Tiesinė algebra ir geometrija bioinformatikams. Paskaitų konspektas.

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8 paskaita. Vektorinės erdvės. Vektorių sistemos.

8.1 Apibrėžimas. *Tegu* $\mathbf{K}-$ *kūnas(arba* $\mathbf{Q}$, *arba* $\mathbf{R}$, *arba* $\mathbf{C}$*). Aibė* V *, kurioje apibrėžta sudėties ir daugybos iš kūno* $\mathbf{K}$ *elemento operacijos, vadinama* ***vektorine erdve****, jeigu su visais* $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$ *ir* $a, b \in \mathbf{K}$ *išpildytos šios sąlygos:*

1. $(\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3)$;
2. $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$;
3. $\exists \mathbf{o} \in V : \mathbf{v} + \mathbf{o} = \mathbf{v}$;
4. $\exists (-\mathbf{v}) : \mathbf{v} + (-\mathbf{v}) = \mathbf{o}$;
5. $a(\mathbf{v}_1 + \mathbf{v}_2) = a\mathbf{v}_1 + a\mathbf{v}_2$;
6. $(a+b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$;
7. $(ab)\mathbf{v} = a(b\mathbf{v})$;
8. $1_K \cdot \mathbf{v} = \mathbf{v}$.

Vektorinės erdvės elementai vadinami vektoriais.

8.2 Pavyzdžiai. 1) Vektoriai plokštumoje ir vektoriai erdvėje sudaro vektorines erdves virš $\mathbf{R}$.

2) Su kiekvienu $n \in \mathbf{N}$ eilučių aibė $\mathbf{K}^n = M_{1,n} = \{(a_1, ..., a_n) : a_i \in \mathbf{K}\}$ ir stulpelių aibė $\mathbf{K}_n = M_{n,1} = \left\{ \begin{pmatrix} a_1 \\ \cdots \\ a_n \end{pmatrix} : a_i \in \mathbf{K} \right\}$ yra vektorinės erdvės virš $\mathbf{K}$.

3) Su visais k ir n visų $k \times n$ matricų vektorinė erdvė $M_{k,n}(\mathbf{K})$ virš kūno $\mathbf{K}$.

4) Kompleksinių skaičių aibė $\mathbf{C}$ yra vektorinė erdvė virš kūno $\mathbf{R}$.

8.3 Apibrėžimas. *Išraišką* $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$ *vadiname vektorių* $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ ***tiesine kombinacija****; čia* $a_1, a_2, ..., a_n \in \mathbf{K}$.

8.4 Apibrėžimas. *Vektorių sistemą* $\mathbf{v}_1, ..., \mathbf{v}_n$ *vadinsime* ***tiesiškai priklausoma****, jei tarp šių vektorių yra toks* $\mathbf{v}_i$ *, kuris yra likusių sistemos vektorių tiesinė kombinacija:* $\mathbf{v}_i = a_1\mathbf{v}_1 + \cdots + a_{i-1}\mathbf{v}_{i-1} + a_{i+1}\mathbf{v}_{i+1} + \cdots + a_n\mathbf{v}_n$*. Jeigu vektorių sistemoje* $\mathbf{v}_1, ..., \mathbf{v}_n$ *nėra vektoriaus, tiesiškai priklausomo nuo likusių, tai šią sistemą vadina* ***teisiškai nepriklausoma****.*

8.5 Pastaba. Susitarta, kad tuščios vektorių sistemos tiesinė kombinacija lygi nuliniam vektoriui: $\underbrace{\mathbf{v}+\cdots+\mathbf{u}}_{0\text{ dėmenų}}=\mathbf{o}$.

8.6 Examples.

1) Every two vectors in the line $\mathbf{R}$ are linearly dependent (one vector is proportional to the other one).

2) Every three vectors a, b, c on a plane $\mathbf{R}^2$ are linearly dependent. Indeed, if a and b are parallel then one of them is a multiple of another one, and so a and b are linearly dependent which implies that all three vectors are linearly dependent. If a and b are not parallel then we know that every vector on the plane, including the vector c, is a linear combination of $a$and b. Thus a, b, c are linearly dependent.

3) If a subset S of R^n consists of more than n vectors then S is linearly dependent. (Prove it!)

4) The set of polynomials $x+1, x^2+x+1, x^2-2x-2, x^2-3x+1$ is linearly dependent. To prove that we need to find numbersa, b, c, d not all equal to 0 such that $a(x+1)+b(x^2+x+1)+c(x^2-2x-2)+d(x^2-3x+1)=0$. This leads to a homogeneous system of linear equations with 4 unknowns and 3 equations. Such a system must have a non-trivial solution by the theorem about homogeneous systems of linear equations.

8.7 Apibrėžimas. *Jeigu* $A\subset\{\mathbf{v}_1, ..., \mathbf{v}_n\}\subset B$ *, tai* A *vadinamas sistemos* $\mathbf{v}_1, ..., \mathbf{v}_n$ *posistemiu, o* $B-$ *viršsistemiu.*

8.8 Vektorių sistemų savybės.

1. Sistema iš vieno vektoriaus $\{\mathbf{v}\}$ yra tiesiškai priklausoma kai $\mathbf{v}=\mathbf{o}$ ir tiesiškai nepriklausoma kai $\mathbf{v}\neq\mathbf{o}$.
2. Tiesiškai priklausomos sistemos $\mathbf{v}_1, ..., \mathbf{v}_n$ viršsistemis yra tiesiškai priklausomas.
3. Tiesiškai nepriklausomos sistemos $\mathbf{v}_1, ..., \mathbf{v}_n$ posistemis yra tiesiškai nepriklausomas.
4. Vektorių sistema, kurioje yra nulinis vektorius, yra tiesiškai priklausoma.
5. Vektorių sistema, kurioje yra du sutampantys vektoriai, yra tiesiškai priklausoma.
6. Vektorių sistema $\mathbf{v}_1, ..., \mathbf{v}_n$ yra tiesiškai priklausoma tada ir tik tada, kai egzistuoja tokie ne visi lygūs nuliui $a_1, ..., a_n\in K$, kad $a_1\mathbf{v}_1+\cdots+a_n\mathbf{v}_n=\mathbf{o}$.

Įrodymas. ($\Rightarrow$)Tegu $\mathbf{v}_1, ..., \mathbf{v}_n-$ tiesiškai priklausoma sistema. Tada egzistuoja $\mathbf{v}_i=a_1\mathbf{v}_1+\cdots+a_{i-1}\mathbf{v}_{i-1}+a_{i+1}\mathbf{v}_{i+1}+\cdots+a_n\mathbf{v}_n$ ir todėl

$$a_1\mathbf{v}_1 + \cdots + a_{i-1}\mathbf{v}_{i-1} + (-1)\,\mathbf{v}_i + a_{i+1}\mathbf{v}_{i+1} + \cdots + a_n\mathbf{v}_n = \mathbf{o}.$$

($\Leftarrow$) Tegu

$$a_1\mathbf{v}_1 + \cdots + a_{i-1}\mathbf{v}_{i-1} + a_i\mathbf{v}_i + a_{i+1}\mathbf{v}_{i+1} + \cdots + a_n\mathbf{v}_n = \mathbf{o}$$

ir $a_i \neq 0$, tada

$$\mathbf{v}_i = -\tfrac{a_1}{a_i}\mathbf{v}_1 - \cdots - \tfrac{a_{i-1}}{a_i}\mathbf{v}_{i-1} - \tfrac{a_{i+1}}{a_i}\mathbf{v}_{i+1} - \cdots - \tfrac{a_n}{a_i}\mathbf{v}_n.$$

Įrodyta.

7. Vektorių sistema $v_1, ..., v_n$ yra tiesiškai nepriklausoma tada ir tik tada, kai iš $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{o}$ turime $a_1 = \cdots = a_n = 0$.

Įrodymas. Akivaizdu.

8.9 Apibrėžimas. *Tegu* $\mathbf{v}_1, ..., \mathbf{v}_n \in V, -$ *vektorių sistema. Vektorinės erdvės* V *poaibis* $\{a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n | a_1, ..., a_n \in K\} = [v_1, ..., v_n]$ *vadinamas vektorių sistemos* $\mathbf{v}_1, ..., \mathbf{v}_n$ ***tiesiniu apvalkalu***. *Vektorių sistema* $\mathbf{v}_1, ..., \mathbf{v}_n$ *vadinama generuojančia šį tiesinį apvalkalą sistema.*

Aišku, kad vektorių sistemos $v_1, ..., v_n$ tiesinis apvalkalas $[\mathbf{v}_1, ..., \mathbf{v}_n]$ yra vektorinė erdvė, o bet koks generuojančios sistemos viršsistemis yra generuojanti sistema.

8.10 Examples.

1. Let $V = R^3$. Let S consist of one non-zero vector A. Then spanA consists of all vectors of the form xA. In other words, span$A$ consists of all vectors which are proportional to A, or span(A) is the set of vector parallel to A.

2. Let $V = R^3, S = (1, 0, 0), (0, 1, 0)$. Then span$(S)$ consists of all vectors of the form $x(1, 0, 0) + y(0, 1, 0) = (x, y, 0)$, that is span$(S)$ consists of all vectors parallel to the (x, y)-plane. More generally, if we take any two non-parallel vectors A and B in R^3 then spanA, B is the subspace of all vectors which are parallel to the plane containing A and B.

3. Let $V = R^3, S = (1, 0, 0), (0, 1, 0), (0, 0, 1)$. Then span$(S)$ coincides with the whole R^3. More generally if $V = R^n$ then$V$ is spanned by the set of basic vectors$(1, 0, ..., 0), (0, 1, ..., 0), ..., (0, 0, ..., 1)$.

8.11 Theorem. Two subsets S_1 and S_2 of a vector space V span the same subspace if and only if every vector of S_1 is a linear combination of vectors of S_2 and every vector of S_2 is a linear combination of vectors of S_1.

The proof is left as an exercise.

8.12 Examples.

1. Vectors $a = (1, 2, 3), b = (0, 1, 2)$,and $c = (0, 0, 1)$ span R^3. Indeed, we know that R^3 is spanned by the vectors $i = (1, 0, 0), j = (0, 1, 0)$ and $k = (0, 0, 1)$.Thus we need to show that the sets a, b, c and i, j, k span the same subspace. By the previous theorem, we need to show that every vector from the first subset is a linear combination of vectors from the second subset and conversely every vector from the second subset is a linear combination of vectors of the first subset. It is clear that a, b, c are linear combinations of i, j, k (as any vector in R^3). So we need to show only that i, j, k are linear combinations of a, b, c. This is easy to check :$i = a - 2b + c; j = b - 2c; k = c$.

2. Polynomials $f_1 = x^2 + 2x + 1, f_2 = x + 1$and $f_3 = x + 2$ in the space of all polynomials P span the subspace P_2 of all polynomials of degree not exceeding 2. Indeed, it is clear that P_2 is spanned by the polynomials $p_1 = 1, p_2 = x$ and $p_3 = x^2$. So we need to show that the setsf_1, f_2, f_3 and p_1, p_2, p_3 span the same subspace. It is clear that f_1, f_2, f_3 are linear combinations of p_1, p_2, p_3. Conversely, it is easy to check that $p_1 = f_3 - f_2; p_2 = 2f_2 - f_3; p_3 = f_1 - 3f_2 + f_3$.

3. Let $a = (1, 2, 3, 4), b = (3, -1, 5, 2), c = (-1, 2, 0, 1), d = (2, 1, 4, 5)$. Determine whether span$\{a, b\}$=span$\{c, d\}$. We need to check if a and $b$are linear combinations of vectors $c$and$d$, and whether$c$ and d are linear combinations of vectors a and b. By definition of a linear combination, the vector a is a linear combination of c and d if there exist x and y such that $a = xc + yd$ This gives the following system of linear equations:

$$\begin{aligned} 1 &= -x + 2y \\ 2 &= 2x + y \\ 3 &= 0x + 4y \\ 4 &= x + 5y. \end{aligned}$$

This system does not have a solution, so a is not a linear combination of c and d, so span$\{a, b\}$ is not equal to spanc, d.

8.13 Apibrėžimas. *Generuojančią sistemą vadiname minimalia, jeigu bet koks jos posistemis nėra generuojanti sistema.*

Tiesiškai nepriklausoma sistema vadinama maksimalia, jeigu bet koks jos viršsistemis yra tiesiškai priklausoma sistema.

Dabar pateiksime svarbias tiesinių apvalkalų savybes.

8.14 Teorema. *Jeigu vektorių sistemos* $\mathbf{v}_1, ..., \mathbf{v}_n$ *vektoriai yra vektorių sistemos* $\mathbf{u}_1, ..., \mathbf{u}_m$ *vektorių tiesinės kombinacijos, t.y.* $\mathbf{v}_1, ..., \mathbf{v}_n \in [\mathbf{u}_1, ..., \mathbf{u}_m]$, *ir vektorius* $\mathbf{v} \in [\mathbf{v}_1, ..., \mathbf{v}_n]$, *tai* $\mathbf{v} \in [\mathbf{u}_1, ..., \mathbf{u}_m]$.

Įrodymas.Paliekame įrodyti studentams.

8.15 Teorema . *Tegu* $\mathbf{v}_1, ..., \mathbf{v}_n-$ *tiesiškai nepriklausoma sistema, o* $\mathbf{v}, \mathbf{v}_1, ..., \mathbf{v}_n-$ *tiesiškai priklausoma sistema. Tada vektorius* $\mathbf{v} \in [\mathbf{v}_1, ..., \mathbf{v}_n]$.

Įrodymas.Paliekame įrodyti studentams.

8.16 Teorema. *Tegu* $\mathbf{v}_1, ..., \mathbf{v}_n-$ *vektorinės erdvės* V *vektorių sistema. Šie trys teiginiai yra ekvivalentūs:*

1. $\mathbf{v}_1, ..., \mathbf{v}_n-$ *tiesiškai nepriklausoma ir vektorinę erdvę* V *generuojanti sistema.*

2. $\mathbf{v}_1, ..., \mathbf{v}_n-$ *maksimali tiesiškai nepriklausoma sistema.*

3. $\mathbf{v}_1, ..., \mathbf{v}_n-$ *minimali vektorinę erdvę* V *generuojanti sistema.*

Įrodymas. 1⇒2.

Jei $\mathbf{v}_1, ..., \mathbf{v}_n-$ generuojanti erdvę V sistema, tai kiek- vienas vektorius $\mathbf{v} \in V$ yra tiesinė vektorių sistemos $\mathbf{v}_1, ..., \mathbf{v}_n$kombinacija, t.y. $\mathbf{v} \in [\mathbf{v}_1, ..., \mathbf{v}_n] \Longrightarrow$

sistema $\mathbf{v}$, $\mathbf{v}_1, ..., \mathbf{v}_n-$ tiesiškai priklausoma $\Longrightarrow$

$\mathbf{v}_1, ..., \mathbf{v}_n-$ maksimali tiesiškai nepriklausoma vektorių sistema.

2⇒3.

Su kiekvienu $\mathbf{v} \in V$ sistema $\mathbf{v}$, $\mathbf{v}_1, ..., \mathbf{v}_n$ yra tiesiškai priklausoma sistema. pagal teiginį 1 $\mathbf{v} \in [\mathbf{v}_1, ..., \mathbf{v}_n] \Longrightarrow$

$\mathbf{v}_1, ..., \mathbf{v}_n-$ generuojanti vektorių sistema. Sakykime, vektorių sistema $\mathbf{v}_2, ..., \mathbf{v}_n$irgi yra vektorinę erdvę V generuojanti sistema$\Longrightarrow$

kiekvienas vektorius $\mathbf{v} \in V$ (tame tarpe ir $\mathbf{v}_1, ..., \mathbf{v}_n$) yra vektorių $\mathbf{v}_2, ..., \mathbf{v}_n$ tiesinė kombinacija$\Longrightarrow$

$\mathbf{v}_1, ..., \mathbf{v}_n \in [\mathbf{v}_2, ..., \mathbf{v}_n] \Longrightarrow$

$\mathbf{v}_1 = a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n \Longrightarrow$

sistema $\mathbf{v}_1, ..., \mathbf{v}_n-$ tiesiškai priklausoma, prieštaravimas prielaidai.

3⇒ 1.

Sakykime, $\mathbf{v}_1 = a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n \Longrightarrow \mathbf{v}_1 \in [\mathbf{v}_2, ..., \mathbf{v}_n]$. Iš teiginio 2 turime, kad kiekvienas vektorinės erdvės V vektorius $\mathbf{v} \in [\mathbf{v}_2, ..., \mathbf{v}_n]$, prieštaravimas prielaidai.

Įrodyta.

8.17 Apibrėžimas. *Vektorinės erdvės V vektorių sistema, tenkinanti vieną iš teoremos sąlygų, vadinama vektorinės erdvės V* ***baze.***

8.18 Positive examples.

1.The set of vectors $V_1 = (1, 0, ..., 0), V_2 = (0, 1, ..., 0), ..., V_2 = (0, ..., 1)$ is a basis of R^n.

2.The set of vectors$(1, 2), (2, 3)$ is a basis of R^2. Indeed, these vectors are linearly independent because they are not proportional. In order to check that R^2 is spanned by these vectors, it is enough to check that $(1, 0)$ and $(0, 1)$ are linear combinations of them (theorem about spans):

$$(1, 0) = -3 \cdot (1, 2) + 2 \cdot (2, 3);$$
$$(0, 1) = 2 \cdot (1, 2) - (2, 3).$$

In fact, every two non-parallel vectors in the plane R^2 form a basis of R^2.

3. The set of polynomials $1, x, x^2$ is a basis of the space of polynomials of degree at most 2.

4.The set of matrices:$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

is a basis of the space of all 2 by 2 matrices. First we need to prove that these matrices are linearly independent. Indeed, if we take a linear combination of these matrices with coefficients a, b, c, d, we get the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This matrix is equal to the zero matrix only if $a = b = c = d = 0$. Second, we need to show that these 4 matrices span the space of all 2 by 2 matrices. Indeed, every 2 by 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the linear combination of our four matrices with coefficientsa, b, c, d.

8.19 Negative examples

1.The set of two vectors $\mathbf{u} = (1, 2, 3)$ and $v = (2, 3, 4)$ is not a basis of R^3. Indeed, although these vectors are linearly independent, they do not span R^3. For example, the vector $(1, 0, 0)$ is not equal to a linear combination of $\mathbf{u}$ and$\mathbf{v}$.

2.The set of four vectors $\mathbf{u} = (1, 2, 3), \mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1)$ is not a basis of R^3. Indeed, although these vectors span R^3 (even $\mathbf{i}, \mathbf{j}, \mathbf{k}$ span R^3), these vectors are not linearly independent because $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

Vektorinės erdvės dimensija.

8.20 Apibrėžimas. *Vektorinės erdvės bazėje esančių vektorių skaičius vadinamas vektorinės erdvės* V ***dimensija****, žymima* $\dim_K V$ *arba* $\dim V$.

8.21 Apibrėžimas. Matricos A **rangu** vadinama matricos eilučių (stulpelių) tiesinio apvalkalo dimensija ir žymima rank$A$. Tai maksimalios tiesiškai nepriklausomos eilučių (stulpelių) sistemos elementų skaičius.

Vektorinės erdvės dimensijos apibrėžimas yra korektiškas. Tai rodo tokia teorema.

8.22 Teorema(tiesinių kombinacijų tiesinis priklausomumas).
Tegu $\mathbf{v}_1, ..., \mathbf{v}_m \in [\mathbf{u}_1, ..., \mathbf{u}_n]$ *, t.y.*

$$\begin{aligned} \mathbf{v}_1 &= a_{11}\mathbf{u}_1 + a_{12}\mathbf{u}_2 + \cdots + a_{1n}\mathbf{u}_n \\ \mathbf{v}_2 &= a_{12}\mathbf{u}_1 + a_{22}\mathbf{u}_2 + \cdots + a_{2n}\mathbf{u}_n \\ &\cdots \\ \mathbf{v}_m &= a_{1m}\mathbf{u}_1 + a_{2m}\mathbf{u}_2 + \cdots + a_{mn}\mathbf{u}_n \end{aligned}$$

ir $m > n$*. Tada* $\mathbf{v}_1, ..., \mathbf{v}_m-$ *tiesiškai priklausoma sistema.*

Įrodymas. Indukcija pagal n.

1. Jeigu $n = 0$, ir $m > n$ tai $\mathbf{v}_1 = \cdots = \mathbf{v}_m = \mathbf{o}$, o sistema, kurioje yra nulinis vektorius - tiesiškai priklausoma.

2. Tegu $n > 0$. Galimi keli atvejai.

i) $a_{11} = a_{12} = \cdots = a_{1m} = 0$. Tada $\mathbf{v}_1, ..., \mathbf{v}_m \in [\mathbf{u}_2, ..., \mathbf{u}_n]$ ir pagal indukcijos prielaidą $\mathbf{v}_1, ..., \mathbf{v}_m-$ tiesiškai priklausoma sistema.

ii) Sakykime ne visi a_{i1} lygūs 0. Tegu $a_{11} \neq 0$. Sudarykime naujus vektorius

$$\begin{aligned} \mathbf{w}_2 &= \mathbf{v}_2 - \tfrac{a_{21}}{a_{11}}\mathbf{v}_1 = \gamma_{22}\mathbf{u}_2 + \cdots \gamma_{2n}\mathbf{u}_n \\ &\cdots \\ \mathbf{w}_m &= \mathbf{v}_m - \tfrac{a_{m1}}{a_{11}}\mathbf{v}_1 = \gamma_{m2}\mathbf{u}_2 + \cdots \gamma_{mn}\mathbf{u}_n, \end{aligned}$$

čia $\gamma_{ij} = a_{ij} - \frac{a_{i1}}{a_{11}}$.

Turime, kad $m - 1$ vektorius $\mathbf{w}_2, ..., \mathbf{w}_m$ reiškiamas $n - 1$ vektoriumi $\mathbf{u}_2, ..., \mathbf{u}_n$. Pagal indukcijos prielaidą vektoriai $\mathbf{w}_2, ..., \mathbf{w}_m-$ tiesiškai priklausomi: egzistuoja ne visi lygūs nuliui tokie $b_2, ..., b_m$ (pvz. $b_i \neq 0$) , kad $b_2\mathbf{w}_2 + \cdots + b_m\mathbf{w}_m = \mathbf{o}$,t.y.

$$\left(-b_2\frac{a_{21}}{a_{11}} - \cdots - b_m\frac{a_{m1}}{a_{11}}\right)\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_m\mathbf{v}_m = \mathbf{o}.$$

Kairėje lygybės pusėje esančioje tiesinėje kombinacijoje ne visi koeficientai lygūs nuliui (pvz. $b_i \neq 0$). Taigi, vektorių sistema $\mathbf{v}_1, ..., \mathbf{v}_m-$ tiesiškai priklausoma sistema.

Įrodyta.

8.23 Išvada. Vektorinės erdvės bazėje esančių vektorių skaičius yra pastovus dydis.

Įrodymas.[....]

8.24 Teorema.

1.S is linearly independent. If S is a linearly dependent set in an n-dimensional space V and V=span(S) then by removing some elements of S we can get a basis of V.

2. If S is a linearly independent subset of V which is not a basis of V then we can get a basis of V by adding some elements to S.

8.25 Examples.

1. Consider the following 5 vectors in $\mathbf{R}^4$:

$(1,2,3,4), (1,1,0,0), (1,2,1,0), (0,1,2,3), (1,0,0,0)$.

It can be shown (check!) that these vectors span $\mathbf{R}^4$. Since $\mathbf{R}^4$ is 4-dimensional (it has the standard basis with 4 vectors), these 5 vectors must be linearly dependent by the theorem about bases. By the theorem about dimension we can through away one of these vectors and get a basis of $\mathbf{R}^4$. By the theorem about throwing away extra elements from a spanning set, we can through away a vector which is a linear combination of other vectors in the set. Let us check that the vector $(1,2,3,4)$ is such a vector. In order to find the linear combination which is equal to this vector, we need to solve the system of linear equation:

$(1,2,3,4) = (1,1,0,0)\cdot x_1 + (1,2,1,0)\cdot x_2 + (0,1,2,3)\cdot x_3 + (1,0,0,0)\cdot x_4$.

This system of equations has the following augmented matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 1| & 1 \\ 1 & 2 & 1 & 0| & 2 \\ 0 & 1 & 2 & 0| & 3 \\ 0 & 0 & 3 & 0| & 4 \end{pmatrix}$$

Using the Gauss-Jordan procedure, we get the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0| & 0 \\ 0 & 1 & 0 & 0| & \frac{1}{3} \\ 0 & 0 & 1 & 0| & \frac{4}{3} \\ 0 & 0 & 0 & 1| & \frac{2}{3} \end{pmatrix}$$

Thus $x_1 = 0, x_2 = \frac{1}{3}, x_3 = \frac{4}{3}, x_4 = \frac{2}{3}$. So the vector $(1, 2, 3, 4)$ can be thrown away. The other vectors, $(1, 1, 0, 0), (1, 2, 1, 0), (0, 1, 2, 3), (1, 0, 0, 0)$, form a basis of $\mathbf{R}^4$. Indeed, they span $\mathbf{R}^4$ by the theorem about throwing away extra elements, and by the theorem about dimension, every four vectors in a 4-dimensional vector space which span the vector space, form a basis of this vector space.

2. Take two vectors $(1, 2, 3, 4), (2, 1, 1, 1)$ in $\mathbf{R}^4$. These vectors are linearly independent because they are not proportional (see the theorem about linearly dependent sets). Thus by the theorem about dimension we can add two vectors and get a basis of $\mathbf{R}^4$. Let us add $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$. Notice that when we add vectors we need to make sure that the added vectors are not linear combinations of the previous vectors. In order to check that the four vectors $(1, 2, 3, 4), (2, 1, 1, 1), (1, 0, 0, 0), (0, 1, 0, 0)$ form a basis of $\mathbf{R}^4$, we need to check only that they are linearly independent, that is the system of equations:

$(0, 0, 0, 0) = (1, 2, 3, 4) \cdot x_1 + (2, 1, 1, 1) \cdot x_2 + (1, 0, 0, 0) \cdot x_3 + (0, 1, 0, 0) \cdot x_4$

has only one, trivial, solution (see the theorem about dimension). This is an homogeneous system with 4 equations and 4 unknowns. We know that this system has only one solution if and only if the matrix of coefficients is invertible (see the second theorem about inverses). And we know that a square matrix is invertible if and only if its determinant is not zero (see the third theorem about determinants). Thus we need to check that the determinant of the matrix of coefficients of our system is not zero. Maple says that

$$\det \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 3 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 \end{pmatrix} = -1$$

Thus, our four vectors form a basis of $\mathbf{R}^4$.