VILNIUS UNIVERSITY

2008

1. Let n be a positive integer. Find
$$\prod_{\substack{1\le x\le n\\(x,n)=1}} x \pmod n.$$
(10 points)

2. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive real numbers such that $a_1 < a_2 < \ldots < a_n,\ b_1 < b_2 < \ldots < b_n$ and
$$\begin{aligned} a_1 + a_2 + \ldots + a_n &= b_1 + b_2 + \ldots + b_n,\\ a_1^2 + a_2^2 + \ldots + a_n^2 &= b_1^2 + b_2^2 + \ldots + b_n^2,\\ &\cdots\cdots\cdots\\ a_1^n + a_2^n + \ldots + a_n^n &= b_1^n + b_2^n + \ldots + b_n^n. \end{aligned}$$
Show that $a_k = b_k$ for every $k = 1, 2, \ldots, n$.
(10 points)

3. Find all positive integers n such that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ which is exactly n-to-one (i. e. for every $y \in \mathbb{R}$ the set $f^{-1}(y)$ has exactly n elements).
(10 points)

4. For a positive real number α, define
$$S(\alpha) = \{[n\alpha] \,:\, n = 1, 2, 3 \ldots\}.$$
Prove that $\{1, 2, 3, \ldots\}$ cannot be expressed by the disjoint union of three sets $S(\alpha)$, $S(\beta)$ and $S(\gamma)$.
(10 points)