Mathematical Competition for Students of the Department of Mathematics and Informatics of Vilnius University, Problems and Solutions, 2019

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PROBLEMS

Problem 1. Show that there are infinitely many pairs of irrational numbers (a, b) such that

$$\frac{2a + 3b + 2019(ab)^3}{(ab+1)^2} \in \mathbb{Q}.$$

Problem 2. Prove that there exist 3×3 matrices A and B such that $ABAB = 0$ and $BABA \neq 0$.

Problem 3. Let S be the set $\{2^k : k \in \mathbb{Z}\}$ and let $f : [1, \infty) \to (0, \infty)$ be a continuous function with the following property: for each $a \in S$ the equation $f(x) = ax^2$ has a solution in $x \in [1, \infty)$.

a) Prove that for each $a > 0$ there exist infinitely many $x > 1$ satisfying $f(x) = ax^2$.

b) Is there a function f as described above which is increasing in $[1, \infty)$?

Problem 4. A group of Facebook contains 2019 members. Some of them are friends. Anybody can send a message to any of its friends. Suppose Angela and Theresa have the largest and the smallest number of friends respectively (any other member of the group has less friends than Angela and more friends than Theresa). Let N be the sum of the friends of Angela and Theresa (joint friends are counted twice). Donald and Vladimir are also members of the group, but they are not friends. Find the smallest possible N such that Donald can always send a message to Vladimir so that at most 3 other members of the group will be able to read it during its transfer.

Each problem is worth 10 points.

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PROBLEMS WITH SOLUTIONS

Problem 1. Show that there are infinitely many pairs of irrational numbers (a, b) such that

$$\frac{2a + 3b + 2019(ab)^3}{(ab + 1)^2} \in \mathbb{Q}.$$

Solution. Fix any $k \in \mathbb{N}$ and select

$$a := \frac{k + \sqrt{2}}{2} \quad \text{and} \quad b := \frac{k - \sqrt{2}}{3}.$$

For each $k \in \mathbb{N}$ the numbers a and b are both irrational, since $\sqrt{2} \notin \mathbb{Q}$. With this choice of a, b, one has $2a + 3b = 2k$ and $ab = (k^2 - 2)/6$. Consequently,

$$S = 2a + 3b + 2019(ab)^3 = 2k + \frac{2019(k^2 - 2)^3}{6^3} \in \mathbb{Q}$$

and $V = (ab + 1)^2 = ((k^2 + 4)/6)^2 \in \mathbb{Q}$. Since $V \neq 0$, the quotient S/V is rational for each $k \in \mathbb{N}$. □

Problem 2. Prove that there exist 3×3 matrices A and B such that $ABAB = 0$ and $BABA \neq 0$.

Solution. Take, for example,

$$A := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then,

$$AB = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad BA = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and hence $ABAB = 0$, but

$$BABA = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a nonzero matrix. □

Problem 3. Let S be the set $\{2^k : k \in \mathbb{Z}\}$ and let $f : [1, \infty) \to (0, \infty)$ be a continuous function with the following property: for each $a \in S$ the equation $f(x) = ax^2$ has a solution in $x \in [1, \infty)$.

a) Prove that for each $a > 0$ there exist infinitely many $x > 1$ satisfying $f(x) = ax^2$.
b) Is there a function f as described above which is increasing in $[1, \infty)$?

Answer: b) Yes.

Solution. a) Fix any $a > 0$. If the assertion a) were false then there exists some $B \geqslant 1$ such that $f(x) \neq ax^2$ for all $x \geqslant B$. Since the function f is continuous, there are two possibilities: $f(x) > ax^2$ for all $x \geqslant B$ or $f(x) < ax^2$ for all $x \geqslant B$.

In the first case, $f(x) > ax^2$ for $x \geqslant B$, we set

$$m := \min_{x \in [1,B]} \frac{f(x)}{x^2}$$

and choose any $a_1 \in S$ smaller than $b := \min(m, a)$. (Clearly, as $b > 0$, there exists $k \in \mathbb{Z}$ for which $2^k < b$.) Then, the inequality $f(x) > a_1 x^2$ holds for all $x \geqslant 1$. In particular, $a_1 \in S$ but the equation $f(x) = a_1 x^2$ has no solution in $x \in [1, \infty)$, a contradiction.

In the second case, $f(x) < ax^2$ for $x \geqslant B$, we set

$$M := \max_{x \in [1,B]} \frac{f(x)}{x^2}$$

and choose any $a_2 \in S$ greater than $\max(M, a)$. Then, $f(x) < a_2 x^2$ for all $x \geqslant 1$, and hence $f(x) = a_2 x^2$ has no solution in $x \in [1, \infty)$.

b) Consider any unbounded sequence $x_1 = 1 < y_1 < x_2 < y_2 < x_3 < y_3 < \dots$ satisfying $y_i > x_i^3$ for each $i \in \mathbb{N}$. Set $f(x_i) = x_i^3$ and $f(y_i) = y_i$ for each $i \in \mathbb{N}$ and extend f by linearity, namely, assume that $f(x)$ has the form $a_i x + b_i$ in each interval $[x_i, y_i]$, $i = 1, 2, 3, \dots$. (Similarly, f is linear in each interval $[y_i, x_{i+1}]$, $i = 1, 2, 3, \dots$.) Clearly, such a function f is continuous, positive and increasing in $[1, \infty)$. Fix any $a > 0$. For each $i \in \mathbb{N}$ one has $f(x_i) = x_i^3$ which is greater than ax_i^2 when i is so large that $x_i > a$. Similarly, $f(y_i) = y_i < ay_i^2$ for each sufficiently large i. Therefore, for each sufficiently large i the function $f(x) - ax^2$ is positive at $x = x_i$ and negative at $x = y_i$. Hence, by the continuity of f, the open interval (x_i, y_i) contains a point ξ_i satisfying $f(\xi_i) = a\xi_i^2$. □

Problem 4. A group of Facebook contains 2019 members. Some of them are friends. Anybody can send a message to any of its friends. Suppose Angela and Theresa have the largest and the smallest number of friends respectively (any other member of the group has less friends than Angela and more friends than Theresa). Let N be the sum of the friends of Angela and Theresa (joint friends are counted twice). Donald and Vladimir are also members of the group, but they are not friends. Find the smallest possible N such that Donald can always send a message to Vladimir so that at most 3 other members of the group will be able to read it during its transfer.

Answer: $N = 2017$.

Solution. First, we will give an example showing that $N = 2016$ is not sufficient. Suppose Donald, Melania and Ivanka are all friends. Suppose all other 2015 members of the group except for the three listed above and Theresa are all friends as well, whereas Theresa has the only friend Angela. In the situation when there are no more friends except for those described above one has $n(A) = 2015$, $n(T) = 1$, $n(D) = n(M) = n(I) = 2$ and $n(X) = 2014$ for any other member of the group X. (Here and below, $n(Z)$ stands for the number of friends of Z.) In this example we have $N = n(A) + n(T) = 2016$ and $n(A) > n(Z) > n(T)$ for any $Z \notin \{A, T\}$, but Donald cannot send a message to Vladimir through the members of the group.

Suppose that $N = 2017$. Then, $n(A) + n(D) > n(A) + n(T) = 2017$, which implies $n(A) + n(D) \geqslant 2018$. Since the whole group, without Angela and Donald, contains 2017 members, either Angela and Donald are friends or they have a joint friend, say, Emmanuel. By the same argument, either Angela and Vladimir are friends or they have a joint friend, say, Alexander. Therefore, using at most three members of the group Donald can always transmit his message to Vladimir. The longest transfer will be as follows:

Donald $\to$ Emmanuel $\to$ Angela $\to$ Alexander $\to$ Vladimir.

(Three other possibilities are even shorter: Donald $\to$ Angela $\to$ Alexander $\to$ Vladimir, Donald $\to$ Emmanuel $\to$ Angela $\to$ Vladimir or Donald $\to$ Angela $\to$ Vladimir.) $\square$