16th LITHUANIAN TEAM CONTEST IN MATHEMATICS

Vilnius, October the 6th 2001

1. Solve the equation

$$2001-3x^2=|3x^2-2001|.$$

2. SoIve the equation

$$8(\cos x)^2+\sin 5x=8(\cos x)^4+1.$$

3. SoIve the equation

$$x^3=4+[x].$$

([x] means the integer part of the number x)

4. Prove the inequality

$$\frac{a^4+b^4+3}{\sqrt{a^4+b^4+2}}>\frac{21}{10}$$

for all real numbers a and b.

5. Find the minimal value of the expression

$$5(a^2+b^2+2c^2)-2(2ab+6ac+bc-2a+3c)$$

when a, b, c are real numbers.

6. Find all pairs of positive numbers satisfying the system of equation

$$\begin{cases} x^{2y+x}=y^{3y-5x}, \\ x^3y=1. \end{cases}.$$

7. Find all possible integers m such that the expression $\sqrt{m^2+m+1}$ an integer too.

8. Find all possible integers n such that 7 divides 4^n-1.

9. Let

$$a_n=\sqrt{|60\sqrt{11}-199|}+\sqrt{60\sqrt{11}+171+n}.$$

Does the sequence $a_1, a_2, a_3, \ldots$ contain at least one positive integer?

10. Find all positive integer values which take ratio of product and sum of two different positive integers.

11. Is it possible a number 1/2 to present as a finite sum $1/n_1^2+1/n_2^2+\ldots+1/n_l^2$ where $n_1, n_2, \ldots, n_l$ are different positive integers?

12. Find all polynomials $p(x)$ such that the equality $p(3x)p(-3x)=81(x^2-1)^2$ holds for all real x.

13. Function $f(x)$ is defined for all real x and takes the real values. Assume that

$$f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}.$$

Does it imply that for all real values the inequality

$$f\left(\frac{x_1+x_2+x_3}{3}\right) \le \frac{f(x_1)+f(x_2)+f(x_3)}{3}$$

holds?

14. The chessboard of size 6×6 is covered by 18 domino stones 2×1. One domino stone covers exactly two fields of the chessboard. Prove that it is possible by one vertical or horizontal cut to divide the chessboard into two not necessarily equal parts without cutting any domino stone.

15. Prove that the unit circle with the centre in the coordinate origin contain infinitely many points with both rational coordinates.

16. In how many parts 2001 lines divide the plane of it is known that no two lines are parallel and no three lines have a point in common?

17. At least how much points of the plane having both integer coordinates covers a square with the side of length 2,1?

18. *M* is a midpoint of the side *AB* of the triangle *ABC*, *O* is the centre of circumscribed circle of triangle ABC and $\angle COM = 90^\circ$. Prove that

$$|\angle ABC - \angle BAC| = 90^\circ.$$

19. It is known that parallel sides of the trapezium are 4 and 16 and the trapezium is such that it is possible to inscribe and circumscribe the circles. Find the radii of these circles.

20. The chord corresponding to an arc of 60° divides the circle into two parts. Into the smaller part the square is inscribed. Find the side of the square if radius of the circle is *R*..