Mathematical Competition for Students (MIFMO) of the Department of Mathematics and Informatics of Vilnius University

2020-01-25

(organized by Paulius Drungilas and Artūras Dubickas)

Problem 1. Let r be a fixed positive number, and let $\mathcal{D}_r$ be a class of functions $f : [0, +\infty) \mapsto \mathbb{R}$ which are continuous in $[0, +\infty)$ and satisfy

$$\lim_{x \to 0+} \frac{f(x)}{x^r} = 0.$$

Find all positive numbers t such that for every $f \in \mathcal{D}_r$ the quotient

$$\frac{f(x) - f(0)}{x^t}$$

tends to a finite limit as $x \to 0+$.

Problem 2. Call an ordered triplet of sets (A, B, C) *nice* if

$$|A \cap B| = |B \cap C| = |C \cap A| = 2$$

and $A \cap B \cap C = \emptyset$. How many ordered triplets (A, B, C) of subsets of the set $\{1, 2, \ldots, 10\}$ are nice?

(Here, $|S|$ denotes the number of elements of S.)

Problem 3. Let a_n, $n = 1, 2, 3, \ldots$, be a sequence of real numbers satisfying $a_1 = 1$ and

$$a_{n+1}(a_n^2 - 3) + 4a_n = \sqrt{3}(a_n^2 + 1)$$

for $n = 1, 2, 3, \ldots$. Find a_{50}.

Problem 4. Do there exist two nonconstant polynomials with real coefficients P and Q satisfying

$$P(x)^5 + P(x)^4 = Q(x)^{11} + 2Q(x)^{10}$$

for all real x?

Each problem is worth 10 points.