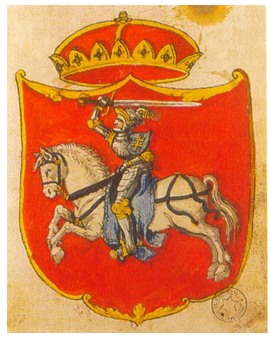
7^{th} Mathematical Contest of Friendship in Honor and Memory of Grand Duchy of Lithuania

27 September 2015



1. Find all pairs of real numbers (x, y) for which the inequality

$$y^2 + y + \sqrt{y - x^2 - xy} \leqslant 3xy$$

holds.

2. Let ω_1 and ω_2 be two circles – with respective centres O_1 and O_2 – that intersect each other in A and B. The line O_1A intersects ω_2 in A and C and the line O_2A inetersects ω_1 in A and D. The line through B parallel to AD intersects ω_1 in B and E. Suppose that O_1A is parallel to DE. Show that CD is perpendicular to O_2C.

3. A table consists of 17×17 squares. In each square one positive integer from 1 to 17 is written; every such number is written in exactly 17 squares. Prove that there is a row or a column of the table that contains at least 5 different numbers.

4. We denote by gcd(...) the greatest common divisor of the numbers in (...). (For example, gcd(4, 6, 8)=2 and gcd(12, 15)=3.) Suppose that positive integers a, b, c satisfy the following four conditions:

   $\gcd(a, b, c)=1$,

   $\gcd(a, b+c)>1$,

   $\gcd(b, c+a)>1$,

   $\gcd(c, a+b)>1$.

   a) Is it possible that $a + b + c = 2015$?

   b) Determine the minimum possible value that the sum $a+b+c$ can take.