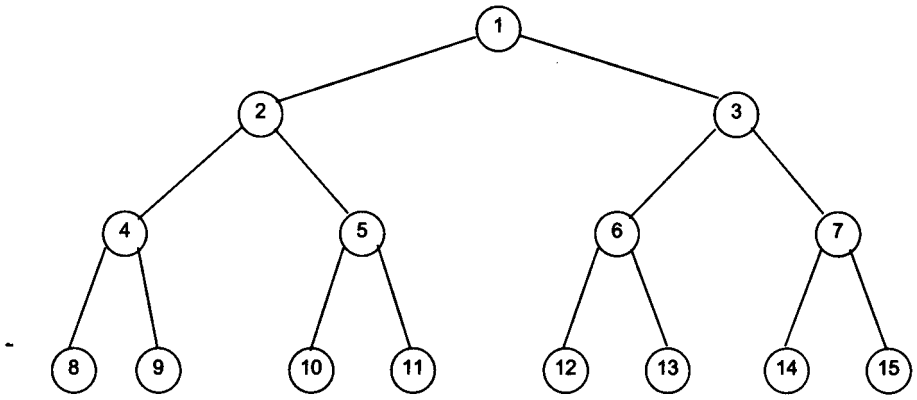
Heaps, Heapsort and Priority Queues

The application of a purely binary tree is a data structure called a *heap.*

Heaps are unusual in the menagerie of tree structures in that they represent trees as arrays rather than linked structures using pointers:



Numbering of a tree's nodes for storage in a array

If we use the number of a node as an array Index, this technique gives us an order in which we can store tree nodes In an array. The tree may be easily reconstructed from the array:

the left child of node number ***k*** has index ***2k*** and the right child has index ***2k + 1***.

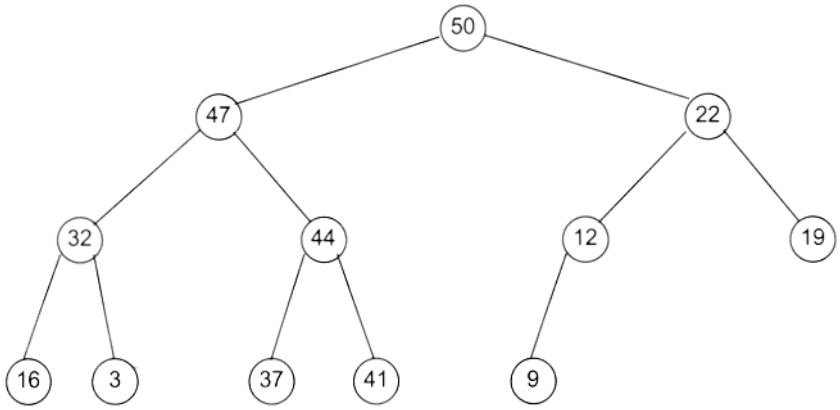
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Any binary tree can be represented in an array in this fashion. This is not an efficient way to store just any binary tree, there would be many array elements that are left empty. ***A heap***, however, is a special kind of binary tree that leaves no gaps in an array implementation:

All leaves are on two adjacent levels.
All leaves on the lowest level occur at the left of the tree.
All levels above the lowest are completely filled.
Both children of any node are again heaps.
The value stored at any node is at least as large as the values in its two children.

The first three conditions ensure that the array representation of the heap will have ***no gaps*** in it. The last two conditions give a heap ***a weak amount*** of order.

The order in which the elements are stored in the array will, in general, not be a sorted representation of the list, although the largest element will be stored at location 1.



In this figure, note that the largest element (50) is the root node, and that the value stored in each node is larger than both of its children. The leaves of the heap occur on two adjacent levels (the third and fourth), and the nodes on the lowest layer fill that layer from the left. We could remove node 9, for example, and still preserve the heap property, but if node 9 remains where it is, we cannot remove any other node in the bottom layer without violating heap property 2 in the list above.

The elements of this heap would be stored in an array in the order:

50	47	22	32	44	12	19	16	3	37	41	9

The most ***useful property*** of a heap is that the largest node in the tree is always at the root.

If the root is extracted from the heap, there is a simple algorithm which can be used to restore the heap condition in the remaining nodes, so that the largest of the remaining nodes is again at the root position.

The most common application of a heap is as ***a priority queue***.

A ***priority queue*** is an ordinary queue, except that each item in the queue has an associated priority. Items with higher priority will ***get processed before*** items with lower priority, even if they arrive in the queue after them.

Priority Queues

In many applications, records with keys must be processed in order, but not necessarily in full sorted order and not necessarily all at once. Often a set of records must be collected, then the largest processed, then perhaps more records collected, then the next largest processed, and so forth. An appropriate data structure in such an environment is one which supports the operations of *inserting* a new element and

deleting the largest element. Such a data structure, which can be contrasted with queues (delete the oldest) and stacks (delete the newest) is called a priority queue.

In fact, the priority queue might be thought of as a generalization of the stack and the queue (and other simple data structures), since these data structures can be implemented with priority queues, using appropriate priority assignments.

Applications of priority queues include simulation systems (where the keys might correspond to "event times" which must be processed in order), job scheduling in computer systems (where the keys might correspond to "priorities" indicating which users should be processed first), and numerical computations (where the keys might be computational errors, so the largest can be worked on first).

It is useful to be somewhat more precise about how to manipulate a priority queue, since there are several operations we may need to perform on priority queues in order to maintain them and use them effectively for applications such as those mentioned above. Indeed, the main reason that priority queues are so useful is their flexibility in allowing a variety of different operations to be efficiently performed on sets of records with keys. We want to build and maintain a data structure containing records with numerical keys *(priorities)* and supporting some of the following operations:

Construct a priority queue from N given items.
Insert a new item.
Remove the largest item.
Replace the largest item with a new item (unless the new item is larger).
Change the priority of an item.
Delete an arbitrary specified item.
Join two priority queues into one large one.

(If records can have duplicate keys, we take "largest" to mean "any record with the largest key value.")

Different implementations of priority queues involve different performance characteristics for the various operations to be performed, leading to cost tradeoffs. Indeed, performance differences are really the only differences that can arise in the abstract data structure concept.

First, we'll illustrate this point by discussing a few elementary data structures for implementing priority queues. Next, we'll examine a more advanced data structure and then show how the various operations can be implemented efficiently using this data structure. We'll then look at an important sorting algorithm that follows naturally from these implementations.

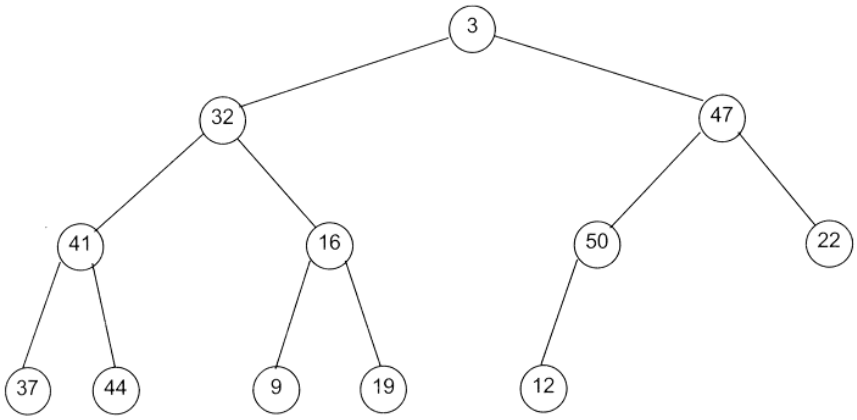
Any priority queue algorithm can be turned into a sorting algorithm by repeatedly using *insert* to build a priority queue containing all the items to be sorted, then repeatedly using *remove* to empty the priority queue, receiving the items in reverse order. Using a priority queue represented as an unordered list in this way corresponds to selection sort; using the ordered list corresponds to insertion sort.

Although heaps are ***fairly sloppy*** in keeping their members in strict order, they are very good at ***rinding the maximum member*** and bringing it to the top of the heap.

Constructing a Heap

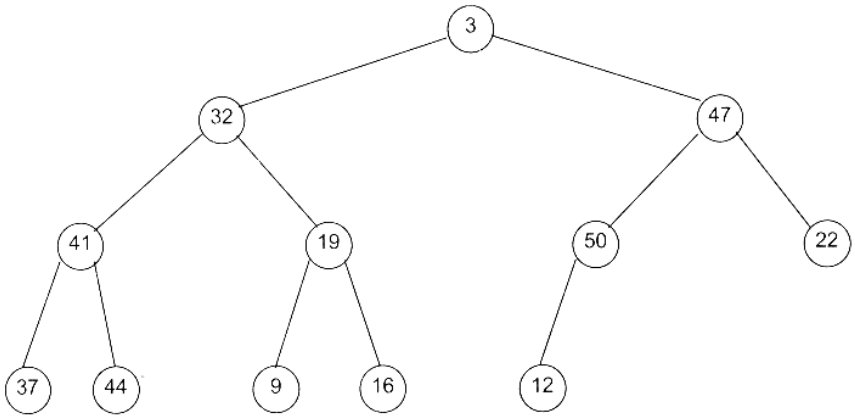
The first step is to organize initial data so that it has the heap properties. To start let's begin with a binary tree by inserting some integers into a binary tree in random order:

The method of constructing a heap begins by considering ***the last node that is not a leaf***. These numbers are ***stored in an array*** by reading across each layer in the tree from left to right, so the last non-leaf node is 50.

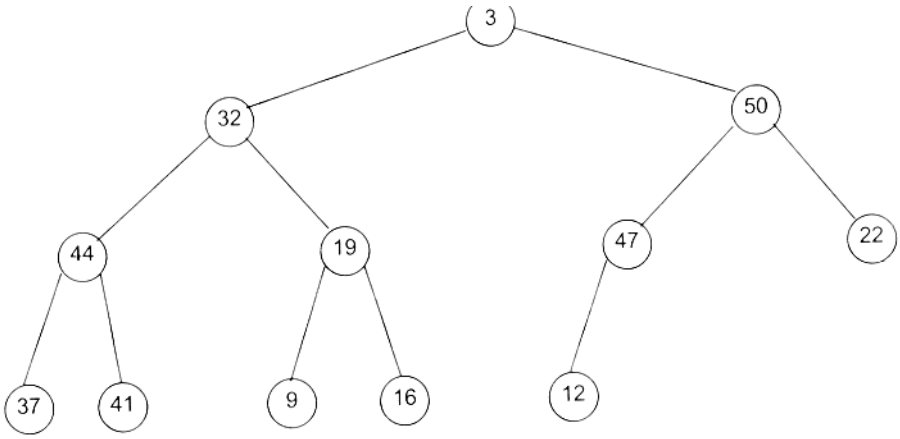


Compare this node to its children. If ***the node is larger*** than its children then it satisfies that heap condition – ***no action*** required.

Move backwards through the tree, examining each node in turn. Looking at node 16, we see that it is larger than 9, but smaller than 19, so it does not satisfy the heap condition. To correct this, we ***swap the node with the larger*** of its two children, to obtain the tree shown:

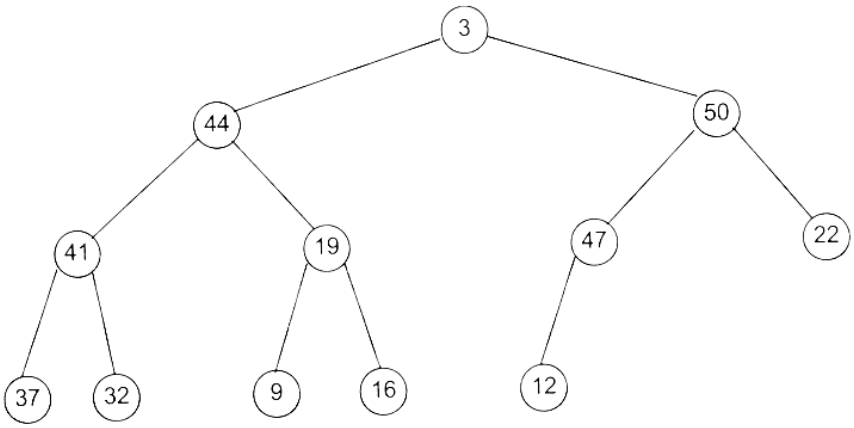


Proceeding backward, we must swap 41 and 44. Then consider 47, we must swap with 50, giving the result:

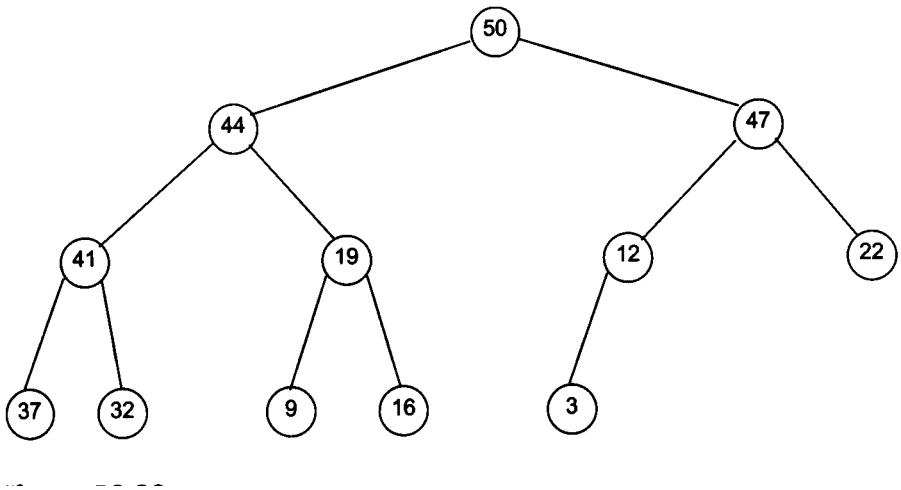


Also check that 47 is acceptable in its new position by comparing it with its children at that location.

Proceeding back to 32, we must swap it with 44. Having done, two children of 32 are 37 and 41, both of which are larger than 32. Tehrefore we must swap 32 with the larger of its children (41), giving the result:



Finally we must consider node 3. It is swapped with the larger of its two children (50), then with 47, and finally with 12, giving the final, properly constructed heap:



Heapsort and Priority Queues

Having created the heap, it can be ***used for a variety*** of applications. The main use for the heap is as ***a priority queue***, in which the largest element is extracted from the root node (and used externally), with the remaining elements being rearranged to form a smaller heap.

A variation on the priority queues provides with another sorting method – ***a heapsort***.

Heapsort ***makes use*** of the fact that a heap is stored in an array. The heap above forms the array (and the size of the heap is 12):

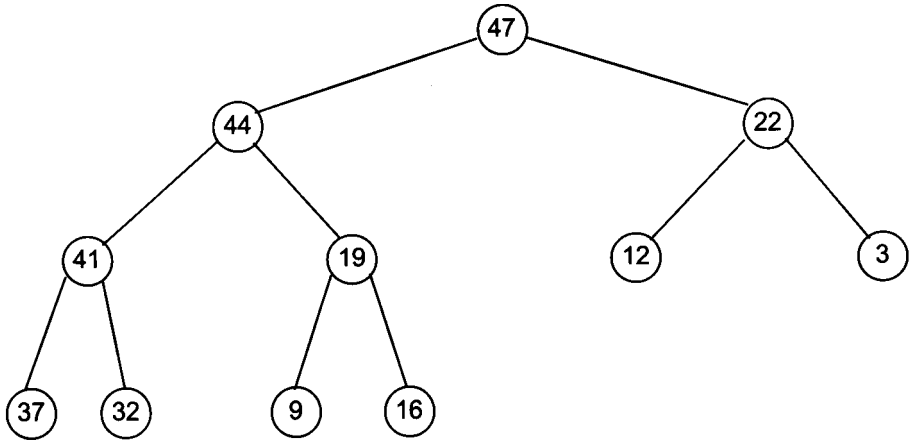
50	44	47	41	19	12	22	37	32	9	16	3

The steps for the algorithm of the heap are:

- remove the root;
- insert the last number of the array in the place of root and make heap condition to be true;
- repeat then for the next root.

Really ***in the array we have to swap*** the root of the heap (the first element of the array, it means 50) with the last element of the heap (it means 3).

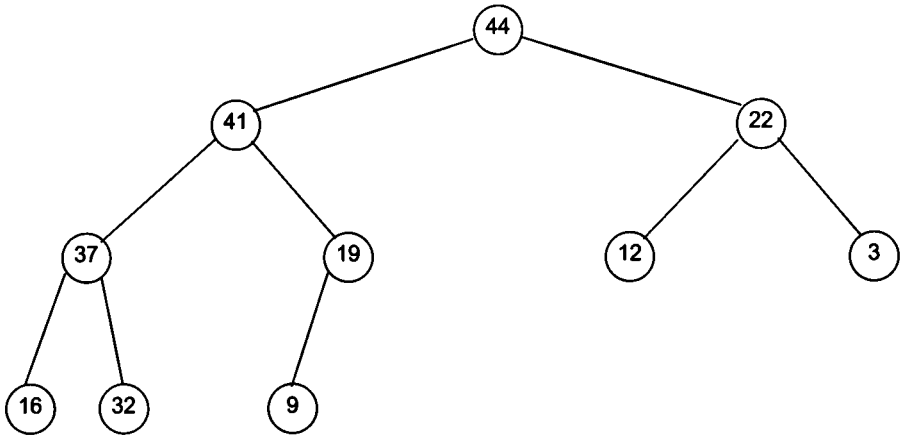
After swaping corresponding array elements, removing (and printing) from the heap the number (50), inserting (3) in the root and moving it down to fulfill the heap condition, we have the heap:



'I'he contents of the array are now (and the heap size is 11):

47	44	22	41	19	12	3	37	32	9	16	50

If we repeat the process with a new root (47), we obtain the heap shown:



'I'he last two numbers in the array are no longer part of the heap, and contain the two largest numbers in sorted order. The array elements are now (with a heap size of 10):

44	41	22	37	19	12	3	16	32	9	47	50

The process continues in the same manner until the heap size has been ***reduced to zero***, at which point the array contains the numbers in sorted order.

If a heap is used purely for sorting, the heapsort algorithm turns out to be an ***O(n log n)*** algorithm.

Although the heapsort is ***only about half*** as efficient as quicksort or mergesort for randomly ordered initial lists.

The same technique can be used to implement a priority queue. Rather than carry the sorting process through to the bitter end, the root node can be extracted (which is the item with highest priority) and rearrange the tree, so that the remaining nodes form a heap with one less element. We need not process the data any further until the next request comes in for an item from the priority queue.

A heap is an efficient way of implementing a priority queue since the maximum number of nodes that need to be considered to restore the heap property when the root is extracted is about twice the depth of the tree. The number of comparisons is around ***2 log N***

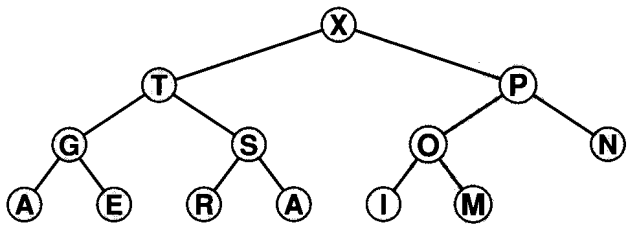
Algorithms on Heaps

The priority queue algorithms on heaps all work by first making a simple structural modification which could violate the heap condition, then traveling through the heap modifying it to ensure that the heap condition is satisfied everywhere.

Some of the algorithms travel through the heap from bottom to top, others from top to bottom. In all of the algorithms,,we'll assume that the records are one-word integer keys stored in an array a of some maximum size, with the current size of the heap kept in an integer N. Note that N is as much a part of the definition of the heap as the keys and records themselves.

To be able to build a heap, it is necessary first to implement the *insert* operation. Since this operation will increase the size of the heap by one, N must be incremented. Then the record to be inserted is put into a *[N],* but this may violate the heap property. If the heap property is violated (the new node is greater than its parent), then the violation can be fixed by exchanging the new node with its parent. This may, in turn, cause a violation, and thus can be fixed in the same way.

For example, if P is to be inserted in the heap above, it is first stored in a *[N]* as the right child of M. Then, since it is greater than M, it is exchanged with M, and since it is greater than 0, it is exchanged with 0, and the process terminates since it is less that X. The heap shown in figure below results.



Inserting a new element (P) into a heap.

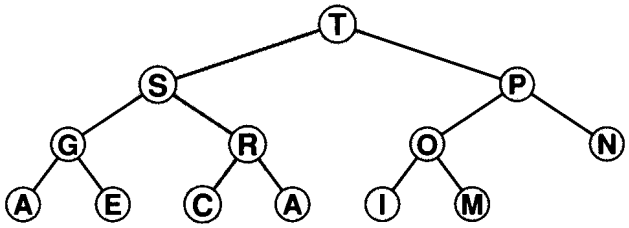
The code for this method is straightforward. In the following implementation, *insert* adds a new item to a *[N],* then calls *upheap (N)* to fix the heap condition violation at *N:*

```
procedure upheap(k: integer);
   var v: integer;
   begin
   v:=a[k]; a[O]:=maxint;
   while a [k div 2 1 < =v do
   begin a [k ]:=a [k div 2 1; k:=k div 2 end;
   a [k ]:=v
   end;
procedure insert(v: integer);
   begin
   N:=N+l; a[N]:=v;
   upheap (N)
   end;
```

If k **div** 2 were replaced by k-1 everywhere in this program, we would have in essence one step of ***insertion sort*** (implementing a priority queue with an ordered list); here, instead, we are "inserting" the new key along the path from *N* to the root. As with insertion sort, it is not necessary to do a full exchange within the loop, because v is always involved in the exchanges.

The *replace* operation involves replacing the key at the root with a new key, then moving down the heap from top to bottom to restore the heap condition. For example, if the X in the heap above is to be replaced with C, the first step is to store C at the root. This violates the heap condition, but the violation can be fixed by exchanging C with T, the larger of the two children of the root.

This creates a violation at the next level, which can again be fixed by exchanging C with the larger of its two children (in this case S). The process continues until the heap condition is no longer violated at the node occupied by C. In the example, C makes it all the way to the bottom of the heap, leaving the heap depicted in figure next:



Replacing the largest key in a heap (with C)

The "***remove the largest***" operation involves almost the same process. Since the heap will be one element smaller after the operation, it is necessary to decrement N, leaving no place for the element that was stored in the last position. But the largest element (which is in a [1]) is to be removed, so the remove operation amounts to *a replace,* using the element that was in a *[N].* The heap shown in figure is the result of removing the T from the heap in figure preceeding by replacing it with the M, then moving down, promoting the larger of the two children, until reaching a node with both children smaller than M.

The implementation of both of these operations is centered around the process of fixing up a heap which satisfies the heap condition everywhere except possibly at the root. If the key at the root is too small, it must be moved down the heap without violating the heap property at any of the nodes touched. It turns out that the same operation can be used to fix up the heap after the value in any position is lowered. It may be implemented as follows:

```
   label 0;
   var i.i. v: integer;
   begin
   v:=a [k
   while k<=N div 2 do
     begin
     j:=k+k;
     if j<N then if a Ul <a U+l if v > =a U ] then goto 0; a[kl:=aU]; k:=j; end;
0: a[k]:=v
   end;
```

```
procedure downheap(k: integer);
label 0;
var i,j. v: integer;
begin
v: =a [k 1;
  while k<=N div 2 do
     begin
     j:=k+k;
     if j<N then if a Ul <a U+l I then j:=j+l;
     if v >=a [ j ] then goto 0;
     a[kl:=aU]; k:=j;
     end;
0: a[k]:=v
   end;
```

This procedure moves down the heap, exchanging the node at position k with the larger of its two children if necessary and stopping when the node at k is larger than both children or the bottom is reached. (Note that it is possible for the node at *k* to

have only one child: this case must be treated properly!) As above, a full exchange is not needed because v is always involved in the exchanges. The inner loop in this program is an example of a loop which really has two distinct exits: one for the case that the bottom of the heap is hit (as in the first example above), and another for the case that the heap condition is satisfied somewhere in the interior of the heap. The goto could be avoided, with some work, and at some expense in clarity.

Now the implementation of the *remove* operation is a direct application of this procedure:

```
function remove: integer;
   begin
   remove:=a [1
   a[l]:=a[N]; N:=N-1;
   downheap (1);
   end;
```

The return value is set from a [I] and then the element from a *[N]* is put into a [1] and the size of the heap decremented, leaving only a call to *downheap* to fix up the heap condition everywhere.

The implementation of the *replace* operation is only slightly more complicated:

```
function replace(v: integer):integer;
        begin
      a[0]:=v;
downheap (0);
replace: =a [0]
end;
```

This code uses a [0] in an artificial way: its children are 0 (itself) and 1, so if *v* is larger than the largest element in the heap, the heap is not touched; otherwise v is put into the heap and a [I] is returned.

The *delete* operation for an arbitrary element from the heap and the *change* operation can also be implemented by using a simple combination of the methods above. For example, if the priority of the element at position k is raised, then *upheap(k)* can be called, and if it is lowered then *downheap(k)* does the job.

Property 1 *All of the basic operations insert, remove, replace, (downheap and upheap), delete, and change require less than 2 Ig N comparisons when performed on a heap of N elements.*

All these operations involve moving along a path between the root and the bottom of the heap, which includes no more than IgN elements for a heap of size *N*. The

factor of two comes from *downheap,* which makes two comparisons in its inner loop; the other operations require only IgN comparisons.

Note carefully that thejoin operation is not included on this list. Doing this operation efficiently seems to require a much more sophisticated data structure. On the other hand, in many applications, one would expect this operation to be required much less frequently than the others.

Indirect Heaps

For many applications of priority queues, we don't want the records moved around at all. Instead, we want the priority queue routine not to return values but to tell us *which* of the records is the largest, etc. This is akin to the "indirect sort" or the "pointer sort". Modifying the above programs to work in this way is straightforward, though sometimes confusing. It will be worthwhile to examine this in more detail here because it is so convenient to use heaps in this way.

Instead of rearranging the keys in the array a the priority queue routines will work with an array p of indices into the array a, such that a *[p [k]*] is the record corresponding to the k^{th} element of the heap, for k between I and *N.* Moreover, we want to maintain another array q which keeps the heap position of the kth array element. This is the mechanism that we use to allow the *(-hange and delete* operations. Thus the q entry for the largest element in the array is 1, and so on.

We start with *p [k]=q [k I=k* for k from I to N, which indicates that no rearrangement has been done. The code for heap construction looks much the same as before:

```
procedure pqconstruct;
  var k: integer;
  begin
  N:=M;
  for k:=l to N do
     begin p [k ]:=k; q [k ]:=k end;
  for k: =M div 2 downto 1 do pqdownheap (k);
  end;
```

(We'll prefix implementations of priority-queue routines based on indirect heaps with "pq" for identification when they are used in later chapters.)

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a[k]	A	S	O	R	T	I	N	G	E	X	A	M	P	L	E
p[k]	10	5	13	4	2	3	7	8	9	1	11	12	6	14	15
a[p[k]]	X	T	P	R	S	O	N	G	E	A	A	M	I	L	E
q[k]	10	5	6	4	2	13	7	8	9	1	11	12	3	14	15

Indirectheap data structures

Now, to modify *downheap* to work indirectly, we need only examine the places where it references a. Where it did a *(-omparison* before, it must now access a indirectly through p. Where it did a *move* before, it must now make the move in p, not a, and it must modify q accordingly. This leads to the following implementation:

```
 procedure pqdownheap (k: integer); label 0;
   var j, v: integer;
   begin
   v:=p [k
   while k<= N div 2 do
     begin
     j:=k+k;
        if j<N then if a [p Ul I <a [p [j+1]] then j:=j+l;
     if a[v]>=a[p[j]] then goto 0;
     p[k]:=pU]; q[pU]]:=k; k:=j;
     end;
0: p [k ]:=v; q [v ]:=k
end;
```

The other procedures given above can be modified in a similar fashion to implement *pqinsert," "pqchange," etc.*

A similar indirect implementation can be developed based on maintaining p as an array of pointers to separately allocated records. In this case, a little more work is required to implement the function of q (find the heap position, given the record).

Advanced Implementations

If thejoin operation must be done efficiently, then the implementations that we have done so far are insufficient and more advanced techniques are needed. Although we don't have space here to go into the details of such methods, we can discuss some of the considerations that go into their design.

By "efficiently," we mean that ajoin should be done in about the same time as the other operations. This immediately rules out the linkless representation for heaps that we have been using, since two large heaps can be joined only by moving all the elements in at least one of them to a large array. It is easy to translate the algorithms we have been examining to use linked representations; in fact, sometimes there are other reasons for doing so (for example, it might be inconvenient to have a large contiguous array). In a direct linked representation, links would have to be kept in each node pointing to the parent and both children.

It turns out that the heap condition itself seems to be too strong to allow efficient implementation of thejoin operation. The advanced data structures designed to solve this problem all weaken either the heap or the balance condition in order to gain the flexibility needed for thejoin. These structures allow all the operations to be completed in logarithmic time.

B-Trees

Multiway Search Trees

The trees that we have considered so far have all been binary (almost) trees: each node can have no more than two children. Since ***the main factor*** in determining the efficiency of a tree is ***its depth***, it is natural to ask if the efficiency ***can be improved by allowing nodes*** to have more than two children. It is fairly obvious that if we allow more than one item to be stored at each node of a tree, the depth of the tree will be less.

In general, however, any attempt to increase the efficiency of a tree by allowing more data to be stored in each node is compriomised by the extra work required to locate an item within the node. To be sure that we are getting any benefit out of ***a multiway search tree***, we should do some ***calculations or simulations for typical data sets*** and compare the results with an ordinary binary tree.

But for the data stored on other media, in external memory, the access time is many times slower than for primary memory. We would like some form of data storage that minimizes the number of times such accesses must be made.

We therefore want a way of searching through the data on disk while satisfying two conditions:

- the amount of data read by each disk access should be close to the page (block, cluster) size;
- the number of disk accesses should be minimized.

If we use a binary tree to store the data on disk, we must access each item of data separately since we do not know in which direction we should branch until we have compared the node's value with the item for which we are searching. This is inefficient in terms of ***disk accesses***.

The ***main solution*** to this problem is to use ***a multiway search tree*** in which to store the data, where the maximum amount of data that can be stored at each node is close to (but does not exceed) the block size for a single disk read operation. We can then load a single node (containing many data items) into RAM and process this data entirely in RAM.

Although there will be some overhead in the sorting and searching operations required to insert and search for data within each node, all of these operations are done exclusively in RAM and are therefore much faster than accessing the hard disk or other external medium.

The B -Tree: a Balanced Multiway Search Tree

It is customary to classify a multiway tree by the maximum number of ***branches*** at each node, rather than the maximum number of items which may be stored at each node. If we use a multiway search tree with ***M*** possible branches at each node, then we can store up to ***M - 1*** data items at each node.
Since multiway trees are primarily used in databases, the data that are stored are usually of a fairly ***complex type***.

In each node of a multiway search tree, we may have up to ***M - 1*** keys labeled

To get the greatest efficiency gain out of a multiway search tree, we need to ensure that most of the nodes contain as much data as possible, and that the tree is as balanced as possible. There are several algorithms which approach this problem from various angles, but the most popular method is the ***B-tree***:

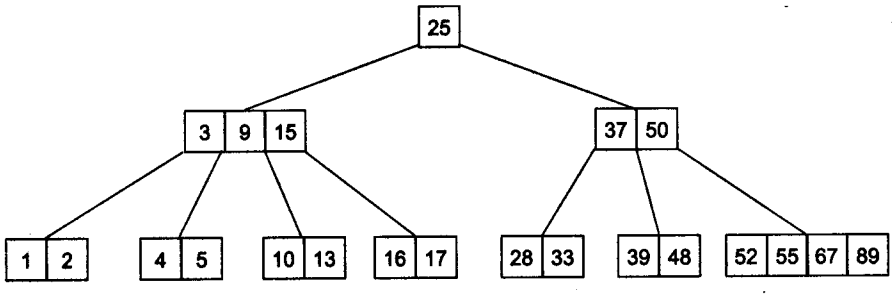
1 a ***B-tree*** is a multiway search tree with a maximum of ***M*** branches at each node. The number *M* is called the ***order*** of the tree.

2 there is ***a single root node*** which may have as few as two children, or none at all if the root is the only node in the tree.

3 at all nodes, except the root and leaf nodes, there must be ***at least half the maximum*** number of children.

4 all leaves are on the ***same level***.

A B-tree of order 5 is shown:



An example of how a ***B-tree*** (or any multiway tree) is searched for an item, which is presented in B-tree and is presented not. Calculating the efficiency of a B-tree.

Constructing a B-Tree

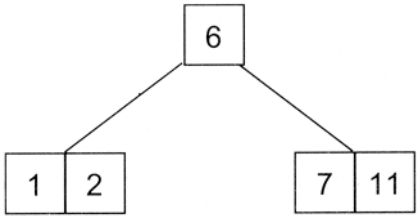
The insertion method for a B-tree is somewhat different to that for the other trees we have studied, since the condition that all leaves be on the same level forces insertion into the upper part of the tree. It is easiest to learn the insertion procedure by example, so we will construct an order-5 B-tree from the list of integers:

1 7 6 2 11 4 8 13 10 5 19 9 18 24 3 12 14 20 21 16

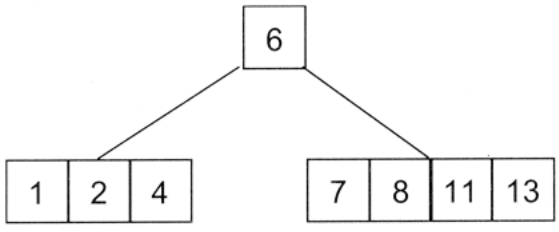
Since each node can have up to five branches, each node can store up to four keys. Therefore, the first four keys can be placed in the root, in sorted order, as shown:

1	2	6	7

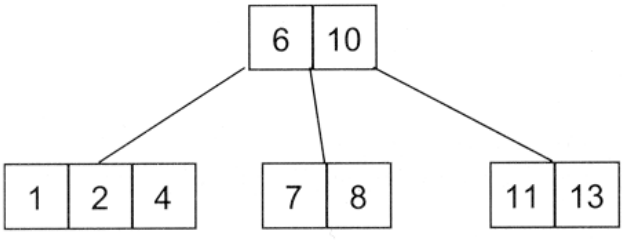
The fifth key, 11, will require the ***creation*** of a new node, since the root is full. In order ***not to violate*** one of the conditions on a B-tree: the root s not allowed to have only ***a single child***, we split the root at its midpoint and create two new nodes, leaving only the middle key in the root. This gives the tree shown:



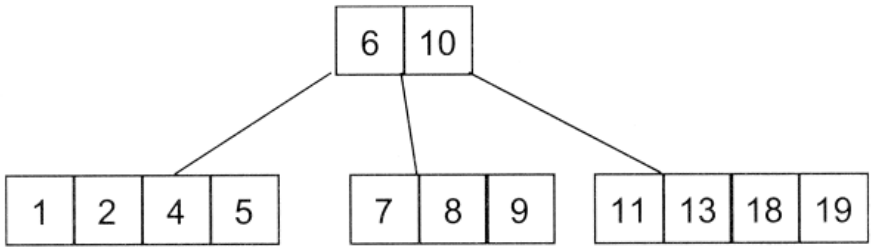
We can add the next three keys wothout having to create any more nodes:



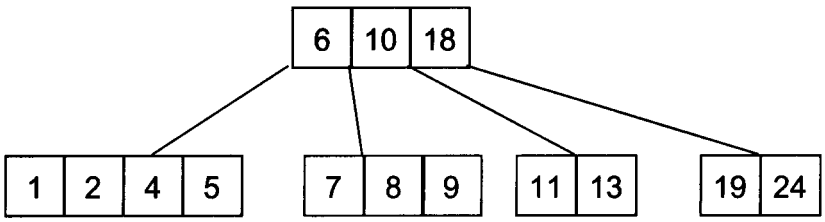
When we wish to add the next key, 10, it would fit into the right child of the root, but this node is full. We split the node, putting the middle key into the node's parent:



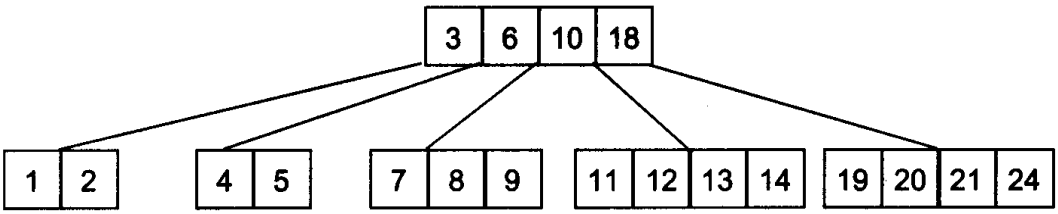
Now we can insert next four keys without any problems:



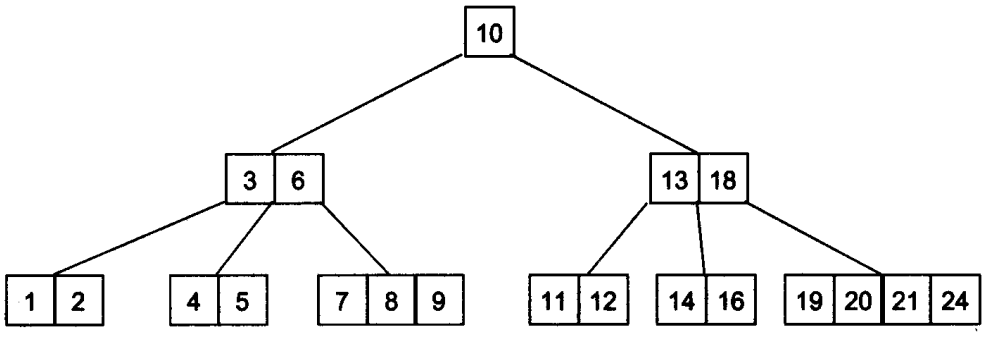
Inserting the key 24 causes another split and increases the number of keys in the root to three:



We can now insert another five keys:



Insertion of the final key, 16, causes the fourth leaf from the left to split, and pushes its middle key, 13, upwards. However, the parent node is also full, so it must split as well, following the same rules. This results in a new root node, and increases the height of the tree to three levels. The completed B-tree is shown:



The algorithm for insertion into a B-tree can be summarized as follows:

1. Find the node into which the new key should be inserted by searching the tree.
2. If the node is not full, insert the key into the node, using an appropriate sorting algorithm.
3. If the node is full, split the node into two and push the middle key upwards into the parent. If the parent is also full, follow the same procedure (splitting and pushing upwards) until either some space is found in a previously existing node, or a new root node is created.

Although the algorithm for insertion may look straightforward on paper, it contains quite a few subtleties which only come out when you try to program it. A program implementing this insertion routine is a nontrivial affair, and could be used as the basis for a programming project.

Data Compression and Huffman Codes

The problem of data compression is of great importance. A lot of different principles and different algorithms were suggested for variuous applications. An example below is the way how image data are encoded in such graphical formats as ***tiff***.

An example: Run-Length Code

Run-length code (RLC or RLE) is normally not an object-oriented image data structure. Originally it was developed for compacting image data, i.e. as a simple storage structure. The structure is well adapted to the relational data model. Generally, a line in its simplest case, is described in terms of its start x- and y-coordinates and its length corresponding to the triplet:

y, x, length

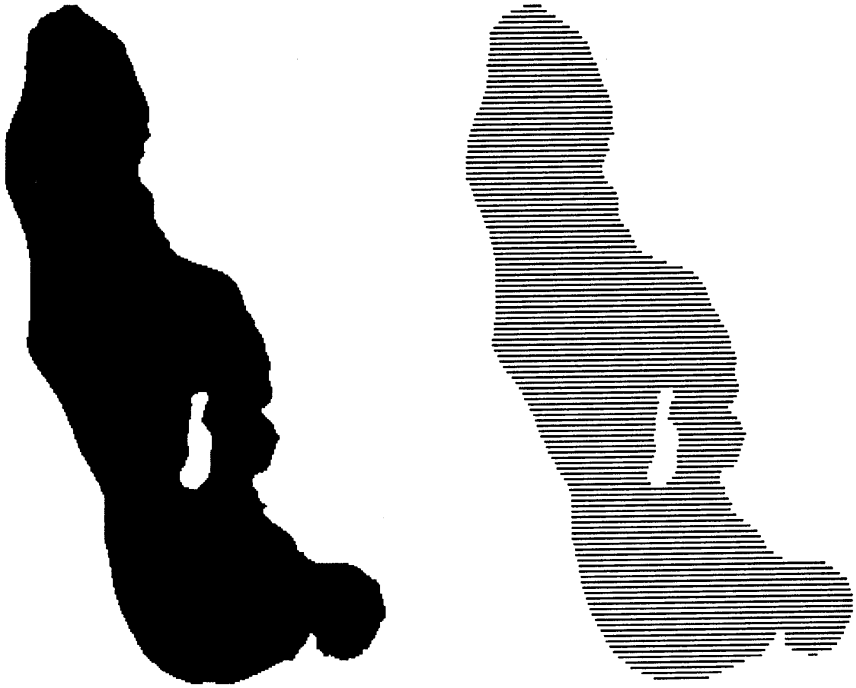
The line records are sorted with respect to their coordinates and stored in sequential order. The structure is homogeneous since the representation of all types of entities including points, lines and entities of extended type are permitted. For point entities the length of a RLC line is by definition set to 1. The line length of a linear object varies depending on its orientation horizontally and vertically. Logically, their representation is not much different from extended objects of general type. Given a RLC database, i.e. a geometric database (GDB), a large number of operations, such as set-theoretical operations, can efficiently be applied.

A geometric database can simply be organized as a flat file with one single access method. The normal access method used ought to be a hierarchical data structure, like a ***B-tree***. A complete access is composed of a single search through the B-tree, directly followed by a sequential access of the flat file corresponding to a predefined interval. The set of records that are read may correspond either to a particular image or to a specific object.

Even if the RLC records permit the storage of logically homogeneous information in the geometric database, further information normally necessary in an ***object-oriented environment*** in order to differentiate the objects from each other. Usually an ***object-id*** *is* necessary as well. Other information that may be useful in some applications is, for instance, the ***entity type***, which gives the following record structure:

y, x, length, type, object-id

In a GIS, for instance, spatial information which does not correspond to objects will appear. An example of this is terrain elevation data. In such cases the ***object-id*** must be dropped and substituted with other information – in a GIS context it could be any kind of characteristic attribute, such as the slope. Consequently, there is no change in the way the information is stored, only in the way it is interpreted.



The process of accessing and displaying RLC data is the part of the data model that deals with how image data are accessed from the geometric database and displayed on the screen. This process must be fast, since the volume of data is often large. In this approach it is attained as a result of the sequential file-access method.

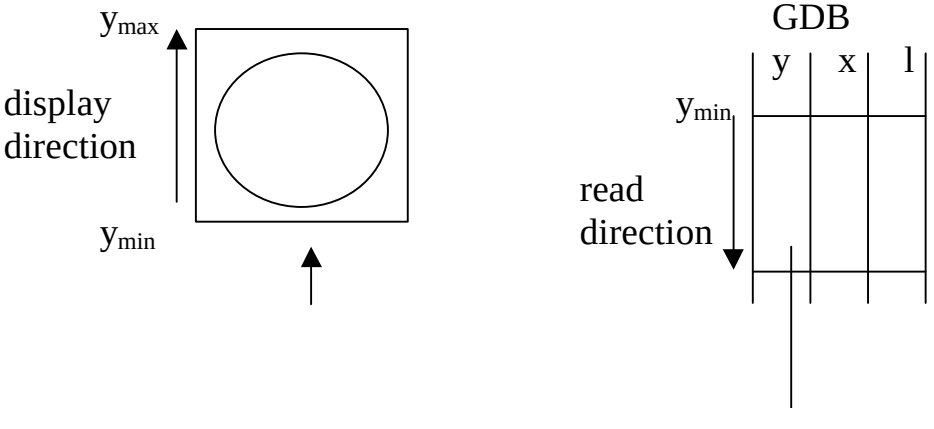


Illustration of the access and display process.

To access and display:

- the first step corresponds to the identification of an interval of the geometric database, defined by the user;
- those records that belong to the interval are simply accessed in sequence and transformed into lines that are drawn on the screen; records outside the image are simply filtered out;
- lines that are on the edge of the image, that is, lines that cross the edge of the image on either the left or the right, are simply cut off;

Advantages:

- the display process is fully sequential, so fast.
- no fill operations are required; all objects are automatically filled at the same pace as they are displayed.

Encoding by changing alphabet

Data compression involves the transformation of a string of characters from some alphabet into a new string that contains the same information, but whose length is smaller than the original string. This requires the design of a code that can be used to uniquely represent every character in the input string.

More precisely, ***a code*** is said to map source messages into ***codewords***.

If ***a fixed-length code*** is used, then all codewords will have the same length. The ASCII code is an example of a fixed-length code that maps 256 different characters into 8-bit codewords.

It is possible to significantly reduce the number of bits required to represent source messages if ***a variable-length code*** is used. In this case, the number of bits required can vary from character to character.

However, when characters are encoded using varying numbers of bits, some method must be used ***to determine the start and end bits*** of a given codeword.

One way to guarantee that an encoded bit string only corresponds to a single sequence of characters is to ensure that no codeword appears as ***a proper prefix*** of any other codeword. Then the corresponding code is ***uniquely*** decodable. A code that has this property is called ***a prefix code*** (or prefix-free code).

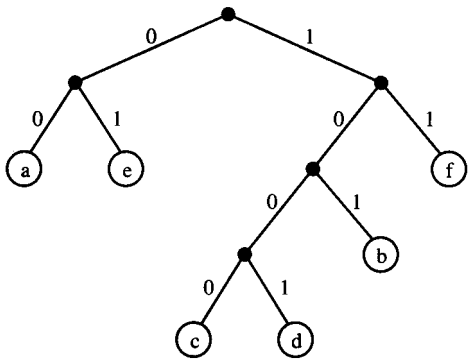
Not only are prefix codes uniquely decodable, they also have the desirable property of being ***instantaneously decodable***. This means that a codeword in a coded message can be decoded without having to look ahead to other codewords in the message.

A technique that can be used to construct optimal prefix codes will be presented on the name of ***Huffman codes***.

This method of data compression involves calculating the frequencies of all the characters in a given message before the message is stored or transmitted. Next, a variable-length prefix code is constructed, with more frequently occurring characters encoded with shorter codewords, and less frequently occurring characters encoded with longer codewords.

The code must be included with the encoded message; otherwise the message could not be decoded.

A convenient way of representing prefix codes is to use a binary tree. Using this approach, each character is stored as ***a leaf in the binary tree***, and the codeword for a particular character is given by ***the path*** from the root to the leaf containing the character. For each bit in the codeword, a "***0***" means "***go to the left child***" and a "***I***" means "***go to the right child***."



The binary tree represents coding scheme:
a = 00, b = 101, c = 1000, d = 1001, e = 01, f = 11

The requirement that all characters reside in the leaves of the tree ensures that no codeword is a prefix of any other codeword. Furthermore, it is quite simple to decode a message using this binary tree representation.

It is useful to think of data compression as an optimization problem in which the goal is to minimize the number of bits required to encode a message. Given a message ***M*** consisting of characters from some alphabet ***G***, and a binary tree ***T*** corresponding to a prefix code for the same alphabet, let $f_M(c)$ denote the frequency

of character ***c*** in ***M*** and $d_T(c)$ the depth of the leaf that stores ***c*** in ***T***. Then the cost (i.e., the number of bits required) to encode ***M*** using ***T*** is

$$C_M(T) = \sum_{c \in \Gamma} f_M(c) d_T(c)$$

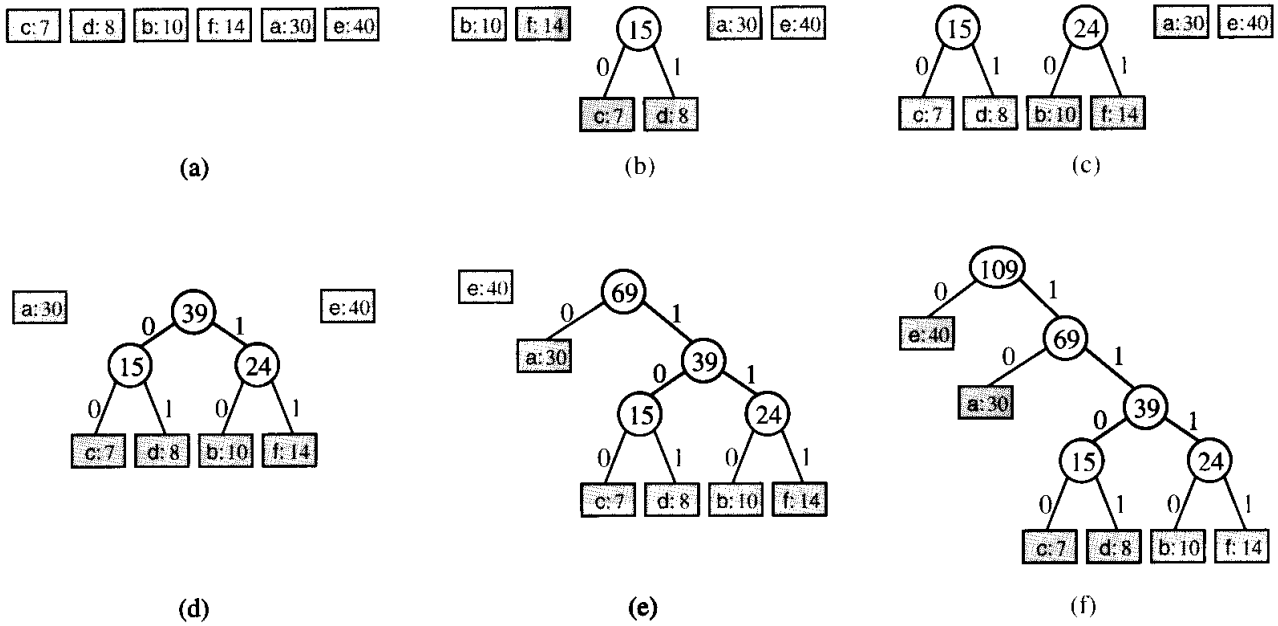
Minimization of this cost function will yield an optimal prefix code for ***M***. The binary tree that represents this optimal prefix code is ***an optimal tree***.

A well-known ***greedy*** algorithm that uses a priority queue to create an optimal prefix code for a message is the resulting optimal tree called a Huffman tree, and the corresponding prefix code is called a Huffman code (the algorithm is greedy because of at each step only the two subtrees of lowest cost are considered):

```
Huffman(vertex set Γ_M)
1   PRIORITY QUEUE P ← Γ_M
2   for i ← 1 to |Γ_M| − 1 do
3       v ← create a new binary tree vertex
4       left[v] ← DeleteMin(P)
5       right[v] ← DeleteMin(P)
6       f_M[v] ← f_M[left[v]] + f_M[right[v]]
7       Insert(P, v)
8   return DeleteMin(P)
```

It is interesting to note that by swapping lines 4 and 5 in Huffman code, aa another different optimal tree can be produced for a message. Thus, optimal prefix codes are not unique for any given message. Notice also that this algorithm will always produce full binary trees, since at each step it always creates a new tree with nonempty left and right subtrees.

For the analysis of Huffman code, we assume that the priority queue ***P*** is irnplemented using a heap. Let us also assume that $|G| = n$. The set G_M can be constructed in ***0 (n)*** time, and that in the worst case $|G_M| = n$. In this case, the loop over lines 2-7 is executed exactly ***n - I*** times. On each iteration three priority queue operations are performed, each requiring ***0 (log n)*** time. So the loop contributes ***0 (n log n)*** to the total running time. The overall running time of Huffman code is therefore ***0 (n log n)***.



The subtrees produced by Huffman code on a message containing the characters ***a, b, c, d, e, f*** with frequencies of ***30, 10, 7, 8, 40, 14***, respectively:

- (a) the initial set of 6 single-vertex sub- trees. The priority of each, which is shown inside the vertex, is the frequency of each character.
- (b)-(e) The subtrees formed during the intermediate stages.
- (f) The final tree.

The binary values encountered on each path from the root to a leaf is the codeword for the character stored in that leaf.

To show that Huffman code produces ***an optimal tree***, we must demonstrate that the sequence of ***greedy*** choices it makes leads to a prefix code that minimizes the cost function C_M given in equation above.

It can be done by induction on n, where n is the size of the vertex set G_M. The base case is trivial; it is easy to verify that Huffman produces an optimal tree when n equals 1 or 2.

The induction hypothesis is that Huffman produces an optimal tree for all messages in which $|G_M|$ is not more than $n - 1$.

The extension step involves that if Huffman produces an optimal tree when $|G_M| = n - 1$, it must also produce an optimal tree when $|G_M| = n$.

Consider a message in which $|G_M|$ equals n, and assume Huffman returns a nonoptimal tree T. Let T_{opt} be an optimal tree for the same message, with x and y being the characters with the first and second lowest frequencies, respectively, in M. We know that x must appear as a leaf with maximal depth in T_{opt} - if this were not

the case, we could exchange this leaf with the lowest leaf in the tree, thereby decreasing C_M and contradicting the optimality of T_{opt}.

Moreover, we can always exchange vertices in T_{opt}, without affecting C_M, so that the vertices storing x and y become siblings. These two vertices must also appear as siblings in T - they are paired together on the first iteration of Huffman. Now consider the two trees T' and T'_{opt} , produced by replacing these two siblings and their parent with a single vertex containing a new character z whose frequency equals $f_M[x] + f_M[y]$.

Note that Huffman would produce T' on the message M' obtained by replacing all occurrences of x and y in M with z. Except for x and y, the depth of every character in these new trees is the same as it was in the old trees. Furthermore, in both T' and T'_{opt} the new character z appears one level higher than both x and y did in T and T_{opt}, respectively. Thus, we have that

$$C_{M'}(T'_{\text{opt}}) = C_M(T_{\text{opt}}) - f_M[x] - f_M[y]$$

$$C_{M'}(T') = C_M(T) - f_M[x] - f_M[y]$$

Since we have assumed that $C_M(Topt) < C_M(T)$, it follows that $C_M\ (T'opt) < C_M\ (T')$. But T' is a Huffman tree in which $|G|$ = n-1, so this latter inequality contradicts our induction hypothesis.