseconds***:



While working with data structures, however, primary memory is managed in a ***complicated*** way too. First, let us make a distinction between ***static*** and ***dynamic*** data structures.

A data structure is said to be ***static*** if a ***fixed*** amount of memory is allocated for that data structure ***before*** program execution (i.e., at compile time), and this amount of memory does not change during program execution.

A ***dynamic*** data structure requires to allocate an amount of memory as it is needed ***during*** program execution. The allocation procedure is referred to as ***dynamic memory allocation.*** With ***dynamic data structure*** the amount of memory that can be used by ADT is not fixed at compile time.



The management of the primary memory is presented as the model above. It is useful to view this model as ***a one-dimensional array*** of storage locations (or bytes) that is divided into three parts. Variables that will persist in memory throughout the execution of the program are allocated in ***static memory.*** The amount of storage allocated to static memory is determined at compile time, and this amount does not change during program execution.

A ***run-time stack is*** maintained by the computer system in low memory. The amount of storage that the run-time stack uses will vary during program execution.

The arrows emanating from the run-time stack indicate that the runtime stack "grows" toward high memory and "shrinks" toward low memory. Each time a procedure is called in a program, an ***activation record*** is created and stored in computer memory on the run-time stack.

This activation record contains storage for all variables declared in the procedure, as well as either a copy of, or a reference to, the parameters that are being passed to the procedure. In addition, an activation record must contain some information that specifies where program execution will resume when the procedure is completed. At the completion of the procedure, the associated activation record will be removed from the run-time stack, and program control will return to the point specified in the activation record.

Finally, the logical model shows ***a free store*** that "grows" toward low memory and "shrinks" toward high memory. Memory that is allocated at run time (i.e., dynamically) is stored on the free store.

Note that the run-time stack and the free store "grow" toward each other in this model. Thus, an obvious ***error situation*** occurs if either too much memory is dynamically allocated without reclaiming it, or too many activation records are created.

While memory allocation and deallocation on the run-time stack are controlled by the ***computer system itself***, such may not be the case for the free store.

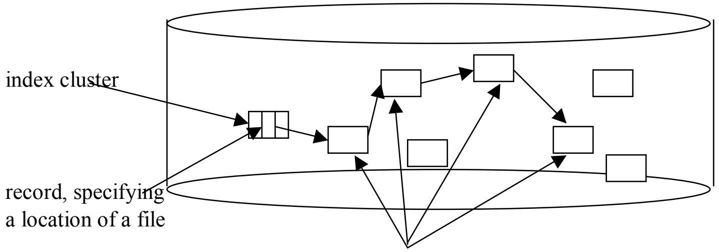
In many programming languages, the responsibility of reclaiming dynamically allocated memory is ***the programmer's***. If the memory is not reclaimed by the programmer, then it will remain on the free store.

This approach to reclaiming dynamically allocated memory is referred to as the ***explicit*** approach. On the other hand, in an ***implicit*** approach to the deallocation of dynamic memory, memory management functions provided by the system are responsible for reclaiming memory as it is no longer needed. The implicit approach is usually called ***garbage collection***.

Secondary memory

This memory also is referred to as a disc-memory (HDD). It consists of units of equal size (so-called pages, or clusters). Data to be stored in this memory has to be organized into files. Clusters are much bigger in size comparing to bytes. Usually a few clusters are allocated to one file.

Also there is an index stored in a specific cluster (as for disc-memory, it's so-called zero-track) and having all files on that disc listed. The read operation (as well as write one) is dealing with clusters, and transfer the whole information of cluster to the main memory.



clusters allocated to a specific file

Furthermore, the access time for main memory is typically orders of magnitude faster than the access time for secondary storage. For this reason, it is preferable to implement data structures in main memory- referred to these as ***internal data structures.***

In general, secondary storage will be used only if a data structure is too large to fit in main memory. Data structures that reside in secondary storage are referred to as ***external data structures.***

Ternary memory

Data, especially multimedia ones, need high capacity. Magnetic disks are too small (20 – 80 GB), and above all they are not changeable. They are useful as work area. The ternary mamory name refers to magnetic tapes, CDs, optical disks, magneto-optic disks, magnetic tapes, video tapes, etc. The specific features of such memory include access time (could last even 1 minute), size of clusters (which are huge), clusters are of different size, etc.

Dependences between capacity, access times and bit costs

Processor	**capacity**	**cost (cents/bit)**	**access**
buffer(cache)	256, 512 KB	≈ 1	≈ 20 ns
main memory	≈ 128 – 1024 MB	≈ 0.1	≈ 1 µs
Disk	≈ 40 GB	$\approx 10^{-4}$	≈ 20 ms
bulk memory	≈ 1000 GB	$\approx 10^{-5}$	≈ 10 s

The multimedia data include: text, images, graphics, sound recordings, video recordings, signals, etc., that are digitalized and stored, their properties can be compared:

Medium	Elements	Configuration	Typical size	Time dependent	Sense
Text	printable characters	sequence	10 KB (5 pages)	no	visual/ acoustic
Graphic	vectors, regions	set	10 KB	no	visual
raster image	pixels	matrix	1 MB	no	visual
Audio	sound/ volume	sequence	600 MB (audio CD)	yes	acoustic
video-clip	raster image/ graphics	sequence	2 GB (30 min.)	yes	visual

The Memory Hierarchy

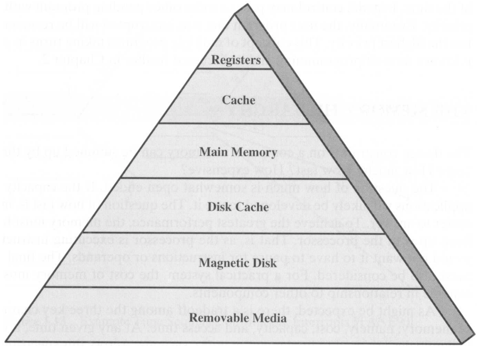
As might be expected, there is a tradeoff among the three key characteristics of memory: namely, cost, capacity, and access time. At any given time, a variety of technologies are used to implement memory systems. Across this spectrum of technologies, the following relationships hold:

- smaller access time, greater cost per bit
- greater capacity, smaller cost per bit
- greater capacity, greater access time.

The way out of this dilemma is not to rely on a single memory component or technology, but to employ a **memory hierarchy.** A typical hierarchy is illustrated below. As one goes down the hierarchy, the following occur:

- decreasing cost per bit
- increasing capacity
- increasing access time
- decreasing frequency of access of the memory by the processor.

Thus, smaller, more expensive, faster memories are supplemented by larger, cheaper, slower memories. The key to the success of this organization is the last item: decreasing frequency of access.



External Memory Algorithms

Data sets in large applications are often too massive to fit completely inside the computer's internal memory. The resulting input/output communication (or I/O) between fast internal memory and slower external memory (such as disks) can be a major performance bottleneck.

The design and analysis of external memory algorithms (also known as EM algorithms or out-of-core algorithms or I/O algorithms). External memory algorithms are often designed using the parallel disk model (PDM). The three machine independent measures of an algorithm's performance in PDM are:

- the number of I/O operations performed,
- the CPU time,
- the amount of disk space used.

PDM allows for multiple disks (or disk arrays) and parallel CPUs, and it can be generalized to handle cache hierarchies, hierarchical memory, and tertiary storage.

Experiments on some newly developed algorithms for spatial databases incorporating these paradigms, implemented using TPIE (Transparent Parallel I/O programming Environment), show significant speedups over currently used methods.

For reasons of economy, general-purpose computer systems usually contain a hierarchy of memory levels, each level with its own cost and performance characteristics. At the smallest scale, CPU registers and caches are built with the fastest but most expensive memory. For internal main memory, dynamic random access memory (DRAM) is typical. At a larger scale, inexpensive but slower magnetic disks are used for external mass storage, and even slower but larger capacity devices such as tapes and optical disks are used for archival storage.

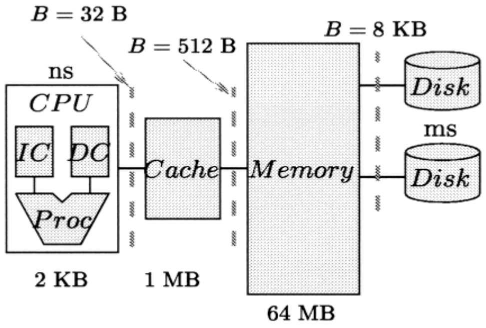


Figure depicts an example of memory hierarchy and its characteristics. The memory hierarchy of a uniprocessor, consisting of registers, data cache, level 2 cache, internal memory, and disk. The B parameter denotes the block transfer size between two adjacent levels of the hierarchy. The size of each memory level is indicated at the bottom.

Most modern programming languages are based upon a programming model in which memory consists of one uniform address space. The notion of virtual memory allows the address space to be far larger than what can fit in the internal memory of the computer. A natural tendency for programmers is to assume that all memory references require the same access time. In many cases, such an assumption is reasonable (or at least doesn't do any harm), especially when the datasets are not large. The utility and elegance of this programming model are to a large extent the reason why it has flourished in the software industry.

However, not all memory references are created equal. Large address spaces span multiple levels of memory hierarchy, and accessing the data in the lowest levels of memory is orders of magnitude faster than accessing the data at the higher levels. For example, loading a register takes on the order of a nanosecond (10^{-9} seconds) whereas the latency of accessing data from a disk is several milliseconds (10^{-3} seconds). The relative difference in access time is more than ***a million***. The Input/Output communication (or simply I/O) between levels of memory is often the bottleneck in applications that process massive amounts of data.

Many computer programs exhibit some degree of locality in their pattern of memory references: Certain data are referenced repeatedly for a while, and then the program shifts attention to other sets of data. Modern operating systems can take advantage of such access patterns by tracking the program's so-called "***working set***", which a vague notion that designates the data items that are being referenced repeatedly.

If the working set is small, it can be cached in very high-speed memory so that access to it is fast. Caching and prefetching heuristics have been developed to reduce the number of occurrences of a ``fault," in which the referenced data item is not in cache and must be retrieved by an I/O from disk.

By their nature, caching and prefetching methods are general-purpose ones, and thus they cannot be expected in all cases to take full advantage of the locality present in a computation.

Some computations themselves are inherently non-local, and even with omniscient cache management decisions they are doomed to perform large amounts of I/O and suffer poor performance. Substantial gains in performance may be possible by incorporating locality directly into the algorithm design and by explicit management of the contents of each level of the memory hierarchy.

Algorithms that explicitly manage data placement and movement are known as ***external memory algorithms***, or more simply EM algorithms. Sometimes the terms ***out-of-core algorithms*** and ***I/O algorithms*** are used.