Mathematical Competition for Students of the Department of Mathematics and Informatics of Vilnius University, Problems and Solutions, 2021

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PROBLEMS

Problem 1. Does there exist an uncountable set of real numbers S such that

$$|s_1 + \cdots + s_n| < 10$$

for every $n \in \mathbb{N}$ and every finite subset $\{s_1, \ldots, s_n\}$ of S?

Problem 2. Find all functions $f : \mathbb{Z} \mapsto \mathbb{Z}$ satisfying

$$f(x + y) = f(x) + f(y) + 2xy$$

for $x, y \in \mathbb{Z}$ and $f(100) = 1000$.

Problem 3. Alex and Sveta play a game in which they take turns filling entries of an initially empty table with 2020 rows and 2021 columns. Alex plays first. At each turn, a player chooses a number and places it in a vacant entry. Alex can only choose numbers of the form $a\sqrt{2}$, with $a \in \mathbb{Z}$, while Sveta can only choose rational numbers. The game ends when all the entries are filled. Alex wins if the rank of the resulting matrix is at least 1011, Sveta wins if the rank is at most 1010. Which player has a winning strategy?

Problem 4. Find all continuous functions $f : [1, 8] \mapsto \mathbb{R}$ satisfying

$$10 \int_1^8 f(x)dx - 15 \int_1^2 f^2(x^3)dx - 30 \int_1^2 f(x^3)dx = 38.$$

Each problem is worth 10 points.

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PROBLEMS WITH SOLUTIONS

Problem 1. Does there exist an uncountable set of real numbers S such that

$$|s_1 + \cdots + s_n| < 10$$

for every $n \in \mathbb{N}$ and every finite subset $\{s_1, \ldots, s_n\}$ of S?

Answer: no.

Solution. Clearly, S is contained in the interval $[-10, 10]$. For every positive integer $n \geqslant 2$ put $S_n = S \cap (10/n, 10/(n-1)]$ and $S_{-n} = S \cap [-10/(n-1), -10/n)$. The set S_n contains at most $n-1$ elements of S, because the sum of n elements of S_n is greater than $n \cdot (10/n) = 10$. Likewise, the set S_{-n} contains at most $n-1$ elements of S. Note that $\cup_{n=2}^{\infty} S_n$ contains all positive elements of S, whereas $\cup_{n=2}^{\infty} S_{-n}$ contains all negative elements of S. Therefore, either S itself (if $0 \notin S$) or $S \setminus \{0\}$ (if $0 \in S$) is the countable union of finite (possibly empty) disjoint sets $S_2 \cup S_{-2} \cup S_3 \cup S_{-3} \cup \ldots$, and hence S is at most countable. □

Problem 2. Find all functions $f : \mathbb{Z} \mapsto \mathbb{Z}$ satisfying

$$f(x + y) = f(x) + f(y) + 2xy$$

for $x, y \in \mathbb{Z}$ and $f(100) = 1000$.

Answer: $f(x) = x^2 - 90x$.

Solution. Set $a = f(1)$. By induction on $n \in \mathbb{N}$ we will show that $f(n) = n^2 - (1-a)n$. This is clearly true for $n = 1$ by $f(1) = 1^2 - (1-a) = a$. Assume that this is true for $n = k$. Then, choosing $x = k$ and $y = 1$ we get

$$f(k+1) = f(k)+f(1)+2k = k^2-(1-a)k+a+2k = k^2+(a+1)k+a = (k+1)^2-(1-a)(k+1),$$

which proves the claim. Inserting $n = 100$ into the formula $f(n) = n^2 - (1-a)n$ we find that

$$1000 = f(100) = 100^2 - (1-a)100 = 9900 + 100a.$$

Hence, $a = -89$. Consequently, $f(x) = x^2 - 90x$ for each $x \in \mathbb{N}$.

Now, we will show that $f(x) = x^2 - 90x$ for each $x \in \mathbb{Z}$. Inserting $x = y = 0$ into the given equality we find that $f(0) = f(0) + f(0)$, so $f(0) = 0$. Thus, the formula holds for $x = 0$. For a negative integer x, by selecting $y = -x$, we get $f(0) = f(x) + f(-x) - 2x^2$. Since $-x \in \mathbb{N}$, we already know that $f(-x) = (-x)^2 - 90(-x) = x^2 + 90x$. Therefore, using $f(0) = 0$ we find that

$$f(x) = 2x^2 - f(-x) = 2x^2 - x^2 - 90x = x^2 - 90x.$$

This completes the proof the formula $f(x) = x^2 - 90x$ for each $x \in \mathbb{Z}$.

Finally, it is easy to verify that $f(x) = x^2 - 90x$ satisfies both conditions, because

$$f(x+y) - f(x) - f(y) - 2xy = (x+y)^2 - 90(x+y) - x^2 + 90x - y^2 + 90y - 2xy = 0$$

for all $x, y \in \mathbb{Z}$ and $f(100) = 100^2 - 90 \cdot 100 = 1000$. □

Problem 3. Alex and Sveta play a game in which they take turns filling entries of an initially empty table with 2020 rows and 2021 columns. Alex plays first. At each turn, a player chooses a number and places it in a vacant entry. Alex can only choose numbers of the form $a\sqrt{2}$, with $a \in \mathbb{Z}$, while Sveta can only choose rational numbers. The game ends when all the entries are filled. Alex wins if the rank of the resulting matrix is at least 1011, Sveta wins if the rank is at most 1010. Which player has a winning strategy?

Answer: Sveta has a winning strategy.

Solution. Set $n = 1010$ and denote the number at ith row and jth column by $m(i,j)$. Here, $i = 1, \dots, 2n$ and $j = 1, \dots, 2n+1$. Let us consider the following $n(2n+1)$ pairs

$$\big(m(2i-1, j), m(2i, j)\big),$$

where $i = 1, \dots, n$ and $j = 1, \dots, 2n+1$. The strategy of Sveta can be as follows:

- If Alex writes a number $a\sqrt{2}$ for $m(2i-1, j)$ then Sveta writes $2a \in \mathbb{Z}$ for $m(2i, j)$.
- If Alex writes $b\sqrt{2}$ for $m(2i, j)$ then Sveta writes $b \in \mathbb{Z}$ for $m(2i-1, j)$.

Following this strategy, as soon as all the entries of the matrix will be filled, for each pair (i, j), where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant 2n+1$, we will have the equality

$$m(2i, j) = \sqrt{2} m(2i-1, j).$$

In particular, if

$$\mathbf{r}_i = \big(m(i, 1), \dots, m(i, 2n+1)\big)$$

is the vector corresponding to the ith row of the matrix then $\mathbf{r}_{2i} = \sqrt{2}\mathbf{r}_{2i-1}$, and so the rows $\mathbf{r}_{2i-1}$ and $\mathbf{r}_{2i}$ are linearly dependent for each $i = 1, \dots, n$. Thus, the rank of the resulting matrix is at most n, and Sveta wins. □

Problem 4. Find all continuous functions $f : [1, 8] \mapsto \mathbb{R}$ satisfying

$$10 \int_1^8 f(x)dx - 15 \int_1^2 f^2(x^3)dx - 30 \int_1^2 f(x^3)dx = 38.$$

Answer: $f(x) = x^{2/3} - 1$ for each $x \in [1, 8]$.

Solution. Using the substitution $x = y^3$ we find that

$$\int_1^8 f(x)dx = \int_1^2 f(y^3)3y^2dy = 3\int_1^2 x^2 f(x^3)dx.$$

Thus, the left hand side of the given equality is equal to

$$15\int_1^2 \Big(2(x^2-1)f(x^3) - f^2(x^3)\Big)dx = 15\int_1^2 \Big((x^2-1)^2 - \big(x^2-1-f(x^3)\big)^2\Big)dx.$$

In view of $I = \int_1^2 \big(x^2-1-f(x^3)\big)^2 dx \geqslant 0$ the left hand side is less than or equal to

$$15\int_1^2 (x^2-1)^2dx = 15\int_1^2 (x^4-2x^2+1)dx = 93-70+15 = 38,$$

which is the right hand side. Since the function f is continuous, equality $I = 0$ holds if and only if $x^2-1-f(x^3) = 0$ for each $x \in [1,2]$. Hence, $f(x^3) = x^2-1$ for each $x \in [1,2]$, and so $f(x) = x^{2/3}-1$ for each $x \in [1,8]$. $\square$