Mathematical Competition for Students of the Department of Mathematics and Informatics of Vilnius University Problems and Solutions

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PROBLEMS

Problem 1. Let S be the set of all real numbers x satisfying $x^4 + 36 \leqslant 13x^2$. Find the largest and the smallest values of $x^3 - 3x$ when $x \in S$.

Problem 2. Prove that for every matrix $A \in M_2(\mathbb{R})$ there exist two matrices $B, C \in M_2(\mathbb{R})$ such that

$$A = B^3 + C^3.$$

(Here, $M_2(\mathbb{R})$ denotes the set of 2×2 matrices with real coefficients.)

Problem 3. Let

$$P(x) = x^{2016} + a_1 x^{2015} + a_2 x^{2014} + \cdots + a_{2015} x + a_{2016}$$

be a polynomial of degree 2016 whose coefficients a_j belong to the set $\{-1, 1\}$ for each $j = 1, 2, \ldots, 2016$. Prove that P has less than 2016 distinct real roots.

Problem 4. Find the value of the limit

$$\lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{1}{(i+j+1)i!j!}.$$

Each problem is worth 10 points.

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PROBLEMS WITH SOLUTIONS

Problem 1. Let S be the set of all real numbers x satisfying $x^4 + 36 \leqslant 13x^2$. Find the largest and the smallest values of $x^3 - 3x$ when $x \in S$.

Answer: The largest value is 18; the smallest value is -18.

Solution. Since $x^4 + 36 - 13x^2 = (x^2 - 4)(x^2 - 9) = (x-3)(x-2)(x+2)(x+3) \leqslant 0$, we find that $S = [-3,-2] \cup [2,3]$. The function $f(x) := x^3 - 3x$ is increasing on $[-3,-2]$ and also increasing on $[2,3]$, since its derivative $f'(x) = 3x^2 - 3 = 3(x^2-1)$ is positive in S. Thus, the largest value of $f(x)$ for $x \in S$ equals $\max\{f(-2), f(3)\} = \max\{-2, 18\} = 18$. The smallest value of $f(x)$ for $x \in S$ is $\min\{f(-3), f(2)\} = \min\{-18, 2\} = -18$. □

Problem 2. Prove that for every matrix $A \in M_2(\mathbb{R})$ there exist two matrices $B, C \in M_2(\mathbb{R})$ such that

$$A = B^3 + C^3.$$

(Here, $M_2(\mathbb{R})$ denotes the set of 2×2 matrices with real coefficients.)

Solution 1. We claim that every upper triangular matrix $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is the cube of some matrix from $M_2(\mathbb{R})$, provided that $ad \neq 0$. Indeed, using the equality

$$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}^3 = \begin{pmatrix} x^3 & y(x^2 + xz + z^2) \\ 0 & z^3 \end{pmatrix}, \tag{1}$$

which holds for all $x, y, z \in \mathbb{R}$, we find that

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}^3 \iff \begin{cases} a = x^3, \\ b = y(x^2 + xz + z^2), \\ d = z^3. \end{cases}$$

For $ad \neq 0$ we can solve it as follows:

$$x = \sqrt[3]{a}, \quad z = \sqrt[3]{d} \quad \text{and} \quad y = \frac{b}{\sqrt[3]{a^2} + \sqrt[3]{ad} + \sqrt[3]{d^2}}.$$

(Note that the denominator is positive, i.e., $\sqrt[3]{a^2} + \sqrt[3]{ad} + \sqrt[3]{d^2} > 0$, if $ad \neq 0$.)

Similarly, by considering the transpose matrices in (1), we see that

$$\begin{pmatrix} x & 0 \\ y & z \end{pmatrix}^3 = \begin{pmatrix} x^3 & 0 \\ y(x^2 + xz + z^2) & z^3 \end{pmatrix}.$$

Hence, every lower triangular matrix $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ is also the cube of some matrix from $M_2(\mathbb{R})$, provided that $ad \neq 0$.

Finally, note that every matrix from $M_2(\mathbb{R})$ is the sum of an upper and a lower triangular matrices with the required property $ad \neq 0$. Indeed, writing

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+t & b \\ 0 & d+t \end{pmatrix} + \begin{pmatrix} -t & 0 \\ c & -t \end{pmatrix} \tag{2}$$

with some sufficiently large $t \in \mathbb{R}$ (e.g., for $t > |a| + |d|$), we have $(a+t)(d+t) \neq 0$ and $t^2 \neq 0$. Hence, by the above, both matrices on the right hand side of (2) are cubes of some matrices B and C from $M_2(\mathbb{R})$, that is, $A = B^3 + C^3$. $\square$

Solution 2. Let $A \in M_2(\mathbb{R})$. By the Cayley-Hamilton theorem,

$$A^2 - aA + dI = \mathcal{O},$$

where a is the trace of A (i.e., the sum of the elements on the main diagonal of A), d is the determinant of A, $\mathcal{O} \in M_2(\mathbb{R})$ is the zero matrix, and $I \in M_2(\mathbb{R})$ is the identity matrix. Hence, $A^2 = aA - dI$. Using this equality, for any $t \in \mathbb{R}$ we deduce that

$$\begin{aligned}
(A+tI)^3 &= A^3 + 3tA^2 + 3t^2A + t^3I \\
&= A(aA - dI) + 3t(aA - dI) + 3t^2A + t^3I \\
&= aA^2 + (3t^2 + 3ta - d)A + (t^3 - 3td)I \\
&= a(aA - dI) + (3t^2 + 3ta - d)A + (t^3 - 3td)I \\
&= (3t^2 + 3ta + a^2 - d)A + (t^3 - 3td - ad)I.
\end{aligned}$$

Thus, denoting $u := 3t^2 + 3ta + a^2 - d$ and $v := t^3 - 3td - ad$, we obtain

$$(A+tI)^3 = uA + vI.$$

Now, choose sufficiently large t for which $u > 0$. Then,

$$A = \frac{1}{u}(A+tI)^3 - \frac{v}{u}I = \left(\frac{1}{\sqrt[3]{u}}(A+tI)\right)^3 + \left(-\sqrt[3]{\frac{v}{u}}I\right)^3,$$

whence the result. $\square$

Problem 3. Let

$$P(x) = x^{2016} + a_1x^{2015} + a_2x^{2014} + \cdots + a_{2015}x + a_{2016}$$

be a polynomial of degree 2016 whose coefficients a_j belong to the set $\{-1, 1\}$ for each $j = 1, 2, \ldots, 2016$. Prove that P has less than 2016 distinct real roots.

Solution. Set, for brevity, $n = 2016$ and assume that for some choice of the coefficients ± 1 the polynomial P has n distinct real roots $\alpha_1, \ldots, \alpha_n$. Then,

$$P(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1}x + a_n = (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n).$$

Consider the sum of squares $S := \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2$. By Vieta's formulas,

$$S = (\alpha_1 + \alpha_2 + \cdots + \alpha_n)^2 - 2(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \ldots \alpha_{n-1}\alpha_n) = a_1^2 - 2a_2 = 1 - 2a_2.$$

Hence, $S \leqslant 1 + 2 = 3$. On the other hand, the arithmetic mean vs geometric mean inequality implies

$$S = \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 \geqslant n(\alpha_1^2\alpha_2^2\ldots\alpha_n^2)^{1/n} = n\left(a_n^2\right)^{1/n} = n.$$

Consequently, $n \leqslant S \leqslant 3$, which is impossible for each $n \geqslant 4$. (In particular, it is impossible for $n = 2016$.) □

Problem 4. Find the value of the limit

$$\lim_{n\to\infty} \sum_{i=0}^{n}\sum_{j=0}^{n} \frac{1}{(i+j+1)i!j!}.$$

Answer: $(e^2-1)/2$.

Solution. Fix $x \geqslant 0$ and $n \in \mathbb{N}$. By the Taylor formula for exponential series with the Lagrange remainder form, we have $e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + r_n(x)\frac{x^{n+1}}{(n+1)!}$, where $1 \leqslant r_n(x) \leqslant e^x$. Thus,

$$\begin{aligned} e^{2x} = e^x \cdot e^x &= \Big(\sum_{i=0}^{n} \frac{x^i}{i!} + r_n(x)\frac{x^{n+1}}{(n+1)!}\Big)\cdot\Big(\sum_{j=0}^{n} \frac{x^j}{j!} + r_n(x)\frac{x^{n+1}}{(n+1)!}\Big) \\ &= \Big(\sum_{i=0}^{n} \frac{x^i}{i!}\Big)\Big(\sum_{j=0}^{n} \frac{x^j}{j!}\Big) + r_n(x)\frac{x^{n+1}}{(n+1)!}\sum_{j=0}^{n}\frac{x^j}{j!} + e^x r_n(x)\frac{x^{n+1}}{(n+1)!} \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n}\frac{x^{i+j}}{i!j!} + r_n(x)\frac{x^{n+1}}{(n+1)!}\Big(e^x + \sum_{j=0}^{n}\frac{x^j}{j!}\Big). \end{aligned}$$

Set

$$\varphi_n(x) := r_n(x)\frac{x^{n+1}}{(n+1)!}\Big(e^x + \sum_{j=0}^{n}\frac{x^j}{j!}\Big).$$

Observe that for each $x \in [0,1]$ and each $n \in \mathbb{N}$ we have

$$0 \leqslant \varphi_n(x) \leqslant r_n(x)\frac{x^{n+1}}{(n+1)!}\Big(e^x + \sum_{j=0}^{\infty}\frac{x^j}{j!}\Big) \leqslant e^x \cdot \frac{1}{(n+1)!}(e^x + e^x) \leqslant \frac{2e^2}{(n+1)!}.$$

Hence, $\lim_{n\to\infty}\int_0^1 \varphi_n(x)dx = 0$.

Integrating from 0 to 1 and using the equality $\int_0^1 x^{i+j}dx = \frac{1}{i+j+1}$, we find that

$$\int_0^1 e^{2x}dx = \int_0^1 \sum_{i=0}^{n}\sum_{j=0}^{n} \frac{x^{i+j}}{i!j!}dx + \int_0^1 \varphi_n(x)dx = \sum_{i=0}^{n}\sum_{j=0}^{n} \frac{\int_0^1 x^{i+j}dx}{i!j!}dx + \int_0^1 \varphi_n(x)dx$$

$$= \sum_{i=0}^{n}\sum_{j=0}^{n} \frac{1}{(i+j+1)i!j!} + \int_0^1 \varphi_n(x)dx.$$

Therefore,

$$\lim_{n\to\infty} \sum_{i=0}^{n}\sum_{j=0}^{n} \frac{1}{(i+j+1)i!j!} = \int_0^1 e^{2x}dx - \lim_{n\to\infty}\int_0^1 \varphi_n(x)dx = \frac{e^{2x}}{2}\Big|_0^1 - 0 = \frac{e^2-1}{2},$$

whence the result. □