Review of R.M.Dudley, R.Norvaiša, *Concrete Functional Calculus*, Springer Monographs in Mathematics (2011) pp.671, ISSN 1439-7382, ISBN 978-1-4419-6949-1

This monograph is a thorough and masterful work on non-linear analysis designed to be read and studied by graduate students and professional mathematical researchers. The overall perspective and choice of material is highly novel and original. It contains much new mathematics that is the authors' own work as well as a lot of other topics that are scattered around in the literature (mostly in research journals) and which are here treated in a very careful systematic way. As the authors point out in their introduction, much of the motivation has come from probability and mathematical statistics, but there is none of the latter to be found here and the former only appears in the final chapter. To get a perspective on the topics covered here, I'll give a chapter by chapter survey.

Chapter 1 is an introduction and overview. It signals a major theme of Chapter 2, which is how best to define an integral? Readers will know that the Riemann integral, which was the first mathematically rigorous approach to this problem, was to a very large extent supplanted in the early twentieth century by the Lebesgue integral which has greater scope and flexibility. However a number of other integration schemes are known to specialists. The authors find that the so-called *Kolmogorov integral* $\int_a^b f d\mu$ is particularly well-suited to much of the work they present later. This is defined similarly to the Lebesgue integral except that the set function μ is not a measure - it is an *interval function* so it is only defined (and is additive) on sub-intervals of $[a, b]$.

In Chapter 3 we meet the concept of *p-variation* which is one of the most important themes in the book. Its worth taking a little time to look into this key idea. Let f be a function from an interval $[a, b]$ to $\mathbb{R}$ and let $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\}$ be a partition of the interval. For $0 < p < \infty$, define $\text{Var}(f, \mathcal{P}) = \sum_{i=1}^n |f(t_i) - f(t_{i-1})|^p$, then we say that f has finite p-variation if $\sup \text{Var}(f, \mathcal{P}) < \infty$ where the supremum is taken over all partitions of $[a, b]$. The case $p = 1$ is the well known concept of *bounded variation* while $p = 2$ is the *quadratic variation* which plays a key role in modern stochastic analysis, e.g. in Itô's formula it captures the deviation from the familiar chain rule of the differential calculus. If $p \geq 1$, the space of functions of finite p-variation forms a Banach space $W_p([a, b])$ under a suitable norm $|| \cdot ||_p$. This chain of ideas also extends to functions taking values in a Banach algebra A where we get the Banach space $W_p([a, b], A)$. We may also consider a generalised concept of Φ-variation where $\Phi(x) = x^p$ is replaced by an arbitrary continuous increasing unbounded function $\Phi : [0, \infty) \to [0, \infty)$.

There is a beautiful relationship between Stieltjes integration and p-variation which is based on the remarkable *Love-Young inequality* (first published by L.C.Young in 1936.) It says (in slightly simplified form) that if $f \in W_p([a,b])$ and $g \in W_q([a,b])$ with $\frac{1}{p} + \frac{1}{q} > 1$ then the Stieltjes integral $\int_a^b f dh$ exists and

$$\left| \int_a^b f dh \right| \leq K ||f||_p ||h||_q,$$

where K turns out to be the Riemann zeta function evaluated at $\frac{1}{p} + \frac{1}{q}$.

Chapters 4 and 5 are quite short and mostly cover quite well known material on basic Banach algebra theory and differentiation in general normed spaces (respectively.) In Chapter 6 we meet another important concept namely that of an *Nemytskii operator.* Let B_1 and B_2 be Banach spaces and S be a non-empty set. Suppose that ψ is a mapping from $B_1 \times S$ to B_2 and that g maps S to B_1. The Nemytskii operator N_ψ is the (non-linear) composition operator $(N_\psi g)(s) = \psi(g(s), s)$. Such an operator is *autonomous* if $\psi(u, s) = F(u)$ in which case we write N_F instead of N_ψ, i.e. $N_F(s) = (F \circ g)(s)$. Much of this chapter is concerned with the questions of differentiability and analyticity of these operators and also when does a Nemytskii operator have nice mapping properties between particular spaces? For example here is a nice relationship between autonomous Nemytskii operators and p-variation. It transpires that N_F maps $W_p([a,b])$ to $W_q([a,b])$ if and only if F is locally α-Hölder continuous on B_1 with $\alpha = \frac{p}{q}$, i.e. $||F(u) - F(v)|| \leq K||u - v||^\alpha$ for all u, v in a closed ball of sufficiently large radius and where $K > 0$. Chapter 7 continues the story of Nemytskii operators, but now within the context of measure spaces (so S is equipped with a σ algebra of subsets $\mathcal{S}$ and a fixed measure μ.) A key theme here is the search for conditions for N_ψ to act (and be Fréchet differentiable) from $L^p(S, \mathcal{S}, \mu)$ to $L^r(S, \mathcal{S}, \mu)$ where $p, r \geq 1$. It turns out that these conditions are closely related to a notion in measurability called a *Shagrin condition* but the details are too involved to recount here.

in Chapter 12 Nemytskii operators are generalised and the authors consider the *two function composition operator* $\mathrm{TC}(F, G) = F \circ G$ where $F : B_1 \to B_2$ and $G : S \to B_1$. Recall that the function F was fixed when we defined autonomous Nemytskii operators, but that is no longer the case here. The authors investigate the action and continuity of TC between various spaces. In particular it turns out that a necessary condition for TC to be differentiable at (F, G) is that N_F is Fréchet differentiable at G.

Chapter 9 is devoted to *product integration*, a subject that traditionally receives far less attention than the usual integral which, however we choose to define it, is essentially a continuous process of summation. Product integrals are defined here with respect to an interval function μ taking values in a (not necessarily commutative) Banach algebra B. So we first define approximants $\mathcal{P}_n(I;\mu)$ with respect to a partition into closed subintervals $\{A_1, A_2, \ldots, A_n\}$ of a closed interval I by

$$\mathcal{P}_n(I;\mu) = (1 + \mu(A_n)) \cdots (1 + \mu(A_2))(1 + \mu(A_1)),$$

(noting that the order is important here) and then pass to the limit (if it exists) by taking finer and finer refinements, to obtain the product integral $\prod_I(1+d\mu)$. The authors also investigate the smoothness of these integrals as operators defined on a suitable space of interval functions. Historically product integrals first arose within the study of differential and integral equations at the beginning of the twentieth century, and in the last part of the chapter these are used to obtain solutions to equations of the form

$$f(t) = y + \int_{[a,t)} d\mu f,$$

where the integral should be understood in the Kolmogorov sense. If μ is additive and suitably continuous then the solution lies in one of the spaces $W_p([a,b], B)$. In Chapter 10, this theme is extended to consider non-linear equations involving Nemytskii operators of the form

$$f(t) = c + \int_{[a,t)} N_\psi f d\mu + \int_{[a,t)} N_\phi f d\nu,$$

where we again utilise the Kolmogorov integral. Existence and uniqueness of solutions is proved using a variant of the classical method of Picard iteration.

In Chapter 11, we pick up the theme of p-variation again and find attractive applications to the convergence of Fourier series. For example it is shown that every continuous function that has finite p-variation for some $0 < p < \infty$ has a uniformly convergent Fourier series. Finally Chapter 12 considers applications of p and Φ-variation within the theory of stochastic processes. A large number of important processes and classes of processes are explored from this viewpoint including Brownian motion, martingales, semimartingales, Lévy processes, Markov processes, fractional Brownian motion and empirical processes. For example it is shown that Brownian motion has finite p-variation (almost surely) if and only if $p \geq 2$. For fractional Brownian motion we need to look at Φ-variation and the "best choice" of Φ is intimately related to the celebrated law of the iterated logarithm.

I hope that my very high regard for this book has radiated out from the enthusiastic way I have discussed the material to be found there. It is a unique account of some key areas of modern analysis which will surely turn out to be invaluable for many researchers in this and related areas.

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