Mathematical Competition for Students of the Department of Mathematics and Informatics of Vilnius University Problems and Solutions

Paulius Drungilas[1], Artūras Dubickas[2] and Jonas Jankauskas[3]

2014-02-08

PROBLEMS

Problem 1. Find all real y for which the equation $x^2 + x\sin(\pi y) + 2\cos(\pi y) = 0$ has two roots of the form $x_1 = \sin z$ and $x_2 = \cos z$, where $z = z(y) \in [0, 1]$.

Problem 2. Suppose $a_0 > a_1 > a_2 > a_3 > \dots$ is a decreasing sequence of positive numbers satisfying $\sum_{k=0}^{\infty} a_k = 1$. Is there a constant C for which the inequality

$$(n+1)^2 \sum_{k=n}^{\infty} a_k^3 \leqslant C$$

holds for each integer $n \geqslant 0$? If so, find the smallest such constant.

Problem 3. Let $a \geqslant 2$ and b be two integers. Prove that the sequence $a^{n^{2014}} + b$, $n = 1, 2, 3, \dots$, contains infinitely many composite numbers. (An integer $n \geqslant 2$ is called *composite* if it is not a prime number.)

Problem 4. Let S be a nonempty set, and let $*$ be an operation which to any $a, b \in S$ assigns some element $a*b \in S$ and satisfies the associativity property $(a*b)*c = a*(b*c)$ for all $a, b, c \in S$. Assume that for each $a \in S$ there is a unique $b = b(a) \in S$ satisfying $a * b * a = a$.

a) Prove that S contains an idempotent. (An element $e \in S$ is called *idempotent* if $e * e = e$.)

b) Prove that S contains a unique idempotent.

Each problem is worth 10 points.

[1]Vilnius University, Department of Mathematics and Informatics, Naugarduko 24, Vilnius LT-03225, Lithuania, http://www.mif.vu.lt/∼drungilas/

[2]Vilnius University, Department of Mathematics and Informatics, Naugarduko 24, Vilnius LT-03225, Lithuania, http://www.mif.vu.lt/∼dubickas/

[3]Vilnius University, Department of Mathematics and Informatics, Naugarduko 24, Vilnius LT-03225, Lithuania, http://www.mif.vu.lt/∼jonakank/ and Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

PROBLEMS WITH SOLUTIONS

Problem 1. Find all real y for which the equation $x^2 + x\sin(\pi y) + 2\cos(\pi y) = 0$ has two roots of the form $x_1 = \sin z$ and $x_2 = \cos z$, where $z = z(y) \in [0, 1]$.

Answer. $y = 2k - 1/2$, where $k \in \mathbb{Z}$.

Solution. Assume that such $z = z(y)$ exists for some $y \in \mathbb{R}$. Then

$$1 = x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = (-\sin(\pi y))^2 - 4\cos(\pi y) = \sin^2(\pi y) - 4\cos(\pi y).$$

This yields $4\cos(\pi y) = \sin^2(\pi y) - 1 = -\cos^2(\pi y)$. Since $\cos(\pi y) \neq -4$, we obtain $\cos(\pi y) = 0$, and thus $\sin(\pi y) = \pm 1$. It follows that one of the roots of the equation $x^2 + x\sin(\pi y) + 2\cos(\pi y) = 0$ must be 0 and the other root, $-\sin(\pi y)$, either -1 or 1. However, for $z \in [0, 1]$, both roots $x_1 = \sin z$ and $x_2 = \cos z$ are nonnegative. Hence, the roots must be 0 and 1, and thus $\sin(\pi y) = -1$. It follows that $\pi y = -\pi/2 + 2\pi k$ with $k \in \mathbb{Z}$, i.e., $y = 2k - 1/2$. Conversely, for $y = 2k - 1/2$, where $k \in \mathbb{Z}$, we have $\cos(\pi y) = 0$ and $\sin(\pi y) = -1$, so the equation is $x^2 - x = 0$. It has two roots $x_1 = 0$ and $x_2 = 1$, so we can select $z = 0 \in [0, 1]$ for each y of the form $2k - 1/2$. □

Problem 2. Suppose $a_0 > a_1 > a_2 > a_3 > \dots$ is a decreasing sequence of positive numbers satisfying $\sum_{k=0}^{\infty} a_k = 1$. Is there a constant C for which the inequality

$$(n+1)^2 \sum_{k=n}^{\infty} a_k^3 \leqslant C$$

holds for each integer $n \geqslant 0$? If so, find the smallest such constant.

Answer. The smallest such constant is $C = 1$.

Solution. Note that

$$a_n \leqslant \frac{1}{n+1} \sum_{k=0}^{n} a_k < \frac{1}{n+1} \sum_{k=0}^{\infty} a_k = \frac{1}{n+1}$$

for $n \geqslant 0$. Hence, for each integer $n \geqslant 0$ we obtain

$$\sum_{k=n}^{\infty} a_k^3 < \sum_{k=n}^{\infty} a_n^2 a_k = a_n^2 \sum_{k=n}^{\infty} a_k \leqslant a_n^2 \sum_{k=0}^{\infty} a_k = a_n^2 < \frac{1}{(n+1)^2},$$

so the required inequality (even strict inequality) holds for $C = 1$. To show that $C = 1$ is the smallest such constant, we assume that the inequality $C_n := (n+1)^2 \sum_{k=n}^{\infty} a_k^3 \leqslant C$ holds for some $0 < C < 1$ and each $n \geqslant 0$. Consider the sequence $a_0 := 1 - \varepsilon$ and $a_n := \varepsilon 2^{-n}$ for $n \in \mathbb{N}$, where $0 < \varepsilon < \min(2/3, 1 - C^{1/3})$. (It is a decreasing sequence of positive numbers satisfying $\sum_{k=0}^{\infty} a_k = 1$.) Inserting $n = 0$ into C_n, by the choice of ε, we find that $C \geqslant C_0 = \sum_{k=0}^{\infty} a_k^3 > a_0^3 = (1 - \varepsilon)^3 > C$, a contradiction. □

Problem 3. Let $a \geqslant 2$ and b be two integers. Prove that the sequence $a^{n^{2014}} + b$, $n = 1, 2, 3, \dots$, contains infinitely many composite numbers. (An integer $n \geqslant 2$ is called *composite* if it is not a prime number.)

Solution. Set $f(n) := a^{n^{2014}} + b$ and assume that there exists $N \in \mathbb{N}$ such that the numbers $f(n)$, $n = N, N+1, \dots$, are all prime. Select any $m \geqslant N$ for which the inequality $a^{m^{2014}} \geqslant |b| + 2$ holds. Then $p = f(m) \geqslant 2$ is a prime number. By Fermat's little theorem, for any positive integers A, d we have

$$A^{p^d} \pmod p \equiv A^{p^{d-1}} \pmod p \equiv \cdots \equiv A^p \pmod p \equiv A \pmod p.$$

Applying this to $A := a^{m^{2014}}$ and $d := 2014$, we find that

$$f(mp) - p = f(mp) - f(m) = a^{(mp)^{2014}} - a^{m^{2014}} = A^{p^d} - A$$

is divisible by p. Hence, $p|f(mp)$. Therefore, the number $f(mp)$ is composite, since $f(mp) > f(m) = p$, and $mp > m \geqslant N$, a contradiction.

Here is an alternative solution. The statement is clear for $b = 0$, so assume that $b \neq 0$. Also, we may assume that the numbers a and b are coprime, since otherwise the result is trivial. Select $m \in \mathbb{N}$ so large that $c = a^{m^{2014}}$ is greater than $|b| + 2$. Since c and b are coprime, the integers $c + b \geqslant 2$ and b are also coprime. Hence, by Euler's theorem, $c^{\varphi(c+b)}$, where $\varphi(m)$ is Euler's function, is equal to 1 modulo $c + b$. Selecting $n = m(\varphi(c+b)k + 1)$, where $k = 1, 2, 3, \dots$, we obtain

$$a^{n^{2014}} = c^{(\varphi(c+b)k+1)^{2014}} = c^{\varphi(c+b)K+1}$$

with $K \in \mathbb{N}$, thus $a^{n^{2014}}$ is c modulo $c+b$. Hence, for each $k \geqslant 2$, the number $a^{n^{2014}} + b = c^{\varphi(c+b)K+1} + b$ is divisible by $c + b \geq 2$ and is greater than $c + b$, so it is a composite number. □

Problem 4. Let S be a nonempty set, and let $*$ be an operation which to any $a, b \in S$ assigns some element $a * b \in S$ and satisfies the associativity property $(a*b)*c = a*(b*c)$ for all $a, b, c \in S$. Assume that for each $a \in S$ there is a unique $b = b(a) \in S$ satisfying $a * b * a = a$.

a) Prove that S contains an idempotent. (An element $e \in S$ is called *idempotent* if $e * e = e$.)

b) Prove that S contains a unique idempotent.

Solution. Take any $a \in S$ and a unique $b \in S$ for which $a = a * b * a$. Then $a * b = a * b * a * b = (a * b) * (a * b)$, so $a * b$ is an idempotent. This proves part *a*). Moreover, as $a * b * a * b * a = a * b * a = a$, in view of the uniqueness of b we must have $b * a * b = b$.

Thus, if b is the unique for a satisfying $a * b * a = a$ then a is also the unique for b satisfying $b * a * b = b$.

To prove part b) let us assume that there are at least two distinct idempotents $x \neq y$ in S. Take $z \in S$ for which $x * y = x * y * z * x * y$. (By the above, we also have $z = z * x * y * z$.) As $y = y * y$ and

$$x * y = x * y * z * x * y = x * y * y * z * x * y = x * y * (y * z) * x * y,$$

by the uniqueness property, we must have $z = y * z$. By a similar argument, $z = z * x$. Hence, $z = z * x * y * z = z * x * z$ and $z = z * x * y * z = z * y * z$. By the uniqueness property, we now obtain $x * y = x = y$, contrary to $x \neq y$. □