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Transformations of formulae of hybrid logic

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Abstract. This paper describes a procedure to transform formulae of hybrid logic $\mathcal{H}(@)$ over transitive and reflexive frames into their clausal form.

Keywords: hybrid logic, clause.

Introduction

In propositional logic resolution calculus works on a set of clauses. However the well-known methods for transforming propositional formulae to sets of clauses can not be directly applied in modal nor hybrid logics – these non-classical logics need a different approach.

In [4, 5] Mints et al describe transformation of formulae into their clausal form for modal logics *S4* and *S5*. A modal literal is defined as formula of the form l, $\Box l$ or $\Diamond l$, where l is a propositional literal. A modal clause is a disjunction of modal literals. In [4] author proves that for every modal logic formula F there exist clauses $D_1, \ldots, D_n$ and a propositional literal l such that sequent $\vdash F$ is derivable in sequent calculus *S4* (and, accordingly, *S5*) if and only if sequent $\Box D_1, \ldots, \Box D_n, l \vdash$ is derivable. This transformation is the basis for the resolution calculus for modal logic *S4* presented in [5]. F is a valid formula if and only if an empty clause is derivable from the set $\{\Box D_1, \ldots, \Box D_n, l\}$.

In this paper we aim to describe a similar transformation for formulae of hybrid logic $\mathcal{H}(@)$ over transitive and reflexive frames. Throughout the paper we will refer to this logic as $\mathcal{H}^{\mathcal{TR}}(@)$. In Section 1 we prove a theorem about subformula replacement in formulae of $\mathcal{H}^{\mathcal{TR}}(@)$ and use this result to describe transformation of formulae in Section 2. To prove things about $\mathcal{H}^{\mathcal{TR}}(@)$ we use the sequent calculus proposed by Braüner in [3] along with two additional rules that make use of the reflexivity and transitivity frame properties of the logic under discussion:

$$\frac{@_a\Diamond a, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ (\text{Refl}) \qquad \frac{@_a\Diamond c, \Gamma \vdash \Delta}{@_a\Diamond b, @_b\Diamond c, \Gamma \vdash \Delta}\ (\text{Trans})$$

For an introduction of hybrid logic and it's properties see [1] and [2].

1 Subformula replacement in $\mathcal{H}^{\mathcal{TR}}(@)$

It is true in propositional logic that if we replace subformula A of some formula $F(A)$ with an equivalent formula B, then $F(A)$ is equivalent to $F(B)$. To put it more briefly,

$(A \equiv B) \to (F(A) \equiv F(B))$. However this statement does not hold in modal nor hybrid logic. In [4] Mints proved that in modal logic *S4* $\Box(A \equiv B) \to (F(A) \equiv F(B))$. We will prove a similar result for $\mathcal{H}^{\mathcal{TR}}(@)$ by first introducing a notion of a *binding nominal*:

Definition 1. A binding nominal of a subformula A in formula $F(A)$ is nominal i, such that A is in the scope of operator $@_i$, and of all such operators $@_i$ has the maximal depth.

For instance, in formula $@_i(\Diamond A \wedge @_j(\Box B \to C))$ subformula A is bound by nominal i whereas subformulae B and C are bound by nominal j.

Theorem 1. *Let F be a formula of $\mathcal{H}^{\mathcal{TR}}(@)$ and let A be some subformula of F bound by nominal i. Then $@_i\Box(A \equiv B)$ implies $F(A) \equiv F(B)$.*

Proof. We will prove by constructing a derivation tree that the following sequent is derivable in sequent calculus of $\mathcal{H}^{\mathcal{TR}}(@)$:

$$@_i\Box\big((A \to B) \wedge (B \to A)\big) \vdash @_s\big((F(A) \to F(B)\big) \wedge \big(F(B) \to F(A)\big)\big)$$

Here s is a new nominal. We will write Γ for $@_i\Box((A \to B) \wedge (B \to A))$ in sequents when it is not used by any rule in order to save space.

After applying rules $(\vdash \wedge)$ and $(\vdash\to)$ in the first two steps the derivation tree branches as follows:

$$\dfrac{\dfrac{\dfrac{\dots}{\Gamma, @_sF(A) \vdash @_sF(B)}}{\Gamma \vdash @_s(F(A) \to F(B))}\,(\vdash\to) \qquad \dfrac{\dfrac{\dots}{\Gamma, @_sF(B) \vdash @_sF(A)}}{\Gamma \vdash @_s(F(B) \to F(A))}\,(\vdash\to)}{\Gamma \vdash @_s((F(A) \to F(B)) \wedge (F(B) \to F(A)))}\,(\vdash \wedge)$$

The two branches are symmetric with respect to interchanging A with B, therefore we will only show derivation of the left branch. It is continued according to the main operation of formulae in the sequent using these rules:

$(\neg)$ $F = \neg G(A)$:

$$\dfrac{\dfrac{\dfrac{\dots}{\Gamma, @_sG(B) \vdash @_sG(A)}}{\Gamma \vdash @_s\neg G(B), @_sG(A)}\,(\vdash \neg)}{\Gamma, @_s\neg G(A) \vdash @_s\neg G(B)}\,(\neg \vdash)$$

$(\wedge)$ $F = (G(A) \wedge H)$:

$$\dfrac{\dfrac{\Gamma, @_sG(A), @_sH \vdash @_sH \qquad \dfrac{\dfrac{\dots}{\Gamma, @_sG(A) \vdash @_sG(B)}}{\Gamma, @_sG(A), @_sH \vdash @_sG(B)}\,(\text{Simp}\vdash)}{\Gamma, @_sG(A), @_sH \vdash @_s(G(B) \wedge H)}\,(\vdash \wedge)}{\Gamma, @_s(G(A) \wedge H) \vdash @_s(G(B) \wedge H)}\,(\wedge \vdash)$$

$(\Box)$ $F = \Box G(A)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dots}{\Gamma, @_tG(A) \vdash @_tG(B)}}{\Gamma, @_s\Box G(A), @_tG(A), @_s\Diamond t \vdash @_tG(B)}\,(\text{Simp}\vdash)}{\Gamma, @_s\Box G(A), @_s\Diamond t \vdash @_tG(B)}\,(\Box \vdash)}{\Gamma, @_s\Box G(A) \vdash @_s\Box G(B)}\,(\vdash \Box)$$

(@) $F = @_t G(A)$:

$$\cfrac{\cfrac{\cfrac{\dots}{\Gamma, @_t G(A) \vdash @_t G(B)}}{\Gamma, @_t G(A) \vdash @_s @_t G(B)}\ (\vdash:)}{\Gamma, @_s @_t G(A) \vdash @_s @_t G(B)}\ (:\vdash)$$

We don't give separate rules for $\vee$, $\rightarrow$ and $\Diamond$ as $G \vee H \equiv \neg(\neg G \wedge \neg H)$, $G \rightarrow H \equiv \neg(G \wedge \neg H)$ and $\Diamond G \equiv \neg\Box\neg G$. The derivation is continued unambiguously by applying one of these rules, and only a single branch is left open each time – the one with subformulae A and B. Since subformula A is bound by nominal i we will encounter operator $@_i$ and by definition of binding nominal this will be the last time the (@) rule is applied. At that point all formulae in the sequent will have the $@_i$ prefix and we will apply (Refl) rule to get:

$$\cfrac{\cfrac{\dots}{@_i\Box((A \rightarrow B) \wedge (B \rightarrow A)), @_i\Diamond i, @_i G(A) \vdash @_i G(B)}}{@_i\Box((A \rightarrow B) \wedge (B \rightarrow A)), @_i G(A) \vdash @_i G(B)}\ (\mathrm{Refl})$$

The sequent is now in the form $\Gamma, @_i\Diamond x, @_x G(A) \vdash @_x G(B)$ and this form will be maintained in the rest of the derivation. The rules for $\neg$ and $\wedge$ do not change prefixes of formulae and we will not encounter the @ operator. For the $\Box$ operator we will use a slightly different rule:

$$\cfrac{\cfrac{\cfrac{\cfrac{\dots}{\Gamma, @_i\Diamond y, @_y G(A) \vdash @_y G(B)}}{\Gamma, @_i\Diamond x, @_y G(A), @_x\Diamond y \vdash @_y G(B)}\ (\mathrm{Trans})}{\Gamma, @_i\Diamond x, @_x\Box G(A), @_x\Diamond y \vdash @_y G(B)}\ (\Box\vdash, \mathrm{Simp})}{\Gamma, @_i\Diamond x, @_x\Box G(A) \vdash @_x\Box G(B)}\ (\vdash\Box)$$

Since formula only has a finite number of operators, subformula A (and B) will be reached and we will complete the derivation as follows:

$$\cfrac{\cfrac{\cfrac{@_x A \vdash @_x A \quad @_x B, @_x A \vdash @_x B}{@_x(A \rightarrow B), @_x A \vdash @_x B}\ (\rightarrow\vdash)}{@_x((A \rightarrow B) \wedge (B \rightarrow A)), @_x A \vdash @_x B}\ (\wedge\vdash, \mathrm{Simp}\vdash)}{@_i\Box((A \rightarrow B) \wedge (B \rightarrow A)), @_i\Diamond x, @_x A \vdash @_x B}\ (\Box\vdash, \mathrm{Simp}\vdash)$$

2 Transformation

In this section we describe how formulae of $\mathcal{H}^{\mathcal{TR}}(@)$ can be transformed to sets of clauses using Theorem 1. A *literal* of hybrid logic $\mathcal{H}^{\mathcal{TR}}(@)$ is a formula of the form $l, \Box l, \Diamond l$ or $@_i l$ where l is a proposition, a nominal or a negation of these, and i is a nominal. A *clause* of hybrid logic is a formula of the form L, $\Box L$ or $@_i L$ where L is a disjunction of hybrid literals.

Formula F is valid if and only if the sequent $\vdash @_s F$ is derivable in sequent calculus $\mathcal{H}^{\mathcal{TR}}(@)$. We will prove the following statement.

Theorem 2. *Let F be a formula of $\mathcal{H}^{\mathcal{TR}}(@)$, A be some subformula of F bound by nominal i, and p be a propositional variable not in F. Then $\Gamma \vdash @_s F(A)$ is derivable if and only if $\Gamma, @_i\Box(p \equiv A) \vdash @_s F(p)$ is derivable.*

Proof. Let us first consider the case that $\mathit{\Gamma} \vdash @_s F(A)$ is derivable. Then we apply the cut rule in the first step to get:

$$\dfrac{\dfrac{\text{our premise}}{\mathit{\Gamma} \vdash @_s F(A)} \qquad \dfrac{\text{derivable by theorem 1}}{@_s F(A), @_i \Box(p \equiv A) \vdash @_s F(p)}}{\mathit{\Gamma}, @_i \Box(p \equiv A) \vdash @_s F(p)}$$

Now let us say that $\mathit{\Gamma}, @_i \Box(p \equiv A) \vdash @_s F(p)$ is derivable. Then there exists a finite derivation tree $\mathit{\Upsilon}$. We can derive $\mathit{\Gamma} \vdash @_s F(A)$ as follows:

$$\dfrac{\dfrac{\text{derivation is trivial}}{\vdash @_i \Box(A \equiv A)} \qquad \dfrac{\mathit{\Psi}}{\mathit{\Gamma}, @_i \Box(A \equiv A) \vdash @_s F(A)}}{\mathit{\Gamma} \vdash @_s F(A)}$$

The subtree $\mathit{\Psi}$ is derived from tree $\mathit{\Upsilon}$ by replacing p with formula A. Since we are replacing a propositional variable (an atom formula) all steps and axioms of the derivation remain correct.

A formula F of $\mathcal{H}^{\mathcal{TR}}(@)$ can be transformed to a set of clauses as follows. We start with a sequent $\vdash @_s F$ and continuously select a subformula A_i containing only a single operation, replace it with a new propositional variable p_i and add a new premise $@_{n_i} \Box(p_i \equiv A_i)$, where n_i is the binding nominal of A_i. By Theorem 2 the new sequent $@_{n_i} \Box(p_i \equiv A_i) \vdash @_s F(p_i)$ is derivable if and only if the original sequent was. We repeat this step to replace every operation in F and derive a sequent of the form:

$$@_{n_1} \Box(p_1 \equiv A_1), @_{n_2} \Box(p_2 \equiv A_2), \ldots, @_{n_k} \Box(p_k \equiv A_k), @_s \neg p_k \vdash$$

Formulae of this sequent are transformed to clauses by converting the equivalences into conjunctive normal form and using $@_i \Box(D' \wedge D'') \equiv @_i \Box D' \wedge @_i \Box D''$.

For example, formula $\Box p \wedge @_b \Diamond q$ is transformed to a set of clauses as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash @_s(\Box p \wedge @_b \Diamond q)}{@_s \Box(r \equiv \Box p) \vdash @_s(r \wedge @_b \Diamond q)}}{@_s \Box(r \equiv \Box p), @_b \Box(t \equiv \Diamond q) \vdash @_s(r \wedge @_b t)}}{@_s \Box(r \equiv \Box p), @_b \Box(t \equiv \Diamond q), @_s \Box(u \equiv @_b t) \vdash @_s(r \wedge u)}}{@_s \Box(r \equiv \Box p), @_b \Box(t \equiv \Diamond q), @_s \Box(u \equiv @_b t), @_s \Box(v \equiv r \wedge u) \vdash @_s v}}{@_s \Box(r \equiv \Box p), @_b \Box(t \equiv \Diamond q), @_s \Box(u \equiv @_b t), @_s \Box(v \equiv r \wedge u), @_s \neg v \vdash}$$

$$\begin{aligned}&\{@_s \Box(\neg r \vee \Box p), @_s \Box(r \vee \Diamond \neg p), @_b \Box(\neg t \vee \Diamond q), @_b \Box(t \vee \Box \neg q),\\ &\;@_s \Box(\neg u \vee @_b t), @_s \Box(u \vee @_b \neg t), @_s \Box(\neg v \vee r), @_s \Box(\neg v \vee u),\\ &\;@_s \Box(v \vee \neg r \vee \neg u), @_s \neg v\}\end{aligned}$$

Conclusions

The described transformation produces clauses of very simple form and can be used to construct efficient resolution calculus for hybrid logic $\mathcal{H}^{\mathcal{TR}}(@)$.

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REZIUMĖ

Hibridinės logikos formulių transformavimas

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Aprašytas tranzityvios ir refleksyvios hibridinės logikos $\mathcal{H}(@)$ formulių transformavimas į disjunktų aibę.

Raktiniai žodžiai: hibridinė logika, disjunktas.