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Inverse method for modal logic S4

Adomas BIRŠTUNAS, Stanislovas NORGĖLA (VU)
e-mail: adomo@takas.lt, stasys.norgela@maf.vu.lt

For the first time an inverse method was defined by S. Maslov in [4] for the classical predicate logic. A.Voronkov defined an inverse method for some modal logics in [5],[6]. We will present an inverse method for propositional modal logic *S4*. We refer to structure of the inverse method defined by S.Katrechko in [1].

We will denote the propositional variables of classical logic by $a, b, c, p, q, p_1, q_1, \ldots$ and the literals by $l, l_1, \ldots$. In this paper we will consider the sequents of modal logic *S4* having the shape of $D_1, \ldots, D_n, \neg p \vdash$, where $D_1, \ldots, D_n$ are formulas having one of the shape: $\Box(a \vee b)$, $\Box(a \vee \neg b)$, $\Box(\neg a \vee \neg b)$, $\Box(a \vee \neg b \vee \neg c)$, $\Box(\neg a \vee \Box b)$, $\Box(a \vee \Diamond\neg b)$. G. Mints in [2] proved, that for every sequent $\vdash F$ of modal logic *S4* we can find a sequent having the shape of $D_1, \ldots, D_n, \neg p \vdash$ such that it is proved if and only if a sequent $\vdash F$ is proved. Since we consider only the sequents having such shape , we can use only these rules of a sequent calculus:

$$(\vee\vdash)\ \frac{F, \Gamma \vdash \qquad G, \Gamma \vdash}{F \vee G \vdash} \qquad (\Diamond\vdash)\ \frac{F, \Box\Gamma \vdash}{\Diamond F, \Box\Gamma, \Delta^0 \vdash} \qquad (\Box\vdash)\frac{\Box F, F, \Gamma \vdash}{\Box F, \Gamma \vdash},$$

where $F,\ G$ are formulas, Γ, Δ $-$ finite (may be empty) sequences of formulas, $\Box\Gamma$ $-$ finite sequence of formulas, in which every formula begins with modal operator $\Box$, Δ^0 $-$ finite sequence of formulas, there none of formulas begins with modal operator $\Box$.

DEFINITION 1. A sequent calculus *S4M* with the axioms $\neg l, l, \Gamma \ \vdash$, $\neg l, \Box l, \Gamma \ \vdash$, $\Diamond\neg l, \Box l, \Gamma \vdash$, where l is a literal of classical logic, and the rules:

$$(\Box\vee\vdash)\ \frac{\Box(F \vee G), F, \Gamma \vdash \qquad \Box(F \vee G), G, \Gamma \vdash}{\Box(F \vee G), \Gamma \vdash} \qquad (\Diamond\vdash)\ \frac{F, \Box\Gamma \vdash}{\Diamond F, \Box\Gamma, \Delta^0 \vdash}$$

we will called *modified sequent calculus* for propositional modal logic *S4*.

Lemma 1. *A sequent $D_1, \ldots, D_n, \neg p \vdash$ is proved in S4 if and only if it is proved in modified sequent calculus S4M.*

We will define a numeral notation. We investigate the sequents $\neg p, D_2, \ldots, D_n \vdash$, where $D_i = \Box(d_{i,1} \vee \alpha_i d_{i,2})$ or $D_i = \Box(d_{i,1} \vee d_{i,2} \vee d_{i,3})$, $i = 2, 3, \ldots, n$ and $d_{i,j}$ are literals; $\alpha_i \in \{\emptyset, \Box, \Diamond\}$. We mark a formula D_i by the number i, and a literal $d_{i,j}$ $-$ by the number i with index j, i.e., i_j. So, we mark by $\alpha_i i_j$ a literal with a modal operator, i.e., $\alpha_i d_{i,j}$. We will not use the modal operator $\Box$ in front of number without index in a numeral notation, because we use only the formulas, whose begin with modal operator $\Box$.

DEFINITION 2. A set of the numbers with indexes (may be with the modal operators), we will call *collection*. A collection consisted of two numbers with indexes whose define a complementary pair ($l, \neg l$; $\neg l, \Box l$; $\Diamond\neg l, \Box l$), we will call *an axiom*. A collection will be called *favorable collection* if it is an axiom or it is derived from other favorable collections by using one of two rules:

1. **Rule B.** If an initial sequent contains a formula $D_i = \Box(d_{i,1} \vee \alpha_i d_{i,2})$ (or $D_i = \Box(d_{i,1} \vee d_{i,2} \vee d_{i,3})$) and if the collections $[A, i_1]$, $[B, \alpha i_2]$ (and $[\Gamma', i_3]$) are favorable collections, then $[A, B]$ ($[A, B, \Gamma']$) will be favorable collections also. The sets A, B, Γ' denote the numbers with indexes which may begin with modal operators and $\alpha_i \in \{\emptyset, \Box, \Diamond\}$,

2. **Rule C.** If a collection $[\Box A, i_2]$ is a favorable collection, then $[\Box A, \Diamond i_2]$ is favorable collection also. We denote a set of numbers with indexes in which all members begin with modal operator $\Box$.

DEFINITION 3. A tree (graph) will be called a deduction tree of a sequent $\neg p, D_2, ..., D_n \vdash$, if all nodes, labelled by the collections, the following conditions hold: a) the root of tree is labelled by a collection correponding to an initial sequent, b) every leaf of tree is labelled by a collection corresponding to a sequent containing a complementary pair, c) if a sequent S corresponds to a node and the sequents $S_1, ..., S_m$ correspond to the nodes placed immediately above of the considered node, then the sequent S is derivable from $S_1, ..., S_m$ by applying one of the rules ($\Box\vee \vdash$) or ($\Diamond \vdash$).

DEFINITION 4. A deduction tree of a sequent $\neg p, D_2, ..., D_n$ will be called *a cleaned deduction tree* , if all nodes are lebelled by the favorable collections and the following conditions hold: a) the root is labelled by an empty collection or by the collection $[1_0]$, b) every leaf is labelled by an axiom, c) if a node is labelled by a collection R and the nodes placed immediately above of the considered node are labelled by the collections $R_1, ..., R_m$, then the collection R is derivable from the collections $R_1, ..., R_m$ by applying one of the rules $B,\ C$.

Lemma 2. *A sequent $\neg p, D_2, ..., D_n \vdash$ is provable in modified sequent calculus S4 if and only if there exists a cleaned deduction tree of this sequent.*

Proof. It is evident that a sequent $\neg p, D_2, ..., D_n \vdash$ is derivable in modified sequent calculus if and only if there exists a deduction tree of this sequent. A deduction tree is the deduction tree of a sequent just written in a numeral notation. We denote a literal $\neg p$ by the number 1_0. We will show that for every deduction tree we can find a cleaned deduction tree. For this reason we use the following cleaning algorithm:

1) We delete all numbers without indexes in every leaf of a deduction tree and all numbers with indexes which are not used in an axiom. After this operation, there will be in the leaves only the collections containing only two members,

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2) Going top-down, we delete all numbers without indexes from every node and all numbers with indexes, which are not used in any node placed immediately above of the considered node. After this operation, every node will satisfy the requirement *c* of the Definition 4.

Thus, we delete all numbers with indexes, therefore an empty collection $[\ ]$ or the collection $[1_0]$ will leave in the root of a deduction tree (the requirement *a* of the Definition 4). We will show that for every cleaned deduction tree we can find a deduction tree in a sequent calculus. Notice that the formulas $D_2, \ldots, D_n$ can be rewritten in every node of a deduction tree of modified sequent calculus of the sequent $\neg p, D_2, \ldots, D_n \vdash$, because there is the modal operator $\Box$ in front of every such formulas. In addition, the unused literals can be deleted. Thus, if the root contains an empty collection $[\ ]$ in a deduction tree, then we add 1_0 (this number represents the literal $\neg p$) to the root and all numbers without indexes $2, 3, \ldots, n$ to every node of tree. If a root contains the collection $[1_0]$ in a cleaned deduction tree, then we simply add all numbers without indexes $2, 3, \ldots, n$ to every node of tree. Lemma is proved.

DEFINITION 5. The sequence of a favorable collections will be called *a derivation of a sequent* $\Gamma \vdash$ *in S4 by inverse method* if it the following conditions hold: every collection in this sequence is an axiom, or it is get from the collections (going to the left of this collection) by applying one of rules (*B* or *C*), and ends with an empty collection $[\ ]$ or the collection $[1_0]$.

Theorem 1. (Theorem about a completeness of the inverse method) *For any sequent* $\vdash F$ *we can find a sequent* $\neg p, D_2, \ldots, D_n \vdash$ *such that* $\vdash F$ *is derivable in S4 if and only if the sequent* $\neg p, D_2, \ldots, D_n \vdash$ *is derivable in S4 by the inverse method.*

Proof. A sequent $\vdash F$ is provable if and only if a corresponding sequent, having the shape $\neg p, D_2, \ldots, D_n \vdash$ is provable in the sequent calculus *S4* (see [2]). According to Lemma 1, we can use a modified sequent calculus *S4M* in which two rules $(\Box\vee \vdash)$, $(\Diamond \vdash)$ are only used. If a sequent $\neg p, D_2, \ldots, D_n \vdash$ is proved in a modified sequent calculus, then, according to Lemma 2, we can find a cleaned deduction tree. If we rewrite this cleaned deduction tree of the form of a sequence, then we will get a proof of a sequent in inverse method.

Assume we have a sequence of collections, in which every collection is an axiom or it is obtained from collections by using one of the rules (*B* or *C*), this sequence ends with an empty collection $[\ ]$ or the collection $[1_0]$. We can construct a cleaned deduction tree by using this sequence. We assign an empty collection $[\ ]$ or the collection $[1_0]$ to the root of tree. If a node is labelled by a collection *C*, then the nodes placed immediately above of the considered node are labelled by the collections from which *C* is obtained by applying one of the rules *B*, *C*. If a node is labelled by an axiom, then it will be a leave of tree. If we can find a cleaned tree of derivation of a sequent, then, according to Lemma 2, this sequent is provable in a modified sequent calculus *S4M*. Theorem is proved.

In other words, an inverse method is a calculus of favorable collections, in which every favorable collection correspond to some provable sequent. So, if we want to prove a sequent using an inverse method, we have to do such steps: 1) find a deductive equivalent sequent having the shape of $\neg p, D_2, \ldots, D_n \vdash$, 2) identify the axioms, 3) show that the collection $[\ ]$ or the collection $[1_0]$ is a favorable collection for a considered sequent. We will consider below the calculus *S4H*.

DEFINITION 6. A sequent calculus with the axioms $\neg l, l, \Gamma \vdash$, where *l* is a literal of classical logic, and the rules:

$$(\vee\vee\vdash)\ \frac{l_1, \Gamma \vdash, \ldots, l_n, \Gamma \vdash}{\Box(l_1 \vee \ldots \vee l_n), \Gamma \vdash} \qquad (\vee\Box\vdash)\ \frac{l_1, \Box(l_1 \vee \Box l_2), \Gamma \vdash \qquad \Box l_2, l_2, \Gamma \vdash}{\Box(l_1 \vee \Box l_2), \Gamma \vdash}$$

$$(\vee\Diamond\vdash)\ \frac{\Box l_1, l_1, \Gamma \vdash \qquad l_2, \Box(l_1 \vee \Diamond l_2), \Gamma^* \vdash}{\Box(l_1 \vee \Diamond l_2), \Gamma \vdash}$$

we will called sequent calculus *S4H* for propositional modal logic *S4*. Γ^* is the list of all formulas of Γ, beginning with $\Box$.

Lemma 3. *Whatever is a formula F of a propositional modal logic, we can find a sequent having the shape of* $\neg p, D_2, \ldots, D_n \vdash$*, such that a sequent* $\vdash F$ *is derivable in S4 if and only if* $\neg p, D_2, \ldots, D_n \vdash$ *is derivable in S4H.*

Lemma 3 is proved in [3]. We will use the more general axioms $\neg l, l, \Gamma \vdash$, $\neg l, \Box l, \Gamma \vdash$, $\Diamond\neg l, \Box l, \Gamma \vdash$, and we can simplify the rules without loss of generality:

$$(\vee\vee\vdash)\ \frac{p_1, \Gamma \vdash \ldots p_n, \Gamma \vdash \qquad \neg q_1, \Gamma \vdash \ldots \neg q_m, \Gamma \vdash}{\Box(p_1 \vee \ldots \vee p_n \vee \neg q_1 \vee \ldots \vee \neg q_m), \Gamma \vdash}$$

$$(\vee\Box\vdash)\ \frac{\neg p, \Box(\neg p \vee \Box q), \Gamma \vdash \qquad \Box q, \Box(\neg p \vee \Box q), \Gamma \vdash}{\Box(\neg p \vee \Box q), \Gamma \vdash}$$

$$(\vee\Diamond\vdash)\ \frac{\Box p, \Box(p \vee \Diamond\neg q), \Gamma \vdash \qquad \neg q, \Box(p \vee \Diamond\neg q), \Gamma^* \vdash}{\Box(p \vee \Diamond\neg q), \Gamma \vdash}$$

We will use the same numeral notation, except for the formula $D_i = \Box(a \vee \Diamond\neg b)$. We will denote the formula $\Diamond\neg b$ by $\Diamond i_2$ and a by $\Box i_1$. Now we will define slightly a different inverse method.

DEFINITION 7. A set of numbers with indexes (may be with modal operators) will be called *a collection.* A collection consisted of two numbers with indexes, which defines a complementary pair, will be called *an axiom.* A collection will be called *a favorable collection* if it is obtained from other favorable collections by using the following rule:

Rule B^***.** If an initial sequent contains a formula $D_i = \Box(d_{i,1} \vee \alpha_i d_{i,2})$ (or $D_i = \Box(d_{i,1} \vee d_{i,2} \vee d_{i,3})$), that is the collections $[A, i_1]$, $[B, \alpha_i i_2]$ (or $[\Gamma, i_3]$) are the favorable collections, then $[A, B]$ ($[A, B, \Gamma]$) will be a favorable collection also. If $\alpha_i = \Diamond$, then all numbers with indexes in *B* must begin with the modal operator $\Box$. We denote by A, B, Γ the sets of numbers with indexes, which may begin with the modal operators also and $\alpha_i \in \{\emptyset, \Box, \Diamond\}$.

DEFINITION 8. A sequence of favorable collections, in which every collection is an axiom or it is obtained from collections by using the rule B^*, and ends with an empty collection or is a collection $[1_0]$, will be called *a derivation of a sequent of modal logic S4H by inverse method.*

Theorem 2. *A sequent of propositional modal logic* $\vdash F$ *is derivable in S4H if and only if it is derivable by inverse method.*

Proof. We can define a tree of derivation and a cleaned tree of derivation similarly to Definitions 3 and 4. In this case, we replace the applications of rules $(\Box\vee\vdash)$, $(\Diamond\vdash)$ by the applications of rules $(\vee\vee\vdash$, $(\vee\Box\vdash)$, $(\vee\Diamond\vdash)$, and the applications of rules *B*,*C* by the applications of B^*. A cleaned tree of derivation will satisfy the requirements of Lemma 2. A relation between a cleaned tree of derivation and a derivation of a sequent by inverse method is a very similar to a relation described in Theorem 1. Theorem is proved.

Theorem 3. *If a set of axioms contains the both axioms* $[\alpha_1, \alpha_2]$ *and* $[\alpha_1, \Box\alpha_2]$*, then an axiom* $[\alpha_1, \alpha_2]$ *can be deleted.*

Theorem 4. *If a set of axioms contain an axiom* $[\alpha_1, \alpha_2]$ *such that: a)* α_1 *(or* α_2*) is a number with index correponding to a literal* l_1 *(or* l_2, l_3*) in a formula of the form* $D = \Box(l_1 \vee l_2)$ *(or* $D = \Box(l_1 \vee l_2 \vee l_3)$*), b) any axiom does not contain a number with index correspoding to a literal* l_2 *or* l_3*. Then such axiom can be deleted also.*

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Atvirkštinis metodas teiginių modalumo logikai S4

A. Birštunas, S. Norgėla

Nagrinėjami du teiginių modalumo logikos *S4* sekvenciniai skaičiavimai ir atitinkami atvirkštinio metodo variantai. Taikomi tam tikro pavidalo formulėms.