Shape of toric surfaces

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Abstract

We present an informal introduction to the theory of toric surfaces from the viewpoint of geometric modeling. Bézier surfaces and many well-known low-degree rational surfaces are found to be toric. Bézier like control point schemes for toric surfaces are defined via mixed trigonometric–polynomial parametrizations. Examples of a wide shape variety are considered: all quadrics, cubic Möbius strip, quartic 'cross-cap', 'pillow' and Dupin cyclides, sextic 'pear' etc.

Keywords: Geometric modeling, Shape, Bezier surfaces, Toric surfaces

1 Introduction

Toric varieties were introduced in the early 1970's in algebraic geometry. Since that time their abstract theory has been rapidly developed and many new important results are achieved. This theory appeared to be tightly related to combinatorics of convex polytopes. This makes the theory of toric varieties very attractive for different kind of applications (e.g. [2]).

There are several reasons why toric surfaces are important to geometric modeling.

1. They are natural generalizations of Bézier surfaces which are widely used in free-form modeling. A similar control point system of toric surfaces allows to model multisided surface patches (see [10], [13]).

2. Non-standard real structures on complex toric surfaces leads to a wide shape possibilities. Many well-known surfaces of low implicit degree are found to be toric.

3. All rational parametrizations of toric surfaces have a constructive description via universal parametrization (see [4], [12], [9]). This is important for data conversion from traditional solid modeling systems that deal with simplest surfaces (which appear to be toric) to NURBS based systems.

4. Recent works [1], [3], [15] show that an implicitization problem for toric surfaces can be solved effectively using variants of $\mathcal{A}$-resultants at least in some important cases.

In this paper we concentrate on the second item. Main notations and preliminaries are introduced in Section 2. It is recommended to skip this section at first and go directly to examples, which are quite elementary. We describe the simplest cases of quadric and cubic toric surfaces in Section 3. Section 4 is devoted to quartic surfaces. It is shown that all quartic Dupin cyclides and all surfaces of revolution with conic generatrix are toric. In Section 5 one modeling example of a "pear shape" using a sextic toric surface is considered. Finally in Section 6 conclusions are discussed.

2 Background

2.1 Notations

Recall that a real n-dimensional projective space $\mathbb{R}P^n$ is defined as a space of all lines in $\mathbb{R}^{n+1}$ that go through the origin $(0,\dots,0)$. We are mostly interested in the case $n = 3$. Points in $\mathbb{R}P^n$ usually are represented by coordinates $(x_0,\dots,x_n)$ of some point on a line except the origin. They are called homogeneous coordinates. Any proportional collection $(tx_0,\dots,tx_n)$, $t \neq 0$, means the same point. There is a natural projection $\mathbb{R}^{n+1} \to \mathbb{R}P^n$ which maps any point to the corresponding line. We will identify the affine space $\mathbb{R}^n$ with the subset in $\mathbb{R}P^n$: $(x_1,\dots,x_n) \mapsto (1,x_1,\dots,x_n)$. So infinite points lie on the hyperplane $x_0 = 0$. Similarly complex projective spaces $\mathbb{C}P^n$ are defined: just change all $\mathbb{R}$ to $\mathbb{C}$ (complex numbers) in the previous definition.

It will be convenient to use homogeneous control points, including control vectors. Hence we write all parametrization formulas in homogeneous setting. Letters E_i, $i = 0,\dots,n$, will denote the standard basis in $\mathbb{R}^{n+1}$. Bernstein polynomials are defined as usual $B_i^k(t) = \binom{k}{i}(1-t)^{k-i}t^i$.

2.2 Complex toric varieties

A projective toric variety can be briefly defined as a monomially parametrized subset in the projective space.

At first consider a simple 1-dimensional case. The map $\mathbb{R} \to \mathbb{R}P^2$, $u_1 \mapsto [1,u_1,u_1^2]$, defines a parametrization of a conic (an affine parabola) but its infinite point $[0,0,1]$ is missed. Using the map $u_0 \mapsto [u_0^2,u_0,1]$ we cover this point but then $[1,0,0]$ is missed. Hence it is necessary to use both variables at the same time $(u_0,u_1) \mapsto$

$[u_0^2, u_0u_1, u_1^2]$. In a more general control point setting we have

$$(u_0, u_1) \mapsto P_0t_0^2 + P_1u_0u_1 + P_2u_2^2. \quad (1)$$

Substituting u_0 and u_1 by $1-t$ and t a Bézier curve is obtained (with control points P_0, $P_1/2$, P_2).

Let us turn to a general case of a d-dimensional variety. The most important particular cases will be $d = 1, 2$. Consider a d-dimensional *lattice* $M = \mathbb{Z}^d$ of points with integer coordinates in space $\mathbb{R}^d$. Define a *lattice polytope* $\Delta \subset \mathbb{R}^d$ as the convex hull of some finite subset in M. Let $\dim \Delta = d$, i.e. Δ is not contained in a hyperplane. Then *facets* (i.e. $(d-1)$-dimensional faces) ϕ_i of Δ are intersections with hyperplanes $h_i(t) = 0$, $i = 1, \ldots, r$. Here we suppose the affine linear forms $h_i(t) = \langle n_i, t\rangle + a_i$ to be normalized: vectors n_i are primitive (i.e. the shortest vectors in this direction with integer coordinates) and inward oriented. We denote by $\widehat{\Delta} = \Delta \cap M$ a set of lattice points of Δ.

Definition 2.1 *A complex projective toric variety T_Δ associated with a lattice polytope Δ, $\widehat{\Delta} = \{m_0, m_1, \ldots, m_N\}$, is a subset in $\mathbb{C}P^N$ parametrized by the following formula*

$$G_\Delta(u_1, \ldots, u_r) = \left[u^{h(m_0)}, u^{h(m_1)}, \ldots, u^{h(m_N)}\right], \quad (2)$$

where $u^{h(m)} = u_1^{h_1(m)}u_2^{h_2(m)} \cdots u_r^{h_r(m)}$. The variables $u_i \in \mathbb{C}$, $i = 1, \ldots, r$, are called facet variables *[2].*

The parametrization G_Δ is undefined on the exceptional subset $G_\Delta^{-1}(0)$, which is contained in a union of intersections of some pairs of coordinate hyperplanes.

Remark 2.2 *It is not difficult to check that it is not necessary to use all facet variables if one wants to cover only a part of the variety. One can start with any collection of primitive vectors n_i. Then one defines affine forms h_i associated to the corresponding supporting hyperplanes and use the same formula* (2)*.*

Let $\{e_1, \ldots, e_d\}$ be a basis in the lattice M and denote by $\mathrm{Conv}\, A$ a convex hull of any subset $A \subset \mathbb{R}^d$.

Example 2.3 *(i) In case of an interval* $\mathrm{I}_k = \mathrm{Conv}\{0, ke_1\}$ $\mathrm{Ex}(\mathrm{I}_k) = \{0\}$ *and* (2) *looks like*

$$G_{\mathrm{I}_k}(u_1, u_2) = [u_2^k, u_1u_2^{k-1}, \ldots, u_1^{k-1}u_2, u_1^k]. \quad (3)$$

This is exactly a homogeneous parametrization $\mathbb{C}P^1 \to \mathbb{C}P^k$ of a rational normal curve.

(ii) Consider a triangle $\triangle_k = \mathrm{Conv}\{0, ke_1, ke_2\}$*. Then* $\mathrm{Ex}(\triangle_k) = \{0\}$ *and the homogeneous coordinates of $T_{\triangle_k}$ coincide with a list of all monomials of total degree k. Hence this is an image of the classical* Veronese embedding $\mathbb{C}P^2 \to \mathbb{C}P^N$, $N = \binom{k+2}{2} - 1$*, i.e., it is exactly the* Veronese surface.

(iii) Consider a rectangle

$$\square_{k,l} = \mathrm{Conv}\{0, ke_1, le_2, ke_1 + le_2\}.$$

Homogeneous coordinates of $T_{\square_{k,l}}$ coincide with a list of all monomials of total degree k (resp. l) in a pair of variables u_1, u_3 (resp. u_2, u_4). Therefore the map $G_{\square_{k,l}}$ can be factored through a product of two complex projective lines, and we get the classical Segre embedding $\mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^N$, $N = kl + k + l$*. Hence $T_{\square_{k,l}}$ is the* Segre surface.

(iv) Let Δ be the standard d-dimensional simplex $\triangle^d = \mathrm{Conv}\{0, e_1, \ldots, e_d\}$*. In this case it will be convenient to use also zero indices. The linear forms $h_0(t) = 1 - t_1 - \cdots - t_d$, $h_i(t) = t_i$, $i = 1, \ldots, d$, define facets of the simplex, and the identity map $G_{\triangle^d}(u_0, u_1, \ldots, u_d) =$* $[u_0, u_1, \ldots, u_d]$*. Hence $T_{\triangle^d} = \mathbb{C}P^d$.*

2.3 Real structures

Let us remind that formally a real variety is a pair (X, c), where X is a complex variety and $c : X \to X$ is some anti-holomorphic involution. The real part $\mathbb{R}_cX$ of X is just the fixed-point set $\{x \in X \mid c(x) = x\}$. The *standard* real structure is defined by the complex conjugation $x_i \mapsto \bar{x}_i$ for all coordinates x_i. Hence the fixed-point set coincides with all points with real coordinates $x_i \in \mathbb{R}$.

In the toric case this leads to the following construction. Fix some lattice polytope Δ with involution c, i.e. some transformation $c : M \to M$, such that $c^2 = \mathrm{id}$ and $c(\Delta) = \Delta$. In general c permutes points of the polytope $\Delta = \{m_0, \ldots, m_N\}$, $c(m_i) = m_{\gamma(i)}$.

Definition 2.4 *A* real projective toric variety $\mathbb{R}_cT_\Delta$ *associated with a lattice polytope Δ with involution c, is the real part of the complex toric variety T_Δ defined by* (2) *with the involution* $c : [x_0, \ldots, x_N] \mapsto [\bar{x}_{\gamma(0)}, \ldots, \bar{x}_{\gamma(N)}]$.

Since $\mathbb{R}_cP^N$ is isomorphic to $\mathbb{R}P^N$ (cf. (5)), we can treat $\mathbb{R}_cT_\Delta$ as a real subvariety in $\mathbb{R}P^N$. The involution $c : M \to M$ permutes the facets ϕ_i of Δ and also the facet variables u_i. Hence we obtain a natural involution on the global parametrization (2).

Therefore, according to our construction $\mathbb{R}_cT_\Delta$ is contained in higher-dimensional space $\mathbb{R}P^N$, where N is equal to the number of lattice points in the polytope Δ. Let us relax slightly Definition 2.1 and consider also projections of toric varieties to lower dimensional spaces, i.e. more general maps $\mathcal{G}_\Delta : U_\Delta \to \mathbb{C}P^n$

$$\mathcal{G}_\Delta(u_1, \ldots, u_r) = \sum_{m \in \widehat{\Delta}} P_m u_1^{h_1(m)} \cdots u_r^{h_r(m)}, \quad (4)$$

where $P_m \in \mathbb{C}^{n+1}$ are homogeneous complex control points. We get $\mathcal{G}_\Delta = G_\Delta$ when P_m are elements of standard basis in $\mathbb{C}^{N+1}$. We refer to this case as a *normal* toric variety.

Let $d = 2$ and $n = 3$ (this is the most interesting for us). In case of a standard real structure facet variables u_i and control points P_m are real. Substituting affine forms $h_i(t)$ instead of facet variables u_i in the formula (2) we get a *toric* surface patch $\mathcal{T}_\Delta(t) = \mathcal{G}_\Delta(h_1(t), \ldots, h_r(t))$,

$t \in \Delta$, defined on the polygon Δ (see details in [10]). In particular, triangles $\triangle_k$ and rectangles $\square_{k,l}$ correspond to Bézier triangular and tensor-product surfaces. From Example 2.3(ii) and (iii) follows that they are just "shadows" of real Veronese and Segre surfaces living in higher-dimensional spaces.

In case of non-standard real structures control points are arranged in complex conjugated pairs (others are real), which can be substituted by real pairs associated with trigonometric parametrizations. Indeed, consider of the conic curve (Example 2.3(i)) and take the reflection of the associated segment $\mathrm{I}_2 = \mathrm{Conv}\{-e_1, e_1\}$ with respect to its central point 0. The associated involutions in the parameter space U_{I_2} and the projective plane $\mathbb{C}P^2$ are as follows: $u_0 \mapsto \bar{u}_1$, $u_1 \mapsto \bar{u}_0$, and $x_0 \mapsto \bar{x}_2$, $x_1 \mapsto x_1$, $x_2 \mapsto \bar{x}_0$. The corresponding fixed point sets are parametrized by $\mathbb{C}^* = \mathbb{C} \setminus 0 \to U_{\mathrm{I}_2}$, $\kappa : z \mapsto (z, \bar{z})$, and the following isomorphism $\iota : \mathbb{R}P^2 \to \mathbb{R}_c P^2 \subset \mathbb{C}P^2$,

$$\iota : (x_0, x_1, x_2) \mapsto (x_0 + \mathrm{i}x_2, x_1, x_0 - \mathrm{i}x_2). \tag{5}$$

Using the inverse isomorphism we get a real part of conic parametrized by non-zero complex numbers

$$\iota^{-1} \circ G_{\mathrm{I}_2} \circ \kappa : \mathbb{C}^* \to \mathbb{R}P^2, \quad z \mapsto [\mathrm{Re}\, z^2, z\bar{z}, \mathrm{Im}\, z^2],$$

Similarly to the standard case we restrict our parameter to the natural one dimensional domain—a circle $|z| = 1$ in a plane of complex numbers $\mathbb{C}$. In a trigonometric form we have $z = \cos\phi + \mathrm{i}\sin\phi$. Finally we get a real parametrization:

$$\phi \mapsto [\cos 2\phi, 1, \sin 2\phi].$$

Since it is a double covering of the conic by the circle, it will be convenient to denote $\alpha = 2\phi$, and use the interval $0 \le \alpha < 2\pi$.

Similar formulas are valid for an arbitrary real rational curve of even degree $2k$ in control point setting. Let $I_{2k} = \mathrm{Conv}\{-ke_1, ke_1\}$ with similar involution $c(e_1) = -e_1$. Then same substitution of u_j by $z \in \mathbb{C}^*$ in (4) gives

$$z \mapsto \sum_{j=-k}^{k} P_j z^{k-j} \bar{z}^{\,k+j},$$

where $P_j = \bar{P}_{-j}$, $i = 0, \ldots, k$, are conjugated pairs of control points. Denoting $P_j^c = (P_{-j} + P_j)/2$ and $P_j^s = (P_{-j} - P_j)/2\mathrm{i}$ we get the trigonometric parametrization with real control points

$$\alpha \mapsto P_0 + \sum_{j=1}^{k} \left(P_j^c \cos(j\alpha) + P_j^s \sin(j\alpha)\right).$$

These trigonometric notations are useful for many surface examples, where we use mostly collections (see Remark 2.2) with three or four vectors. All conjugated pairs will give $\cos$ / $\sin$ functions and pairs of opposite real vectors will give Bernstein polynomials.

2.4 Implicit degrees and singular points

Here we consider only case $d = 2$. If an affine transformation L of a plane $\mathbb{R}^2$ preserves the lattice $L(\mathbb{Z}^2) = \mathbb{Z}^2$ then it is called an *affine unimodular transformation*. It is easily seen that L is a composition of some translation by a lattice vector and a linear transformation, which has a matrix with integer entries and the determinant ± 1. The transformation L preserves area Vol_2, which is normalized assuming $\mathrm{Vol}_2(\triangle_1) = 1$. Note that $\mathrm{Vol}_2\,\Delta$ is an integer number for every lattice polygon Δ. It is twice as large as standard area in $\mathbb{R}^2$.

The following remarkable property is well-known in the theory of toric varieties.

Theorem 2.5 *The implicit degree of a normal toric surface T_Δ is equal to* $\mathrm{Vol}_2(\Delta)$.

The implicit degree of an arbitrary real toric surface cannot exceed $\mathrm{Vol}_2(\Delta)$. We refer to [14] for a relatively elementary proof.

Now we are going to consider singular points of toric surfaces. As usual we call a point on a surface singular if there is no tangent plane at this point. So it is similar to the vertex of a cone, or a corner point of a "pillow" in Fig 2. Consider a toric surface parametrized by (4). It is easy to check that its control points P_m labeled by vertices m of the polygon Δ lie on the surface. We call them *vertices* of the toric surface. For every vertex m of the lattice polygon Δ we define a *lattice triangle* Θ_m with three vertices: m and the two nearest lattice points on the adjacent boundary edges.

Theorem 2.6 *Singular points on a normal toric surface T_Δ can occur only in vertices. A vertex associated with a vertex point $m \in \Delta$ is non-singular if and only if* $\mathrm{Vol}_2(\Theta_m) = 1$.

The proof and more detailed classification of singular points can be found in [10]. Of course non-normal toric surfaces can have much more singular points, that appear as double points of the projection from the corresponding normal one. Therefore, Theorem 2.6 says that "intrinsicly singular" points are only vertex control points P_m which can be easily detected from the polygon Δ.

2.5 List of lattice polygons

We call lattice polygons Δ and Δ' *equivalent* if $L(\Delta) = \Delta'$ for some unimodular transformation L. Equivalent polygons define the same toric surfaces (just different parametrizations).

Simple examples of lattice polygons having $\mathrm{Vol}_2 \le 4$ are shown in Fig. 1. Examples of $\triangle_k$ and $\square_{k,l}$ are shown as shaded triangles and rectangles in Fig. 1. They correspond to Bézier surfaces. The polygons are sorted by the area. In fact the full list for of such polygons: any other lattice polygon with this property will be unimodular equivalent to one of them. Also they are not equivalent

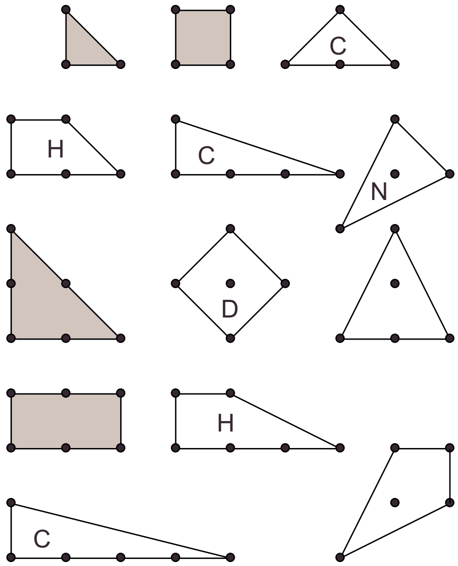

Figure 1: Lattice polygons of area ≤ 4.

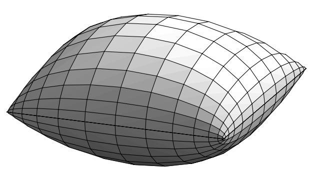

Figure 2: A “pillow”.

to each other, since they have different area or different number of singular vertices. The standard real structures on toric surfaces associated to these polygons are as follows (see [10] for details). In the first row we see a triangle, which means just a projective plane, and two cases of quadrics: double ruled and a cone. On the second row we can see four cubic surfaces: a kind of Hirzebruch surface (H) (see [5] for a definition), a cone over rational cubic (C) and a cubic with 3 singular points (N). Other polygons correspond to different quartic surfaces. For example, the square (D) means a quartic in the form of “pillow” Fig. 2 with 4 lines intersecting in 4 singular points.

In the following sections we will concentrate on non-standard real cases of these surfaces.

3 Quadric and cubic surfaces

In Fig. 3 we can see three lattice polygons with reflections (their axes are marked by dotted lines). Here we list the associated toric surfaces, their collections of normals ν (Remark 2.2) and their control point expressions.

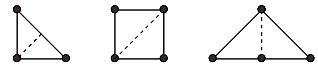

Figure 3: Lattice polygons with reflections.

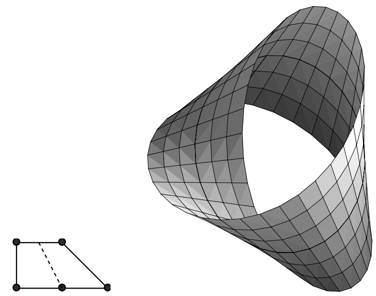

Figure 4: Möbius strip.

1. A plane, $\nu = \{e_1, e_2, e_1 + e_2\}$,

$$\begin{array}{ccc} P_1^c\, t\cos\phi & & \\ +\,P_0 & + & P_1^s\, t\sin\phi \end{array}$$

These are the polar coordinates if

$$P_0 = E_0, \quad P_1^c = E_1, \quad P_1^s = E_2.$$

2. An oval quadric, $\nu = \{e_1 + e_2, -e_1 - e_2, e_1 - e_2, -e_1 + e_2\}$,

$$\begin{array}{ccc} P_1^c B_1^2(t)\cos\phi & + & P_2 B_2^2(t) \\ +\,P_0 B_0^2(t) & + & P_1^s B_1^2(t)\sin\phi. \end{array}$$

This is a sphere if, for example, the control points are

$$P_0 = -E_3, \quad P_1^c = E_1, \quad P_1^s = E_2, \quad P_2 = E_3.$$

The detailed theory of biangle patches on the oval quadric with four control points is developed in [7].

3. A quadratic cone, $\nu = \{e_1 - e_2, -e_1 - e_2, e_2, -e_2\}$,

$$\begin{array}{ccc} & P_1(1-t) & \\ +\,(P_0^c\cos\alpha & +\,P_0 & +\,P_0^s\sin\alpha)t \end{array}$$

The cubic Hirzebruch surface H (Fig. 1) can model the famous Witney umbrella [10]. It naturally lives in $\mathbb{R}P^4$ and has a topology of the Klein bottle. Therefore it is not strange that one can find some Möbius strip on H, which can be parametrized using a toric structure with a reflection as in Fig. 4. Indeed, consider the collection $\nu = \{e_1, e_2, -e_1 - e_2\}$ and the corresponding expression

$$\begin{array}{ccccc} P_0^c t\cos\phi & + & P_0^s t\sin\phi & & \\ +\,P_1^c\cos 2\phi & + & P_1 & + & P_1^s\sin 2\phi \end{array}$$

with control points

$$\begin{array}{rcl} P_0^c & = & E_2 - E_3, \quad P_0^s = E_2 + E_3, \\ P_1 & = & E_0, \quad P_1^c = E_1, \quad P_1^s = E_2. \end{array}$$

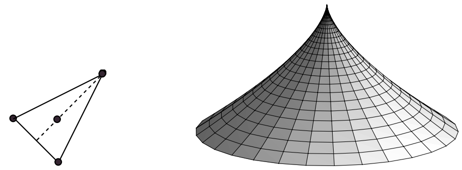

Figure 5: A cubic of revolution.

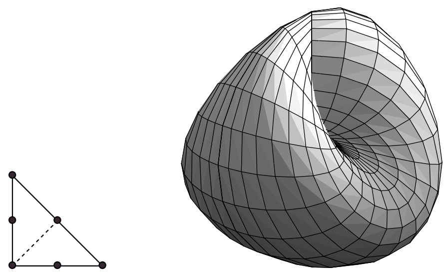

Figure 6: Cross cap.

Restricting parameters to the domain $0 \le \phi < 2\pi$, $0 \le t \le 1/2$, we obtain a nice Möbius strip (Fig. 4). This is a patch of cubic surface—the most minimal representation of the Möbius strip respectively to algebraic degree.

A reflection on a lattice triangle with $\mathrm{Vol}_2 = 3$ (see Fig. 5) corresponds to the parametrization of a special cubic with only one singular point (equal to P_0) as follows

$$\begin{array}{ccc} & & P_0 B_0^3(t) \\ + P_3^c B_3^3(t)\cos\phi & + & P_2 B_2^3(t) \\ & + & P_3^s B_3^3(t)\sin\phi. \end{array}$$

The collection $\nu = \{e_1+e_2, e_1-e_2, -e_1-e_2, -e_1+e_2\}$ is used here.

4 Quartic surfaces

4.1 Steiner surfaces

The normal toric surface associated with $\triangle_2$ is in the well-known Veronese surface in $\mathbb{R}P^5$ isomorphic to a projective plane $\mathbb{R}P^2$. All its real structures are equivalent. Nevertheless we show that it is worth considering the lattice triangle $\triangle_2$ with a reflection and its associated parametrization. An interesting case of such surface is called a "Cross cap" and its natural trigonometric parametrization is described in [6] (see Fig. 6). Here we show that it is essentially a toric surface associated with $\triangle_2$. Indeed, the expression

$$\begin{array}{lll} P_2^c B_2^2(t)\cos 2\phi & & \\ + P_1^c B_1^2(t)\cos\phi & + P_2 B_2^2(t) & \\ + P_0 B_0^2(t) & + P_1^s B_1^2(t)\sin\phi & + P_2^s B_2^2(t)\sin 2\phi \end{array}$$

with the control points

$$P_0 = P_2 = E_0, P_1^c = 0, P_1^s = E_1, P_2^c = E_3, P_2^s = E_2.$$

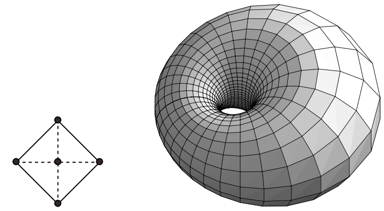

Figure 7: Ring cyclide.

gives exactly the Cross cap from the book [6]!

4.2 Dupin cyclides

Consider the Dupin cyclide $\mathrm{Q}_{(a,b,d)}$ defined by the quartic equation in homogeneous coordinates of $\mathbb{R}P^3$

$$\left(x_1^2+x_2^2+x_3^2+(b^2-d^2)x_0^2\right)^2 = 4x_0^2\left((ax_1-cdx_0)^2+b^2x_2^2\right),$$

where $c^2 = a^2 - b^2$. It has the parametrization by two pairs of homogeneous parameters $u_0 : u_1$ and $v_0 : v_1$ (cf. [11])

$$\begin{aligned} \mathrm{Q}(u_0,u_1,v_0,v_1) = &\left(d(u_0^2-u_1^2) - a(u_0^2+u_1^2)\right)C_0 \\ &+\left(c(v_0^2-v_1^2) - d(v_0^2+v_1^2)\right)C_1, \end{aligned} \quad (6)$$

where C_0, C_1 are 4-dimensional vectors

$$\begin{aligned} C_0 &= \left(v_0^2+v_1^2, a(v_0^2-v_1^2), 2bv_0v_1, 0\right), \\ C_1 &= \left(u_0^2-u_1^2, c(u_0^2+u_1^2), 0, 2bu_0u_1\right). \end{aligned}$$

In order to show that $\mathrm{Q}_{(a,b,d)}$ is toric we have to distinguish three different cases depending on the number of singular points. All fans will be equal to $\{e_1, -e_1, e_2, -e_2\}$.

Smooth case: $c < d < a$ (ring cyclide). We substitute the variables in (6) by trigonometric functions

$$\mathrm{Q}\left(\frac{\sin\frac{\alpha}{2}}{\sqrt{a+d}}, \frac{\cos\frac{\alpha}{2}}{\sqrt{a-d}}, \frac{\cos\frac{\beta}{2}}{\sqrt{d-c}}, \frac{\sin\frac{\beta}{2}}{\sqrt{d+c}}\right)$$

and obtain the expression

$$\begin{array}{ccccc} & & P_\beta^c\cos\beta & & \\ + & P_\alpha^c\cos\alpha & + \quad P_1 & + & P_\alpha^s\sin\alpha \\ & & + \quad P_\beta^s\sin\beta & & \end{array}$$

with the following control points

$$\begin{aligned} P_\alpha^c &= \frac{1}{a^2-d^2}(a\,E_0 + cd\,E_1), \\ P_\alpha^s &= \frac{b}{\sqrt{a^2-d^2}}\,E_3, \\ P_1 &= \frac{b^2}{(a^2-d^2)(d^2-c^2)}\,(d\,E_0 + ac\,E_1), \\ P_\beta^c &= \frac{1}{d^2-c^2}(c\,E_0 + ad\,E_1), \\ P_\beta^s &= \frac{b}{\sqrt{d^2-c^2}}\,E_2. \end{aligned}$$

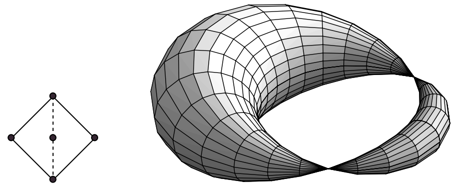

Figure 8: 2-horn cyclide.

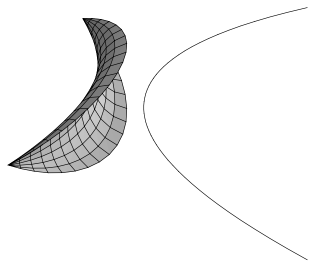

Figure 9: A biangle patch of 2-horn cyclide.

Hence this is a toric surface associated with a lattice square (marked by (D) in Fig. 1) with a central symmetry, which is a composition of two reflections (marked by dotted lines in Fig. 7).

Case with two singular points. Let us consider the case of a 2-horn cyclide, when $d < c$ (spindle cyclide case $a < d$ is similar). Substituting in (6) the appropriate trigonometric and linear functions

$$\mathrm{Q}\left(\frac{\sin\frac{\alpha}{2}}{\sqrt{a+d}}, \frac{\cos\frac{\alpha}{2}}{\sqrt{a-d}}, \frac{1-t}{\sqrt{c-d}}, \frac{t}{\sqrt{c+d}}\right)$$

we get the expression

$$\begin{array}{ccccc} & & P_0B_0^2(t) & & \\ + & (P^c_\alpha\cos\alpha & + \quad P_1 & + & P^s_\alpha\sin\alpha)B_1^2(t) \\ & & + \quad P_2B_2^2(t), & & \end{array}$$

where three control points P^c_α, P^s_α, P_1 have the same formulas as in the previous case, and P_i, $i = 0, 2$, have the form

$$P_i = \frac{1}{c^2-d^2}(c\,E_0 + ad\,E_1) + \frac{(i-1)b}{\sqrt{c^2-d^2}}\,E_2.$$

In fact they are singular points of the surface. The geometric meaning of this parametrization is clear. The surface is swept by a Bézier curve with fixed endpoints. The middle point traces some conic (see Fig. 9). Moreover, this is a toric surfaces associated with the same lattice square as in ring case but with only one reflection (marked by a dotted line in Fig. 8).

Case with one singular point. We consider only 1-horn cyclide case $d = c$ (case $d = a$ is similar). As earlier substituting in (6) we get

$$\mathrm{Q}\left(\frac{\sin\frac{\alpha}{2}}{\sqrt{a+d}}, \frac{\cos\frac{\alpha}{2}}{\sqrt{a-d}}, \frac{1}{\sqrt{2d}}, \frac{t}{\sqrt{2d}}\right)$$

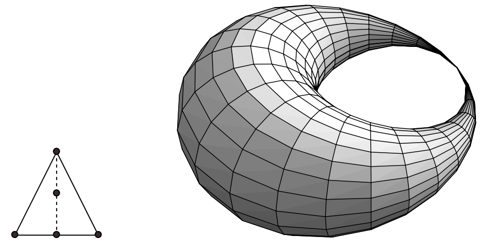

Figure 10: 1-horn cyclide.

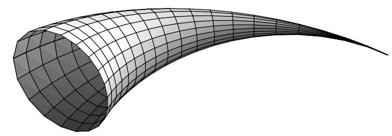

Figure 11: A part of 1-horn cyclide.

$$\begin{array}{ccccc} & & = \quad P_0B_0^2(t) & & \\ & & + \quad P_1B_1^2(t) & & \\ + & (P^c_\alpha\cos\alpha & + \quad P_1 & + & P^s_\alpha\sin\alpha)B_2^2(t) \end{array}$$

with the control points

$$\begin{aligned} P_0 &= \frac{1}{2d}(E_0 + a\,E_1), \\ P_1 &= \frac{1}{2d}(E_0 + a\,E_1 + b\,E_2), \\ P_2 &= \frac{1}{d}\,(d\,E_0 + b\,E_2) + \frac{d}{a^2-d^2}\,(E_0 + a\,E_1). \end{aligned}$$

The remaining two control points P^c_α, P^s_α, have the same formulas as in the previous cases. In Fig. 10 we can see the corresponding lattice triangle with reflection and the full surface. A part of 1-horn cyclide parametrized by $0 \le \alpha < \pi$, $0 \le t \le 1$ is shown in Fig. 11.

4.3 Surfaces of revolution

The ring cyclide $\mathrm{Q}_{(4,4,3)}$ (so $c = \sqrt{a^2-b^2} = 0$) is a torus surface. If its control point P_1 goes down to the new position $P_1' = P_1 - 0.8E_3$ then the surface is slightly deformed as in Fig. 12. The resulting surface still has a rotational symmetry but its plane section through the axis of rotation deforms from a circle to some ellipse. It is easy to check that this type of toric structure gives us all surfaces of revolution with conic generatrix C, which does not intersect with the symmetry axis. Cases when C intersects the axis in two points or in one point (i.e., it is a touching point) correspond to the same toric structures as 2-horn and 1-horn cyclides respectively.

5 One shape modeling example

Finally let us consider a sextic toric surface associated with the lattice hexagon with reflection as in Fig. 13. It has the

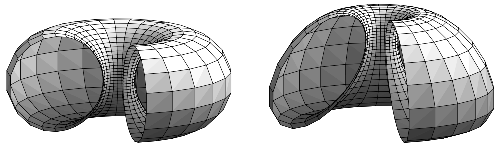

Figure 12: Torus and its deformation.

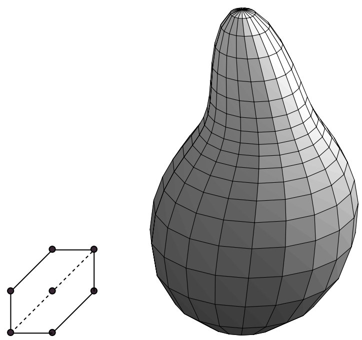

Figure 13: A toric surface of degree 6.

parametrization ($\nu = \{\pm(e_1 + e_2), \pm(e_1 - e_2)\}$)

$$\begin{array}{ccc} & P_1^c B_1^4(t)\cos\phi & +P_0 B_0^4(t) \\ +\,P_3^c B_3^4(t)\cos\phi & +P_2 B_2^4(t) & +P_1^s B_1^4(t)\sin\phi \\ +\,P_4 B_4^4(t) & +\,P_3^s B_4^3(t)\sin\phi. & \end{array}$$

If we put these seven control points into the following positions

$$\begin{array}{rcl} P_0 & = & E_0 - 1.5E_3, \quad P_1^c = 1.5E_1, \quad P_1^s = 1.5E_2, \\ P_2 & = & 2(E_0 - 0.2E_1 + 0.35E_3), \quad P_3^c = 0.5E_1, \\ P_3^s & = & 0.5E_2, \quad P_2 = E_0 + 1.5E_3, \end{array}$$

we get a surface looking like a pear. How can one come to this result in a natural way?

The modeling method has several simple steps:

1. model a flat quartic Bézier curve C_+ choosing finite control points P_i, $i = 0, 2, 4$, on the z-axis and control vectors P_i, $i = 1, 3$, parallel to x-axis;

2. add the complementary Bézier curve C_- with the same control points but the opposite control vectors (therefore the union $C_+ \cup C_-$ is the whole quartic rational curve);

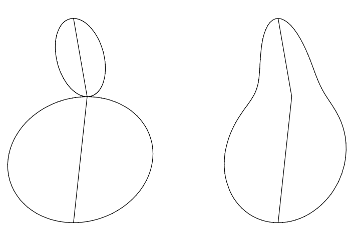

Figure 14: A "pear shape" modeling.

3. denote $P_i^c = P_i$, $i = 1, 3$, and add the vectors P_i^s, $i = 1, 3$, with the same lengths and parallel to y-axis;

4. we already have all seven control points of a toric surface associated with the lattice hexagon;

5. it is easy to check that it is a surface of revolution, so we can move the middle point P_2 slightly (in order to achieve a nice incline of the "axis" as in Fig. 14 right side).

We can make further improvements to the "pear shape" moving control vectors P_i^c and P_i^s, $i = 1, 3$, around. This means that we work with two ellipsoids (see Fig. 14 left side) that are associated with two subsquares of the lattice hexagon.

6 Conclusions

We have shown that toric surfaces have many attractive features.

1. They are natural generalizations of Bézier surfaces.

2. Arbitrary toric surfaces have more variety of implicit degrees, allowing to parametrize without basepoints a large class of rational surfaces.

3. Many well-known surfaces of low implicit degree are found to be toric, i.e. they have natural control point structures.

4. Considering also non-standard real structures one gets additional possibilities for shape modeling. This leads to trigonometric parametrizations.

5. The natural mixed trigonometric–polynomial parametrizations cover the geometry of many interesting toric surfaces much better than traditional rational–polynomial approach.

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