Mathematical Competition for Students of the Department of Mathematics and Informatics of Vilnius University Problems and Solutions

Paulius Drungilas[1], **Artūras Dubickas**[2] **and Jonas Jankauskas**[3]

2013-02-23

PROBLEMS

Problem 1. Evaluate the integral

$$\int_0^\pi \frac{x \sin x}{1+\cos^2 x}\, dx.$$

Problem 2. A polynomial $p(x) \in \mathbb{R}[x]$ is called *positive* if $p(y) > 0$ for each $y \in \mathbb{R}$. Suppose that $p(x) \in \mathbb{R}[x]$ is positive. Prove that the polynomials

$$p(x) - p'(x) + \frac{p''(x)}{2!} - \frac{p'''(x)}{3!} + \cdots$$

and

$$p(x) + p'(x) + p''(x) + p'''(x) + \cdots$$

are both positive.

Problem 3. Define a *selfish* set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subset is selfish.

Problem 4. For each positive integer n, let d_n denote the greatest common divisor of the four entries of the matrix

$$\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}^n + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(For example, $d_1 = \gcd(4,2,4,4) = 2$.) Prove that $\lim_{n\to\infty} d_n = \infty$.

Each problem is worth 10 points.

[1]Vilnius University, Department of Mathematics and Informatics, Naugarduko 24, Vilnius LT-03225, Lithuania, http://www.mif.vu.lt/∼drungilas/

[2]Vilnius University, Department of Mathematics and Informatics, Naugarduko 24, Vilnius LT-03225, Lithuania, http://www.mif.vu.lt/∼dubickas/

[3]Vilnius University, Department of Mathematics and Informatics, Naugarduko 24, Vilnius LT-03225, Lithuania, http://www.mif.vu.lt/∼jonakank/ and Department of Mathematics, Simon Fraser University, 8888 University Drive, Burnaby, British Columbia, Canada V5A 1S6

PROBLEMS WITH SOLUTIONS

Problem 1. Evaluate the integral
$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx.$$

Answer. The integral is equal to $\pi^2/4$.

Solution. Put $f(x) := \operatorname{arctg}(\cos x)$ and observe that $f'(x) = -\sin x/(1+\cos^2 x)$. Integrating by parts, we obtain
$$I := \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx = -\int_0^{\pi} x\, f'(x)\, dx = -x f(x)\Big|_0^{\pi} + \int_0^{\pi} f(x)\, dx.$$
Since $f(\pi) = \operatorname{arctg}(-1) = -\pi/4$ and $f(x) = -f(\pi - x)$ for each $x \in [0,\pi]$, we find that
$$I = -\pi \cdot f(\pi) + 0 \cdot f(0) + 0 = \frac{\pi^2}{4},$$
as claimed.

Here is another variation of this proof. By changing the variable x into $\pi - x$, we obtain
$$I := \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\, dx.$$
Adding both these integrals, we deduce that
$$2I = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}\, dx = -\pi \int_0^{\pi} f'(x)\, dx = -\pi(f(\pi) - f(0)) = -\pi\Big(-\frac{\pi}{4} - \frac{\pi}{4}\Big) = \frac{\pi^2}{2},$$
whence the result.

Finally, we shall give an alternative proof (without the immediate introduction of the function $f(x)$ as above or using the fact that $\sin x\, dx = -d\cos x$). Fix a positive real number $\varepsilon < \pi/2$. For each $x \in [\varepsilon, \pi - \varepsilon]$ we have
$$\frac{1}{1+\cos^2 x} = 1 - \cos^2 x + \cos^4 x - \cos^6 x + \cdots = \sum_{k=0}^{\infty} (-1)^k \cos^{2k} x.$$
Note that the above series converge uniformly in $[\varepsilon, \pi-\varepsilon]$. Hence
$$I(\varepsilon) := \int_{\varepsilon}^{\pi-\varepsilon} \frac{x \sin x}{1+\cos^2 x}\, dx = \int_{\varepsilon}^{\pi-\varepsilon} \sum_{k=0}^{\infty} (-1)^k x \sin x \cos^{2k} x\, dx$$
$$= \sum_{k=0}^{\infty} (-1)^k \int_{\varepsilon}^{\pi-\varepsilon} x \sin x \cos^{2k} x\, dx = \sum_{k=0}^{\infty} (-1)^{k+1} \int_{\varepsilon}^{\pi-\varepsilon} x \left(\frac{\cos^{2k+1} x}{2k+1}\right)' dx$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1} \left(x \cos^{2k+1} x \Big|_{\varepsilon}^{\pi-\varepsilon} - \int_{\varepsilon}^{\pi-\varepsilon} \cos^{2k+1} x\, dx \right).$$

Observing that $\int_{\varepsilon}^{\pi-\varepsilon} \cos^{2k+1} x\, dx = 0$ for each integer $k \geqslant 0$ and

$$x\, \cos^{2k+1} x \,\Big|_{\varepsilon}^{\pi-\varepsilon} = (\pi - \varepsilon)\cos^{2k+1}(\pi - \varepsilon) - \varepsilon \cos^{2k+1} \varepsilon = -\pi \cos^{2k+1} \varepsilon,$$

we obtain

$$I(\varepsilon) = \pi \sum_{k=0}^{\infty} \frac{(-1)^k \cos^{2k+1} \varepsilon}{2k+1} = \pi \operatorname{arctg}(\cos \varepsilon).$$

Therefore,

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx = \lim_{\varepsilon \to 0} I(\varepsilon) = \lim_{\varepsilon \to 0} \pi \operatorname{arctg}(\cos \varepsilon) = \pi \operatorname{arctg}(1) = \frac{\pi^2}{4}.$$

□

Problem 2. A polynomial $p(x) \in \mathbb{R}[x]$ is called *positive* if $p(y) > 0$ for each $y \in \mathbb{R}$. Suppose that $p(x) \in \mathbb{R}[x]$ is positive. Prove that the polynomials

$$p(x) - p'(x) + \frac{p''(x)}{2!} - \frac{p'''(x)}{3!} + \cdots$$

and

$$p(x) + p'(x) + p''(x) + p'''(x) + \cdots$$

are both positive.

Proof. Consider the Taylor expansion of the polynomial $p(z)$ at $z = x$:

$$p(z) = p(x) + p'(x)(z - x) + \frac{p''(x)}{2!}(z - x)^2 + \frac{p'''(z)}{3!}(z - x)^3 + \cdots .$$

Putting $z = x - 1$ into this expansion we obtain

$$p(x - 1) = p(x) - p'(x) + \frac{p''(x)}{2!} - \frac{p'''(x)}{3!} + \cdots ,$$

so the polynomial on the right hand side is equal to $p(x - 1) > 0$ for each $x \in \mathbb{R}$. Therefore, it is positive.

Note that the degree of the positive polynomial $p(x)$ is either zero (in which case there is nothing to prove) or an even positive integer. Moreover, the leading coefficient of $p(x)$ is a positive real number and coincides with the leading coefficient of the polynomial

$$g(x) := p(x) + p'(x) + p''(x) + p'''(x) + \cdots .$$

Since $\lim_{x\to-\infty} g(x) = \lim_{x\to+\infty} g(x) = +\infty$, the polynomial $g(x)$ attains its global minimum at some point, say, at $x = x_0$. Then $x = x_0$ is also a local minimum point, thus, by Fermat's theorem, $g'(x_0) = 0$. Therefore, it remains to prove that for each $y \in \mathbb{R}$ satisfying $g'(y) = 0$ we have $g(y) > 0$. Indeed, in view of

$$g'(x) = p'(x) + p''(x) + p'''(x) + \cdots = g(x) - p(x)$$

we obtain $g(y) = p(y) + g'(y) = p(y) > 0$, since the polynomial p is positive. Thus $g(x) \in \mathbb{R}[x]$ is positive. □

Problem 3. Define a *selfish* set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subset is selfish.

Answer. The number of subsets is F_n, the nth Fibonacci number.

Solution. Let f_n denote the number of minimal selfish subsets of $\{1, 2, \ldots, n\}$. We have $f_1 = 1$ and $f_2 = 1$. We claim that $f_n = f_{n-1} + f_{n-2}$ for $n \geqslant 3$. Indeed, for $n \geqslant 3$ the number of minimal selfish subsets of $\{1, 2, \ldots, n\}$ not containing n is equal to f_{n-1}. On the other hand, for any minimal selfish set containing n, by removing n from the set and subtracting 1 from each remaining element, we obtain a minimal selfish subset of $\{1, 2, \ldots, n-2\}$. (Note that 1 could not have been an element of the set, because the set $\{1\}$ is itself selfish.) Conversely, any minimal selfish subset of $\{1, 2, \ldots, n-2\}$ gives rise to a minimal selfish subset of $\{1, 2, \ldots, n\}$ containing n, by the inverse procedure. Hence the number of minimal selfish subsets of $\{1, 2, \ldots, n\}$ containing n is f_{n-2}. If follows that $f_n = f_{n-1} + f_{n-2}$ for each $n \geqslant 3$, which together with the initial values $f_1 = f_2 = 1$ implies that $f_n = F_n$. □

Problem 4. For each positive integer n, let d_n denote the greatest common divisor of the four entries of the matrix

$$\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}^n + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(For example, $d_1 = \gcd(4, 2, 4, 4) = 2$.) Prove that $\lim_{n\to\infty} d_n = \infty$.

Proof. Denote

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}.$$

By induction on n one can see easily that there exist positive integers a_n, b_n such that

$$A^n = \begin{pmatrix} a_n & b_n \\ 2b_n & a_n \end{pmatrix}.$$

In fact, $a_{n+1} = 3a_n + 4b_n$ and $b_{n+1} = 2a_n + 3b_n$ for each $n \in \mathbb{N}$, so $a_n, b_n \to \infty$ as $n \to \infty$.

Note that $a_n^2 - 2b_n^2 = \det A^n = (\det A)^n = (9-8)^n = 1$. By the definition of d_n,

$$d_n = \gcd(a_n + 1, b_n, 2b_n, a_n + 1) = \gcd(a_n + 1, b_n).$$

Thus

$$2d_n^2 = 2\gcd(a_n + 1, b_n)^2 = \gcd(2(a_n + 1)^2, 2b_n^2) = \gcd(2(a_n + 1)^2, a_n^2 - 1)$$

is divisible by $a_n + 1$, and hence $2d_n^2 > a_n$. From $\lim_{n\to\infty} a_n = \infty$ we conclude that $\lim_{n\to\infty} d_n = \infty$. $\square$