Comparison of Different Multisided Patches Using Algebraic Geometry

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Abstract. Different constructions of multisided surface patches (due to Sabin, Hosaka-Kimura, Warren, Loop-DeRose, etc.) are studied via considering base points of their parametrizations. This analysis shows hidden interrelations between various cases and enables to find new efficient control point schemes in more general situations. In particular, toric patches are introduced.

§1. Introduction

The problem of smooth filling of m-sided holes arises in many modeling situations. It is solved using various methods: recursive subdivision, surface splitting, data blending and control point schemes. We consider here only the cases when a m-sided patch is defined via control points as a single piece bounded by Bézier curves of degree n. M. Sabin [11] introduced 3- and 5-sided patches bounded by conics ($n = 2$) and suitable for an inclusion in B-spline surface. Hosaka and Kimura [2] proposed the same type of patches with $n = 3$. Zheng and Ball [15] extended the previous constructions to arbitrary degree n. In the same fashion 6-sided patches were constructed (see [2, 12, 15]). Unfortunately, these 6-sided patches seems to be nonrational. Loop and DeRose [9] introduced rational S-patches, and used them in [10] for building Sabin and Hosaka–Kimura-like patches ($n = 2, 3$) with arbitrary number of sides m. As far as we know Warren was the first to introduce the method of blowing up base points (well-known in algebraic geometry) to the CAGD community. He used it in [14] for creating 5-, 6-sided patches. Analysis of mentioned approaches and the convex combination method (cf. Gregory [1]) shows that m-sided patches for $m > 4$ should be rational. Hence it is natural to use theoretical results from algebraic geometry concerning rational surfaces. The method of base points enabled Karčiauskas [3] to build well structured rational 5-sided patch with actually the same properties as the original Warren hexagon. In [4] these patches are used for creating 5- and 6-sided Sabin

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and Hosaka–Kimura-like surface patches with boundary curves of arbitrary degree n. Similar patches over a regular m-gon for any m (except 4) and for arbitrary n are obtained in [5] also using the base point method. The patches in [4, 5] have lower degree parametrization than previous ones. We call them T-patches. Moreover, the base points method is good for building bridges between various approaches, especially in pentagonal case. In 6-sided case the relations are more complicated. On the other hand, it appeared that this hexagonal patch belongs to a special class of so-called toric surfaces, which were studied in detail in algebraic geometry. First applications of toric varieties in CAGD were demonstrated by Warren [14] and Krasauskas [7].

In this paper we describe initially hidden interrelations between pentagonal Sabin, Hosaka–Kimura and Loop–DeRose patches via the T-patch concept. Six-sided patches are considered using both base points and toric methods. Five- and six-sided cases are actually most important (beside triangular and rectangular patches) in geometric modeling and at the same time most convenient from the algebraic geometry point of view. Here we only outline results. Full proofs can be found in papers [4, 5, 8] of the authors. Relations between triangular Sabin, Hosaka–Kimura and Loop–DeRose patches are described in [5]. Algebraic version of convex combination patches is presented in [6].

§2. Notations and Definitions

In order to consider several variants of multisided patches defined via control points, we recall the most general concept of a rational patch.

Definition 1. *A* rational surface patch *is a mapping $F : D \to \mathbb{R}^k$ defined on a domain $D \subset \mathbb{R}^2$ by the formula*

$$F(\boldsymbol{t}) = \frac{\sum_{q\in\mathcal{I}} w_q \boldsymbol{p}_q f_q(\boldsymbol{t})}{\sum_{q\in\mathcal{I}} w_q f_q(\boldsymbol{t})}, \tag{1}$$

where polynomial functions f_q labeled by some set $\mathcal{I}$ are called basis functions, *the points $\boldsymbol{p}_q \in \mathbb{R}^k$ are* control points *and the numbers w_q are their* weights.

The Sabin and Hosaka–Kimura-like patches (see [2, 4, 5, 10, 11, 15]) behave like tensor product surfaces along their boundaries, and can be connected smoothly with surrounding rectangular patches. We denote a patch of this type by SHK_m^n, where m is a number of boundary curves and n is their degree.

Let $\boldsymbol{w}_0, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{m-1}$ be the vertices of a regular m-gon with a center $\boldsymbol{w}$ and let n be a fixed natural number. For each triangle with the vertices $\boldsymbol{w}, \boldsymbol{w}_s, \boldsymbol{w}_{s+1}$, $0 \le s \le m-1$, the points

$$\boldsymbol{w}_{ij}^s = \frac{i}{n}\boldsymbol{w} + \frac{j}{n}\boldsymbol{w}_{s+1} + \frac{n-i-j}{n}\boldsymbol{w}_s, \quad i,\, j \ge 0,\; i+j \le n, \tag{2}$$

linked together form a triangulation of an m-gon (see Fig. 1). The set of all its vertices is denoted by $\mathcal{L}_m^n$. It is convenient to enumerate them by the triples

$$(s,i,j), \quad 0 \le s \le m-1,\; 0 \le i \le n,\; 0 \le j \le n-i,$$

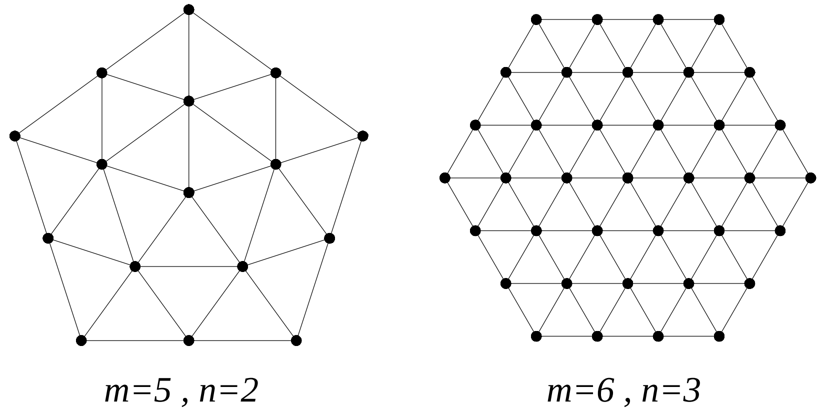

Fig. 1. Control point schemes of T-patches.

where triples $(s, i, n-i)$ and $(s+1, i, 0)$ are identified (the first index s is treated in a cyclic fashion). Indices s, i, j correspond to labeling in the formula (2). The graphs $\mathcal{L}_m^n$ define a combinatorial structure on the control point nets of T-patches.

The domain of some patches is a regular m-gon. In this case we assume linear functions have inward-oriented normal vectors. For $0 \le s \le m-1$, we write $\hat{l}_s$ for the function defining a line $\overline{\boldsymbol{w}_s \boldsymbol{w}_{s+1}}$. An intersection of the lines $\overline{\boldsymbol{w}_{s-1} \boldsymbol{w}_s}$ and $\overline{\boldsymbol{w}_{s+1} \boldsymbol{w}_{s+2}}$ is denoted by $\boldsymbol{b}_s$. By $\bar{l}_s$ we denote a function defining a line $\overline{\boldsymbol{b}_{s-1} \boldsymbol{b}_s}$.

Using the blowing up method (see [3, 13]) a 5-sided patch is defined via basis functions vanishing simultaneously at the two vertices $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ of the domain triangle $\Delta \boldsymbol{v}_0 \boldsymbol{v}_1 \boldsymbol{v}_2$. A 6-sided patch is defined via basis functions vanishing simultaneously at all three vertices. In these cases we denote by l_0, l_1, l_2 the barycentric coordinates of a point with respect to the triple $\boldsymbol{v}_0$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$. The infinite points corresponding to the lines $\overline{\boldsymbol{v}_0 \boldsymbol{v}_1}$ and $\overline{\boldsymbol{v}_0 \boldsymbol{v}_2}$ are denoted by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ respectively.

Definition 2. *A function f has a zero of multiplicity μ at a point $\boldsymbol{p}$ if it vanishes at $\boldsymbol{p}$ together with all partial derivatives up to the order $\mu - 1$. A point $\boldsymbol{p}$ is a* base point *of multiplicity μ of a rational map (1) if all basis functions f_q have a zero of multiplicity μ at $\boldsymbol{p}$.*

For a set of planar points $\mathcal{X} = \{\boldsymbol{p}_0, \dots, \boldsymbol{p}_s\}$, we denote by $\mathcal{P}(k, \mu, \mathcal{X})$ the linear space of polynomials of degree k which have zero of multiplicity μ at all points $\boldsymbol{p}_0, \dots, \boldsymbol{p}_s$.

§3. T-patches

Defining 5- and 6-sided T-patches, we set $\mathcal{I} = \mathcal{L}_5^n$ and $\mathcal{I} = \mathcal{L}_6^n$ respectively. Various type of basis functions for 5- and 6-sided patches are defined using the following scheme. Assume there are $m+1$ functions $h_0, h_1, \dots, h_{m-1}, h$ $(m = 5, 6)$ and positive numbers k_{ij}^n, $0 \le i \le n$, $0 \le j \le n-i$, satisfying the symmetry conditions $k_{ij}^n = k_{i,n-i-j}^n$. For $q = (s, i, j) \in \mathcal{L}_m^n$, the functions f_q

are defined by the formula

$$f_q = k_{ij}^n h_s^{n-i-j} h_{s+1}^j h^i. \tag{3}$$

Now we specify the functions h_s, h.

Definition 3. *Five-sided T_5^n-patch and six-sided T_6^n-patch are defined over a triangle via the formulas*

$$T_5^n : \begin{cases} h_0 = l_0^2, \quad h_1 = l_0 l_1 (l_0 + l_1), \quad h_2 = l_1^2 l_2, \quad h_3 = l_1 l_2^2, \\ h_4 = l_0 l_2 (l_0 + l_2), \quad h = l_0 l_1 l_2, \end{cases} \tag{4}$$

$$T_6^n : \begin{cases} h_0 = l_0^2 l_1, \quad h_1 = l_0 l_1^2, \quad h_2 = l_1^2 l_2, \quad h_3 = l_1 l_2^2, \\ h_4 = l_0 l_2^2, \quad h_5 = l_0^2 l_2, \quad h = l_0 l_1 l_2, \end{cases}$$

A five-sided $\widetilde{T}_5^n$-patch and six-sided $\widetilde{T}_6^n$-patch are defined over a regular pentagon and hexagon, respectively, via

$$\widetilde{T}_5^n : h_s = \hat{l}_{s+1} \hat{l}_{s+2}^2 \hat{l}_{s+3} \bar{l}_s, \quad s = 0, 1, \ldots, 4, \quad h = \prod_{s=0}^{4} \hat{l}_s, \tag{5}$$

$$\widetilde{T}_6^n : h_s = \hat{l}_{s+1} \hat{l}_{s+2}^2 \hat{l}_{s+3}^2 \hat{l}_{s+4}, \quad s = 0, 1, \ldots, 5, \quad h = \prod_{s=0}^{5} \hat{l}_s.$$

If $k_{0j}^n = \binom{n}{j}$, the boundary curves are Bézier curves of degree n. So the boundary curves are integral if their weights are equal to 1, though the patches are rational for any choice of the other weights.

From the designers point of view, it is convenient when a cyclic change of the input data does not change a patch as an image in $\mathbb{R}^3$. The $\widetilde{T}_5^n$- and $\widetilde{T}_6^n$-patches are symmetric by definition. The T_5^n- and T_6^n-patches are also symmetric (see [4, 5]). Their cyclic reparametrizations are given by the birational transformations of the domain triangle (Cremona transformations) of order 5 and 6 respectively.

Remark 4. *It is shown in [4] that T_5^n- and T_6^n-patches give the same class of the surfaces as $\widetilde{T}_5^n$- and $\widetilde{T}_6^n$-patches. So we actually have two kinds of parametrizations of 5- and 6-sided surfaces. The $\widetilde{T}_5^n$- and $\widetilde{T}_6^n$-patches can be easier handled using standard methods, since they are defined over traditional symmetric domain. The T_5^n- and T_6^n-patches are more convenient from the algebraic geometry point of view. For example, the latter approach gives the third type of parametrization of T-patches, which is suitable for an efficient plotting: T_5^n-patch can be represented as a collection of three Bézier patches of bidegree $(2n, 2n)$; T_6^n-patch can be represented as a collection of six Bézier patches of the same bidegree.*

The principles of blowing up and plotting T_5^n-patches are shown in Fig. 2.

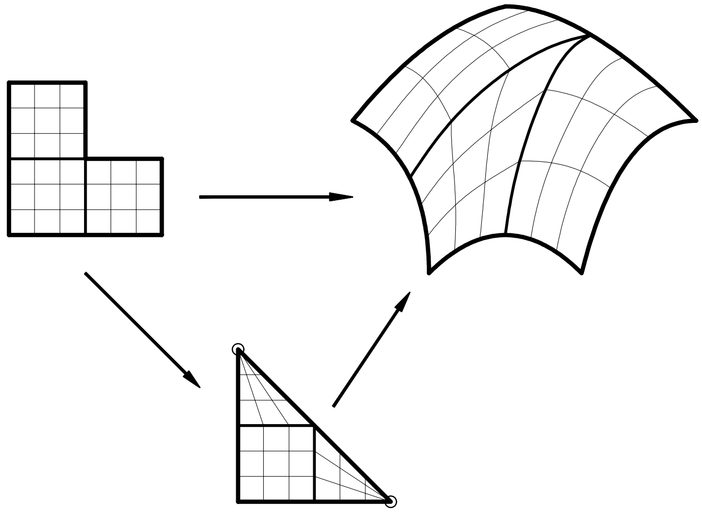

Fig. 2. Blowing up and plotting T_5^n-patches.

Lemma 5. *The basis functions of all T-patches are linearly independent. Moreover, the spaces $\mathcal{P}(3n, n, \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{e}_1, \boldsymbol{e}_2\})$, $\mathcal{P}(3n, n, \{\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2\})$, $\mathcal{P}(5n, 2n, \{\boldsymbol{b}_s, s = 0, \ldots, 4\})$ are generated by the basis functions of the T_5^n-, T_6^n- and $\widetilde{T}_5^n$-patches respectively.*

Lemma 5 enables us to establish algebraic relations between different surfaces.

§4. Interrelations Between Pentagonal Patches

We denote by U_5 a surface in $\mathbb{R}^5$ defined via equations $x_s - 1 + x_{s+2}x_{s+3} = 0$, $s = 1, \ldots, 5$ (the index s is treated in a cyclic fashion). This surface was introduced by Sabin [11]. A domain D for the pentagonal patches from [2, 11, 15] is a region in U_5 with $x_s \geq 0$, $s = 1, \ldots, 5$.

The interrelation mappings are defined via formula (1) assuming, that $\mathcal{I} = \mathcal{L}_5^1$, $k_{00}^1 = 1$ and all weights are equal to 1. We set for a simplicity $\boldsymbol{p}_s = \boldsymbol{p}_{s00}$, $\boldsymbol{p} = \boldsymbol{p}_{000}$, and denote $\boldsymbol{r}_0 = (0,1,1,1,0)$, $\boldsymbol{r}_1 = (0,0,1,1,1)$,..., $\boldsymbol{r}_4 = (1,1,1,0,0)$, $\boldsymbol{r} = (2/3, \ldots, 2/3)$ ($\boldsymbol{r}_s$ are the corner points of the Sabin domain). By $\boldsymbol{c}$ is denoted a barycenter of the triangle $\Delta \boldsymbol{v}_0\boldsymbol{v}_1\boldsymbol{v}_2$.

Definition 6. *Define rational mappings $H_5, \widetilde{H}_5 : \mathbb{R}^2 \to \mathbb{R}^2$ and $G_5, \widetilde{G}_5 : \mathbb{R}^2 \to \mathbb{R}^5$ as follows. H_5 and G_5 are defined fixing basis functions (4) with $k_{10}^1 = 5(\sqrt{5} - 1)/2$ and taking control points $\boldsymbol{p}_s = \boldsymbol{w}_s$, $\boldsymbol{p} = \boldsymbol{w}$ and $\boldsymbol{p}_s = \boldsymbol{r}_s$, $\boldsymbol{p} = \boldsymbol{r}$ respectively. $\widetilde{H}_5$ and $\widetilde{G}_5$ are defined fixing functions (5) with $k_{10}^1 = 3(\sqrt{5} + 1)/2$ and taking control points $\boldsymbol{p}_0 = \boldsymbol{v}_0$, $\boldsymbol{p}_1 = \boldsymbol{v}_1$, $\boldsymbol{p}_2 = \boldsymbol{v}_1$, $\boldsymbol{p}_3 = \boldsymbol{v}_2$, $\boldsymbol{p}_4 = \boldsymbol{v}_2$, $\boldsymbol{p} = \boldsymbol{c}$ and $\boldsymbol{p}_s = \boldsymbol{r}_s$, $\boldsymbol{p} = \boldsymbol{r}$ respectively.*

Theorem 7. *The mappings G_5 and $\widetilde{G}_5$ define parametrizations of the surface U_5. They map triangular and regular pentagonal domains respectively onto the Sabin domain. Moreover, $G_5 \circ \widetilde{H}_5 = \widetilde{G}_5$, $\widetilde{H}_5 \circ H_5 = \mathrm{id}$, $H_5 \circ \widetilde{H}_5 = \mathrm{id}$.*

Corollary 8. *Five-sided Sabin [11] and Hosaka-Kimura [2] patches can be represented as T_5^3- and T_5^4-patches respectively.*

Proof: The basis functions of the Sabin and Hosaka–Kimura patches are special polynomials of degree 12 and 20 respectively, which sum to 1 on U_5. Calculations (with MAPLE) give that their compositions with G_5 have the form Ag_r and Bg'_p respectively, where A, B are some rational functions, $g_r \in \mathcal{P}(9, 3, \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{e}_1, \boldsymbol{e}_2\})$, $g'_p \in \mathcal{P}(12, 4, \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{e}_1, \boldsymbol{e}_2\})$. Now the proof follows from Lemma 5. $\square$

Notice, SHK$_5^n$-patches in [4] can be represented as T_5^n-patches.

Let $\mathcal{I} = \{1, 2, 3, 4, 5\}$, $f_s = \hat{l}_s\hat{l}_{s+1}\hat{l}_{s+2}$, $s \in \mathcal{I}$, $\boldsymbol{p}_1 = (1, 0, 0, 0, 0), \ldots, \boldsymbol{p}_5 = (0, 0, 0, 0, 1)$. If all weights are equal to 1, the formula (1) defines a map $L : \mathbb{R}^2 \to \mathbb{R}^5$. An image of the map L is denoted by U'_5. The surface U'_5 is used in [9] for a definition of 5-sided S-patches. A domain of 5-sided S-patch is a regular pentagon.

Proposition 9. *A five-sided S-patch of depth n over regular pentagon can be represented as $\widetilde{T}_5^n$-patch.*

Proof: The basis functions of an S-patch of depth n (see [9]) are the compositions of the map L with the homogeneous polynomials of degree n. They are polynomials in $\mathcal{P}(3n, n, \{\boldsymbol{b}_s, s = 0, \ldots, 4\})$. Multiplication of the basis functions by C^n, where $C = 0$ defines a circle going through the points $\boldsymbol{b}_s$, does not change the patch. New polynomials are in $\mathcal{P}(5n, 2n, \{\boldsymbol{b}_s, s = 0, \ldots, 4\})$. Hence the original S-patch can be represented as $\widetilde{T}_5^n$-patch. $\square$

We have seen, that Sabin and Hosaka–Kimura patches can be considered as the patches over a regular domain or over the Sabin domain in U_5. Similarly, an S-patch can be considered over the domain in U'_5 with nonnegative coordinates. We call it a Loop–DeRose domain.

Proposition 10. *There exists a mapping $p : U_5 \to U'_5$, which maps the Sabin domain on to the Loop–DeRose domain and $L = p \circ \widetilde{G}_5$.*

Proof: Define p as a composition of the projective transformation

$$\begin{aligned} y_i &= x_i + x_{i+2} + x_{i+4} - a(x_{i+1} + x_{i+3}) + a - 2 \quad i = 1, \ldots, 5, \\ y_0 &= (3 - 2a)(x_1 + x_2 + x_3 + x_4 + x_5 + 2a). \end{aligned}$$

$(a = (\sqrt{5} + 1)/2)$ with the projection from a point on U_5: $p : (y_0, \ldots, y_5) \mapsto (y_1/y_0, \ldots, y_5/y_0)$. $\square$

From the algebraic geometry point of view, the surface U_5 is more universal in the algebraic constructions than U'_5. As a confirmation of this property, we have that SHK$_5^2$- and SHK$_5^3$-patches in [10] can be represented only as T_5^5- and T_5^6-patches, respectively.

Remark 11. *The surface U_5 plays a key role in the theory of 5-sided patches. It would be interesting to investigate deeper geometric properties of U_5. Here are two of them: 1) as a surface in $\mathbb{R}P^5$ it contains 10 lines; 2) exactly 5 conics go through a generic point of U_5.*

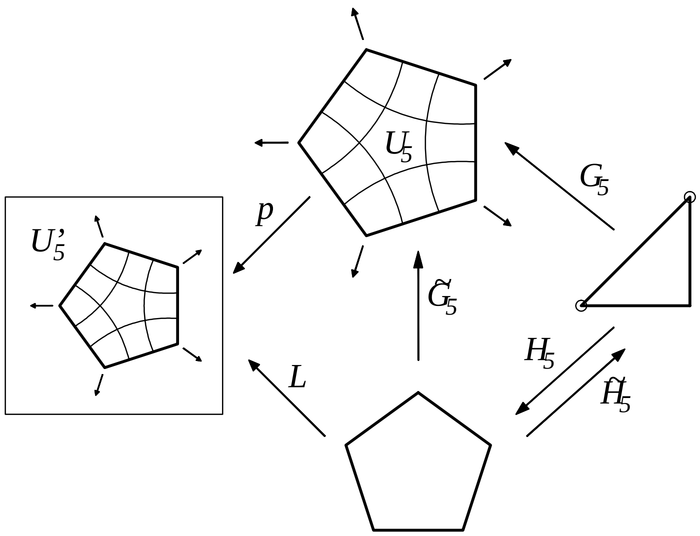

Fig. 2. Interrelations of 5-sided patches.

A schematic of the interrelations between 5-sided patches is shown in Fig. 3.

§5. Toric Patches

Here we present several results about toric patches obtained in [8]. Some details can be found also in [16].

Consider a lattice $\mathbb{Z}^2$ of points with integer coordinates in the real affine plane $\mathbb{R}^2$. We call a convex polygon $\Delta \subset \mathbb{R}^2$ a lattice polygon if its vertices are in the lattice $\mathbb{Z}^2$. Edges δ_i of Δ define lines $h_i(\boldsymbol{t}) = \langle \boldsymbol{n}_i, \boldsymbol{t} \rangle + a_i = 0$, with inward oriented normal vectors $\boldsymbol{n}_i$, $i = 1, \ldots, r$. We choose $\boldsymbol{n}_i$ to be primitive lattice vectors, i.e. the shortest vectors with integer coordinates in the given direction.

Denote by $\widehat{\Delta} = \Delta \cap \mathbb{Z}^k$ a set of lattice points of the polygon Δ. It is easy to see that $h_i(\boldsymbol{m})$ is a non-negative integer for all $i = 1, \ldots, r$ and $\boldsymbol{m} \in \widehat{\Delta}$.

Definition 12. *A toric patch associated with a lattice polygon Δ is a rational patch T_Δ with a domain $D = \Delta$ and basis functions*

$$f_{\boldsymbol{m}} = c_{\boldsymbol{m}} h_1^{h_1(\boldsymbol{m})} h_2^{h_2(\boldsymbol{m})} \cdots h_r^{h_r(\boldsymbol{m})} \tag{6}$$

indexed by lattice points $\boldsymbol{m} \in \widehat{\Delta}$. Here $c_m > 0$ are some coefficients which may vary from case to case.

Example 13. *Bézier surfaces and the Warren hexagon [13] are toric:*

1) *If Δ is a triangle with vertices $(0,0)$, $(d,0)$ and $(0,d)$, then T_Δ with $c_{(i,j)} = d!/(i!j!(d-i-j)!)$ is exactly a rational Bézier triangle of degree d, which parameter domain is scaled d times.*
2) *If Δ is a rectangle with four vertices $(0,0)$, $(d_1,0)$, (d_1,d_2) and $(0,d_2)$, then T_Δ with coefficients $c_{(i,j)} = \binom{d_1}{i}\binom{d_2}{j}$ is a tensor product surface of bidegree (d_1,d_2) with a scaled parameter domain $[0,d_1] \times [0,d_2]$.*

3) *Let Δ be a hexagon Δ_6 (see Fig. 4) then T_Δ with appropriate coefficients $c_{\boldsymbol{m}}$ is the Warren 6-sided patch denoted by $\widetilde{T}_6^1$ in Section 3.*

Toric patches have similar properties as Bézier surfaces. They are *affine invariant*, and have *convex hull property.* Every edge δ_i of the lattice polygon Δ corresponds to a boundary rational Bézier curve with control points $\boldsymbol{m} \in \widehat{\delta_i} = \delta_i \cap \mathbb{Z}$. In particular, its degree is equal to an 'integer length' of the edge δ_i.

The following property is in some sense similar to the affine invariance of the domain for Bézier surfaces.

Lemma 14. *(Unimodular invariance of the domain.) Let two lattice polygons be related via some affine unimodular transformation $L(\Delta) = \Delta'$ (i.e. L preserves the lattice $\mathbb{Z}^2$). Then toric patches T_Δ and $T_{\Delta'}$ with the same control points and weights are just reparametrizations of each other: $T_\Delta = T_{\Delta'} \circ L$.*

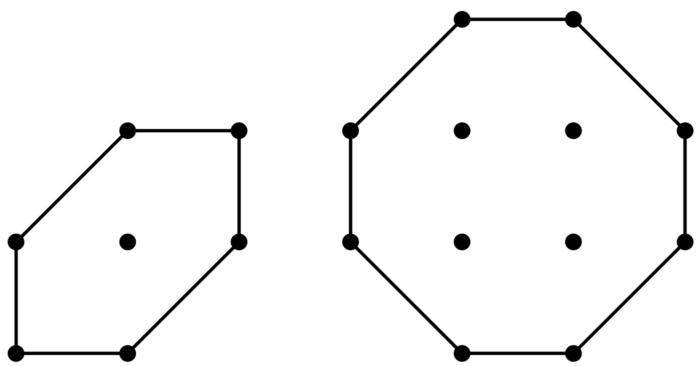

Fig. 4. Examples of lattice polygons.

In Fig. 4 we see a lattice hexagon Δ_6 and an octagon Δ_8. Since they have 6- and 4-sided symmetry, corresponding toric patches T_Δ for $\Delta = \Delta_6, \Delta_8$ have the same symmetry.

Corollary 15. *For $m = 3, \ldots, 8$, the only symmetric (in the sense of Section 3) toric patches may be 3-, 4- and 6-sided, for example, Bézier triangles, tensor product surfaces of degree (d, d) and the Warren hexagon $T_{\Delta_6} = T_6^1$. In particular, the 5-sided T_5^n-patch cannot be toric.*

Proof: These numbers correspond to cyclic subgroups in the group $\mathrm{SL}_2(\mathbb{Z})$ of unimodular linear transformations of the lattice $\mathbb{Z}$. $\square$

It is clear that an affine unimodular transformation L preserves area, since $\det L = \pm 1$. It is convenient to use so-called *normalized area* which is twice as large as the usual area in $\mathbb{R}^2$, since then area(Δ) is always integer for lattice polygons Δ. The following result is well-known in the theory of toric varieties (see [14] for an elementary proof).

Theorem 16. *The* implicit degree $\deg T_\Delta$ *of a toric patch* T_Δ *does not exceed* area(Δ). *It is equal to* area(Δ) *when the control points are in general position.*

For example, $\deg T_{\Delta_6} = 6$ *and* $\deg T_{\Delta_8} = 14$ *(see Fig. 4).*

Consider now the more general parametrization of a toric patch $F' : \mathbb{R}^r_{\geq 0} \to \mathbb{R}^k$ defined as in (1) via basis functions

$$f'_{\boldsymbol{m}}(u_1, u_2, \ldots, u_r) = c_{\boldsymbol{m}} u_1^{h_1(\boldsymbol{m})} u_2^{h_2(\boldsymbol{m})} \cdots u_r^{h_r(\boldsymbol{m})}, \quad \boldsymbol{m} \in \Delta.$$

Definition 12 is obtained substituting variables u_i by affine forms h_i. Although the domain $\mathbb{R}^r_{\geq 0}$ has dimension r, the image of F' is 2-dimensional in all cases (cf. [8]). Hence, using various substitutions, one can get different interesting parametrizations of the same patch. The simplest piecewise substitution

$$\Phi_i(u, v) = F'(\underbrace{1, \ldots, 1}_{i-1}, u, v, 1, \ldots, 1), \quad i = 1, \ldots, r-1$$
$$\Phi_r(u, v) = F'(v, 1, \ldots, 1, u), \quad 0 \leq u, v \leq 1,$$

defines a subdivision of the toric patch into r tensor product pieces. This directly generalizes the Warren hexagon subdivision [13].

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