Mathematical Competition for Students of the Department of Mathematics and Informatics of Vilnius University, Problems and Solutions, 2020

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PROBLEMS

Problem 1. Let r be a fixed positive number, and let $\mathcal{D}_r$ be a class of functions $f : [0, +\infty) \mapsto \mathbb{R}$ which are continuous in $[0, +\infty)$ and satisfy

$$\lim_{x\to 0+} \frac{f(x)}{x^r} = 0.$$

Find all positive numbers t such that for every $f \in \mathcal{D}_r$ the quotient

$$\frac{f(x) - f(0)}{x^t}$$

tends to a finite limit as $x \to 0+$.

Problem 2. Call an ordered triplet of sets (A, B, C) *nice* if

$$|A \cap B| = |B \cap C| = |C \cap A| = 2$$

and $A \cap B \cap C = \emptyset$. How many ordered triplets (A, B, C) of subsets of the set $\{1, 2, \ldots, 10\}$ are nice?

(Here, $|S|$ denotes the number of elements of S.)

Problem 3. Let a_n, $n = 1, 2, 3, \ldots$, be a sequence of real numbers satisfying $a_1 = 1$ and

$$a_{n+1}(a_n^2 - 3) + 4a_n = \sqrt{3}(a_n^2 + 1)$$

for $n = 1, 2, 3, \ldots$. Find a_{50}.

Problem 4. Do there exist two nonconstant polynomials with real coefficients P and Q satisfying

$$P(x)^5 + P(x)^4 = Q(x)^{11} + 2Q(x)^{10}$$

for all real x?

Each problem is worth 10 points.

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PROBLEMS WITH SOLUTIONS

Problem 1. Let r be a fixed positive number, and let $\mathcal{D}_r$ be a class of functions $f : [0, +\infty) \mapsto \mathbb{R}$ which are continuous in $[0, +\infty)$ and satisfy

$$\lim_{x\to 0+} \frac{f(x)}{x^r} = 0.$$

Find all positive numbers t such that for every $f \in \mathcal{D}_r$ the quotient

$$\frac{f(x) - f(0)}{x^t}$$

tends to a finite limit as $x \to 0+$.

Answer: $t \in (0, r]$.

Solution. The limit condition on f as $x \to 0+$ means that for each $\varepsilon > 0$ (in particular, for $\varepsilon = 1$) there exists a real $\delta > 0$ such that $|f(x)| < \varepsilon x^r = x^r$ when $0 < x < \delta$. Thus, by the continuity of the function $f(x)$ (and so $|f(x)|$) at the point $x = 0$, it follows that

$$|f(0)| = \lim_{x\to 0+} |f(x)| \leqslant \lim_{x\to 0+} x^r = 0.$$

This implies $f(0) = 0$ for all $f \in \mathcal{D}_r$, and hence

$$\frac{f(x) - f(0)}{x^t} = \frac{f(x)}{x^t} = \frac{f(x)}{x^r} \cdot x^{r-t}$$

for $x > 0$.

If $t = r$ then $x^{r-t} = 1$, so, by the limit condition on f, the limit in question is 0 for each $f \in \mathcal{D}_r$. Similarly, for $t \in (0, r)$, by the same condition combined with $\lim_{x\to 0+} x^{r-t} = 0$, we conclude that the limit in question is 0 for $t \in (0, r)$ too. On the other hand, in the case when $t > r$, for a counterexample we can take, for instance, $g(x) = x^{(r+t)/2}$. Then, $g \in \mathcal{D}_r$, but the quotient

$$\frac{g(x) - g(0)}{x^t} = \frac{x^{(r+t)/2} - 0}{x^t} = x^{(r-t)/2}$$

tends to infinity as $x \to 0+$. □

Problem 2. Call an ordered triplet of sets (A, B, C) *nice* if

$$|A \cap B| = |B \cap C| = |C \cap A| = 2$$

and $A \cap B \cap C = \emptyset$. How many ordered triplets (A, B, C) of subsets of the set $\{1, 2, \ldots, 10\}$ are nice?

(Here, $|S|$ denotes the number of elements of S.)

Answer: 4838400.

Solution. As $(A\cap B)\cap(B\cap C)=A\cap B\cap C=\emptyset$, the two elements in $A\cap B$ are distinct from the two elements in $B\cap C$. By the same argument, both of these pairs of two elements are distinct from the two elements in $C\cap A$. There are $\binom{10}{2}$ ways to choose the two elements of $A\cap B$, then $\binom{8}{2}$ ways to choose the two elements of $B\cap C$, and then $\binom{6}{2}$ ways to choose the two elements of $C\cap A$. Finally, $10-2-2-2=4$ elements not yet assigned to any of the sets A,B,C may appear in exactly one of A,B,C or in none of them. There are thus 4 ways to pick where to assign each of those four remaining elements, for a total of

$$\binom{10}{2}\cdot\binom{8}{2}\cdot\binom{6}{2}\cdot 4^4=45\cdot 28\cdot 15\cdot 4^4=4838400$$

such ordered triplets (A,B,C). □

Problem 3. Let a_n, $n=1,2,3,\ldots$, be a sequence of real numbers satisfying $a_1=1$ and

$$a_{n+1}(a_n^2-3)+4a_n=\sqrt{3}(a_n^2+1)$$

for $n=1,2,3,\ldots$. Find a_{50}.

Answer: $a_{50}=2-\sqrt{3}$.

First solution. Write $a_n=\tan(\pi\theta_n)$ with a unique $\theta_n\in(-1/2,1/2)$. Clearly, $\theta_1=1/4$. Assume that $a_n\neq\pm\sqrt{3}$ (or, equivalently, $\theta_n\neq\pm1/3$) for some $n\in\mathbb{N}$. Then, using

$$\begin{aligned}\tan(\pi\theta_{n+1})=a_{n+1}&=\frac{\sqrt{3}(a_n^2+1)-4a_n}{a_n^2-3}=\frac{(\sqrt{3}a_n-1)(a_n-\sqrt{3})}{(a_n+\sqrt{3})(a_n-\sqrt{3})}=\frac{\sqrt{3}a_n-1}{a_n+\sqrt{3}}\\&=\frac{a_n-\tan(\pi/6)}{a_n\tan(\pi/6)+1}=\frac{\tan(\pi\theta_n)-\tan(\pi/6)}{\tan(\pi\theta_n)\tan(\pi/6)+1}=\tan(\pi(\theta_n-1/6)),\end{aligned}$$

we find that for every such n

$$\theta_{n+1}=\begin{cases}\theta_n-1/6 & \text{if}\quad \theta_n\in(-1/3,1/2),\\ \theta_n+5/6 & \text{if}\quad \theta_n\in(-1/2,-1/3).\end{cases}$$

Therefore, step by step, we calculate $\theta_1=1/4$, $\theta_2=1/12$, $\theta_3=-1/12$, $\theta_4=-1/4$, $\theta_5=-5/12$, $\theta_6=5/12$. Afterwards, the sequence is continued periodically (with period 6) in view of $\theta_7=\theta_6-1/6=1/4=\theta_1$. (In particular, $\theta_n\neq\pm1/3$ for each $n\in\mathbb{N}$, so $a_n\neq\pm\sqrt{3}$.) Now, from $\theta_{n+6}=\theta_n$ we get $\theta_{50}=\theta_{50-6}=\cdots=\theta_{50-8\cdot6}=\theta_2=1/12$. Using

$$\tan(\pi/12)=\frac{\sin(\pi/12)}{\cos(\pi/12)}=\frac{2\sin(\pi/12)\cos(\pi/12)}{2\cos^2(\pi/12)}=\frac{\sin(\pi/6)}{1+\cos(\pi/6)}=\frac{\frac{1}{2}}{1+\frac{\sqrt{3}}{2}}=2-\sqrt{3},$$

we derive that $a_{50}=\tan(\pi\theta_{50})=\tan(\pi/12)=2-\sqrt{3}$. □

Second solution. Here is direct solution (without tangent function). By the formula

$$a_{n+1} = \frac{\sqrt{3}(a_n^2+1) - 4a_n}{a_n^2 - 3} = \frac{(\sqrt{3}a_n - 1)(a_n - \sqrt{3})}{(a_n + \sqrt{3})(a_n - \sqrt{3})} = \frac{\sqrt{3}a_n - 1}{a_n + \sqrt{3}},$$

we find that

$$a_2 = 2 - \sqrt{3},\ a_3 = -2 + \sqrt{3},\ a_4 = -1,\ a_5 = -2 - \sqrt{3},\ a_6 = 2 + \sqrt{3},\ a_7 = 1 = a_1.$$

Consequently, the sequence a_n, $n = 1, 2, 3, \dots$, is periodic with period 6, and hence $a_{50} = a_{50-6} = \cdots = a_{50-8\cdot 6} = a_2 = 2 - \sqrt{3}$. □

Problem 4. Do there exist two nonconstant polynomials with real coefficients P and Q satisfying

$$P(x)^5 + P(x)^4 = Q(x)^{11} + 2Q(x)^{10}$$

for all real x?

Answer: No.

Solution. Suppose such P and Q exist. Since the integers 5 and 11 are relatively prime, by the degree consideration, there exists $n \in \mathbb{N}$ such that $\deg P = 11n$ and $\deg Q = 5n$. By taking the derivatives of both sides of the given identity, we find that

$$P'(x)(5P(x) + 4)P(x)^3 = Q'(x)(11Q(x) + 20)Q(x)^9.$$

Suppose $\alpha \in \mathbb{C}$ is a root of the polynomial $5P(x) + 4$, i.e., $P(\alpha) = -4/5$. Then,

$$Q(\alpha)^{10}(Q(\alpha) + 2) = Q(\alpha)^{11} + 2Q(\alpha)^{10} = P(\alpha)^5 + P(\alpha)^4 = \frac{4^4}{5^5} \neq 0,$$

so $Q(\alpha) \neq 0$. Thus, each root of the polynomial $5P(x) + 4$ must be a root of the polynomial $Q'(x)(11Q(x) + 20)$. However, by the fundamental theorem of algebra, $5P(x) + 4$ has $11n$ roots in $\mathbb{C}$ (counted with multiplicities), whereas $Q'(x)(11Q(x) + 20)$ has only $5n - 1 + 5n = 10n - 1 < 11n$ roots in $\mathbb{C}$ (counted with multiplicities), a contradiction. □