Mathematical Competition for Students of the Department of Mathematics and Informatics of Vilnius University, Problems and Solutions, 2022

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PROBLEMS

Problem 1.

a) Prove that there exist infinitely many integers n such that n, $n+1$ and $n+2$ are each the sum of two squares of integers.

b) Does there exist an integer n such that each of the numbers n, $n+1$, $n+2$ and $n+3$ is the sum of two squares of integers?

Problem 2. Find the value of

$$\int_0^{\pi} \cos^{2022}(x)\cos(100x)dx.$$

Problem 3. Let $f : (\mathbb{Z}, \mathbb{Z}) \to \mathbb{Z}$ be a function satisfying

$$f(x,x) = f(y,y)$$

for all $x, y \in \mathbb{Z}$ and

$$f(x, f(y,z)) = f(x,y) + z$$

for all $x, y, z \in \mathbb{Z}$. (Here, $\mathbb{Z}$ denotes the set of all integers.)

Find all possible values of $f(1000, 2022)$.

Problem 4. A subset of a group of students is called an *ideal company* if

(i) it contains at least one girl and at least one boy;

(ii) each boy of this subset likes every girl of this subset;

(iii) nobody can be added to this subset without violation of rule (ii).

Find the maximal number of ideal companies in a group of 10 girls and 20 boys.

Each problem is worth 10 points.

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PROBLEMS WITH SOLUTIONS

Problem 1.

a) Prove that there exist infinitely many integers n such that n, $n+1$ and $n+2$ are each the sum of two squares of integers.

b) Does there exist an integer n such that each of the numbers n, $n+1$, $n+2$ and $n+3$ is the sum of two squares of integers?

Answer: b) there is no such integer.

Solution. a) Select $n = (2t^2+1)^2 - 1$ with an arbitrary integer t. Then,

$$n = 4t^4 + 4t^2 + 1 - 1 = 4t^4 + 4t^2 = (2t^2)^2 + (2t)^2,$$

$n+1 = (2t^2+1)^2 = (2t^2+1)^2 + 0^2$ and $n+2 = (2t^2+1)^2 + 1^2$.

Second solution of part a). It is well known that the Pell equation $x^2 - 2y^2 = 1$ has infinitely many solutions in positive integers (x, y). Select $n = 2y^2 = y^2 + y^2$. Then, $n+1 = 2y^2 + 1 = x^2 = x^2 + 0^2$ and $n+2 = x^2 + 1^2$.

b) The square of an integer is clearly congruent to 0 or 1 modulo 4. Consequently, the sum of two squares of integers is congruent to 0, 1 or 2 modulo 4. However, among any four consecutive integers, exactly one is congruent to 3 modulo 4, and therefore it can't be expressed by the sum of two squares of integers. □

Problem 2. Find the value of

$$\int_0^\pi \cos^{2022}(x)\cos(100x)dx.$$

Answer: $2^{-2022}\binom{2022}{961}\pi$.

Solution. We will prove that the given integral I is equal to $2^{-2022}\binom{2022}{961}\pi$.

Firstly, for each positive integer n, from

$$\big(2\cos(x)\big)^{2n} = (e^{ix}+e^{-ix})^{2n} = \sum_{k=0}^{2n}\binom{2n}{k}e^{i(2k-2n)x} = \binom{2n}{n} + \sum_{k=0}^{n-1}\binom{2n}{k}2\cos((2n-2k)x)$$

it follows that

$$\cos^{2n}(x) = 2^{-2n}\binom{2n}{n} + 2^{1-2n}\sum_{k=0}^{n-1}\binom{2n}{k}\cos((2n-2k)x).$$

Inserting $n = 1011$ into this formulae we derive that

$$\cos^{2022}(x) = \sum_{j=0}^{1011} c_j\cos(2jx),$$

where $c_j \in \mathbb{R}$ for each $j = 0, 1, \ldots, 1011$. In particular, for $j = 50$ (as $2j = 2n - 2k$ and so $k = n - j = 1011 - 50 = 961$) we find that

$$c_{50} = 2^{-2021}\binom{2022}{961}.$$

Secondly, note that

$$\int_0^{\pi} \cos(2kx)dx = \frac{\sin(2kx)}{2k}\Big|_0^{\pi} = 0$$

for each nonzero integer k. Hence, for every nonnegative integer $j \neq 50$ we have

$$\int_0^{\pi} \cos(2jx)\cos(100x)dx = \int_0^{\pi} \frac{\cos((2j+100)x) + \cos((2j-100)x)}{2}dx = 0.$$

Summarizing, we find that

$$\begin{aligned} I &= \int_0^{\pi} \cos^{2022}(x)\cos(100x)dx = \int_0^{\pi} \sum_{j=0}^{1011} c_j \cos(2jx)\cos(100x)dx \\ &= \sum_{j=0}^{1011} c_j \int_0^{\pi} \cos(2jx)\cos(100x)dx = c_{50}\int_0^{\pi} \cos^2(100x)dx \\ &= c_{50}\int_0^{\pi} \frac{1+\cos(200x)}{2}dx = \frac{\pi}{2}c_{50} = 2^{-2022}\binom{2022}{961}\pi, \end{aligned}$$

which gives the claimed value $2^{-2022}\binom{2022}{961}\pi$ for the given integral I. □

Problem 3. Let $f : (\mathbb{Z}, \mathbb{Z}) \to \mathbb{Z}$ be a function satisfying

$$f(x, x) = f(y, y)$$

for all $x, y \in \mathbb{Z}$ and

$$f(x, f(y, z)) = f(x, y) + z$$

for all $x, y, z \in \mathbb{Z}$. (Here, $\mathbb{Z}$ denotes the set of all integers.)

Find all possible values of $f(1000, 2022)$.

Answer: -1022.

Solution. Set $a = f(0, 0) \in \mathbb{Z}$. Then, $f(x, x) = f(y, y) = a$ for all $x, y \in \mathbb{Z}$. Note that

$$a + x = f(x, x) + x = f(x, f(x, x)) = f(x, a) = f(x, f(y, y)) = f(x, y) + y.$$

This implies that $f(x, y) = a + x - y$ is the only possible function satisfying both conditions. Using this expression and the second condition we find that

$$a+x-y+z = f(x,y)+z = f(x,f(y,z)) = f(x,a+y-z) = a+x-(a+y-z) = x-y+z,$$

so $a = 0$ is the only possibility. The function $f(x, y) = x - y$ clearly satisfies both conditions on f. Thus, $f(1000, 2022) = 1000 - 2022 = -1022$ is the only possibility. □

Problem 4. A subset of a group of students is called an *ideal company* if

(i) it contains at least one girl and at least one boy;
(ii) each boy of this subset likes every girl of this subset;
(iii) nobody can be added to this subset without violation of rule (ii).

Find the maximal number of ideal companies in a group of 10 girls and 20 boys.

Answer: 1023.

Solution. The number of ideal companies cannot exceed the number of nonempty subsets of girls, which is $2^{10} - 1 = 1023$. Indeed, by rule (iii), if an ideal company I contains a nonempty subset of girls S, then $I \setminus S$ must contain all the boys that like all the girls from S, so the boys in I are uniquely determined by the set of girls S.

Now, we will give an example with exactly 1023 ideal companies. Let us label all 10 girls by integers from 1 to 10, and some 10 boys by integers from 1 to 10. Assume that the remaining 10 boys like all the girls. We label each of them by the same integer 11. Finally, suppose that for each $i \in \{1, \dots, 10\}$ the boy with label i likes all the girls except for the girl with label i. Then, to each nonempty subset of girls, say $S \subseteq \{1, 2, \dots, 10\}$, we add all the boys with labels in $\{1, \dots, 10, 11\} \setminus S$, and denote such a group by $I(S)$. Then, $I(S)$ satisfies (i), (ii), (iii) for each nonempty set S, and so each $I(S)$ is an ideal company. (In particular, all 10 girls and 10 boys with label 11 is an ideal company.) This completes the proof, because $I(S) \neq I(S')$ if $S \neq S'$. $\square$