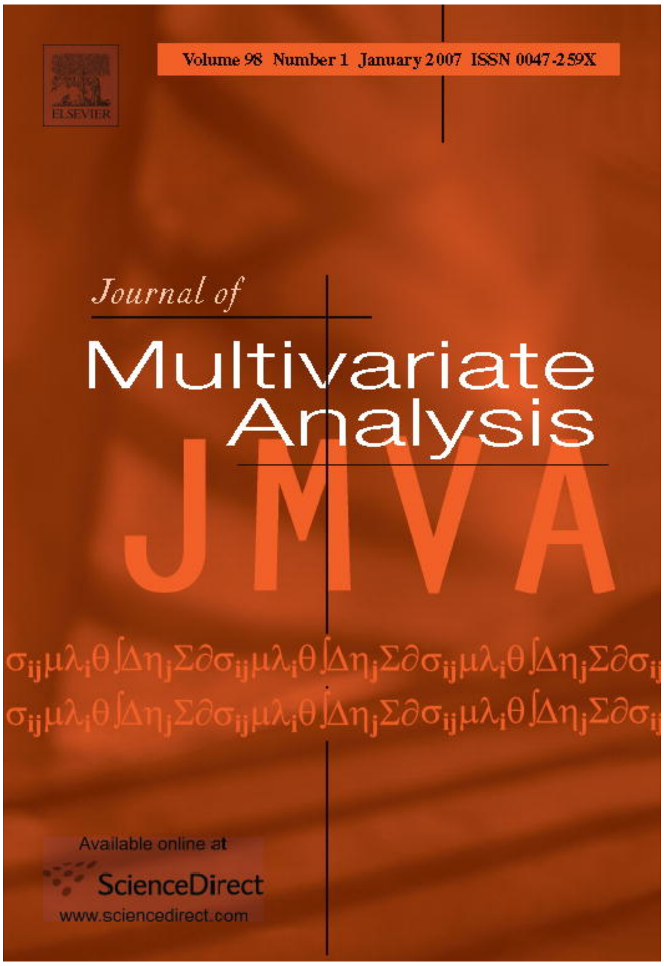
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On unit roots for spatial autoregressive models☆

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Abstract

In this paper we consider the unit root problem for one rather simple autoregressive model $Y_{t,s} = aY_{t-1,s} + bY_{t,s-1} + \varepsilon_{t,s}$ on a two-dimensional lattice. We show that the growth of variance of $Y_{t,s}$ is essentially different from corresponding growth in the unit root case for $AR(1)$ or $AR(2)$ time series models. We also show that the dimension of the lattice plays an important role: the growth of variance of autoregressive field on a d-dimensional lattice is different for $d = 2$, 3 and $d \geqslant 4$.

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1. Introduction

The most simple autoregressive model can be formulated as follows. We say that a real-valued process $\{X_t, t \in \mathbf{Z}\}$ follows an autoregressive model of order p, $1 \leqslant p \leqslant \infty$ (and denote it by $AR(p)$), if

$$X_t = \sum_{k=1}^{p} a_k X_{t-k} + \varepsilon_t, \tag{1}$$

where a_k, $k \geqslant 1$, are nonrandom coefficients, and $\{\varepsilon_t, t \in \mathbf{Z}\}$ is a sequence of independent identically distributed (i.i.d.) random variables. Such and more general models with different

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assumptions on innovations ε_t are deeply investigated and generalized to the multidimensional setting ($X_t \in R^d$, and a_k are $d \times d$ matrices) and even to the case of Banach-valued random elements X_t with a_k being linear bounded (or compact) operators (see, for example, [7] and references therein).

A different direction of generalization is obtained when, instead of the one-dimensional parameter t, we consider a multidimensional index $\bar{t}$. One can consider a real-valued random field $Y_{\bar{t}}$ with $\bar{t} \in \mathbf{Z}^d$ satisfying the relation

$$Y_{\bar{t}} = \sum_{\bar{k} \in \Lambda} a_{\bar{k}} Y_{\bar{t}-\bar{k}} + \varepsilon_{\bar{t}}, \tag{2}$$

where $\bar{t} = (t_1, \ldots, t_d)$, $\bar{k} = (k_1, \ldots, k_d)$, $\bar{t} - \bar{k} = (t_i - k_i, i = 1, \ldots, d)$, and Λ is some subset of $\mathbf{Z}^d \setminus \{\bar{0}\}$. It seems that, in the literature, the most commonly used name for such fields is spatial autoregressive process (see, for example, [1] and references therein). By analogy with the standard notation in time series $AR(p)$ or $ARMA(p, q)$ we denote such a process by $SAR(d, \Lambda)$ showing two main parameters d and Λ. While in time series there is the natural notion of "past" and "future", we must admit that, in the case of multidimensional index, there is no such natural notion. Therefore, it is not easy to say which sets Λ could be considered as natural. One possible way is to consider the requirement $\Lambda \subset (\mathbf{Z}^+)^d \setminus \{\bar{0}\}$ which leads to the quarter-plane autoregressive models. For example, the sets

$$\Lambda_p = \left\{ \bar{b} \in \mathbf{Z}^d,\ b_i \geqslant 0, i = 1, \ldots, d, 0 < \sum_{i=1}^{d} b_i \leqslant p \right\}$$

satisfy this condition. Another class of examples of *SAR* processes are nonsymmetric half-plane models, in which the set Λ is defined in a more complicated way. Mathematical theory of these processes is very well developed (see, for example, [9,11,12], and references there). Here it is relevant to note that a major part of research on *SAR* models is devoted to the case $d = 2$, mainly due to the fact that most applications (image recognition, segmentation and restoration, texture models, etc.) deal with models on a two-dimensional lattice. One can also mention the time–space autoregression models that can be formally considered as *SAR* models, but they are specific in the sense that one coordinate of indices is separated and denotes time, while the others are used to index variables in "space" (or in fixed locations; in this case the term "panel data" is used). In such models one usually uses a lag in time and lag in space, which is generally defined by the so-called weight matrix (see [8] or recent paper [10], where such models are discussed). It is also necessary to mention the important class of the conditional and intrinsic autoregressions (see [3,4,13], and references there), but since it seems that there is no direct relation of these papers with the topic of our note, we do not introduce these notions.

There is an enormous literature devoted to *SAR* processes, mainly in statistics and engineering. The problems considered for *SAR* processes are the same as those in time series: the existence of stationary solutions, fitting the data to the model, estimation of parameters of the model, etc. However, as noted in one of the first papers on autoregressive models with a two-dimensional lattice [17], there are some new effects and complications absent in time-series analysis.

In this note, we consider a rather specific problem which demonstrates such a new effect. We take a simple spatial model with a "unit root" and consider the growth of the variance of the autoregressive process, which satisfies the model under consideration. By unit roots for model (2) we mean values of the parameters $a_{\bar{k}}$ for which there are no stationary solutions. We start with

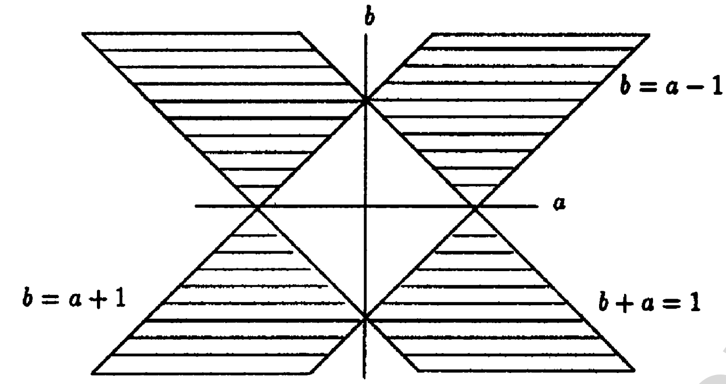

Fig. 1. Unit roots.

the simple model $SAR(2, \Lambda_1)$, that is (for simplicity we write $a = a_{10},\ b = a_{01}$),

$$Y_{t,s} = aY_{t-1,s} + bY_{t,s-1} + \varepsilon_{t,s}. \tag{3}$$

The model

$$Y_{t,s} = aY_{t-1,s} + bY_{t,s-1} - abY_{t-1,s-1} + \varepsilon_{t,s} \tag{4}$$

can be considered as being more simple one, see the recent papers [5,6] and references therein. The reason is that this model can be reduced to two "one-dimensional" autoregressions. Note that, for model (4) written in the form $p(L_1, L_2)Y_{t,s} = \varepsilon_{t,s}$, where L_1 and L_2 are lag operators (defined by the relations $L_1X_{t,s} = X_{t-1,s},\ L_2X_{t,s} = X_{t,s-1}$), polynomial $p(z_1, z_2)$ can be factorized in the form

$$p(z_1, z_2) = 1 - az_1 - bz_2 + abz_1z_2 = (1 - az_1)(1 - bz_2).$$

Therefore, in the case of a unit root $a = b = 1$, the problem of growth of the variance of $Y_{s,t}$ is trivial, since (with appropriate boundary conditions) $Y_{s,t} = \sum_{i,j=1}^{t,s} \varepsilon_{i,j}$, see [6]. A similar situation is in the case of a unit root in one direction ($a = 1, |b| < 1$ or $|a| < 1, b = 1$).

In this paper we always assume that $\varepsilon_{t,s}, (t, s) \in \mathbf{Z}^2$ are i.i.d. random variables with $E\varepsilon_{1,1} = 0,\ \ E\varepsilon_{1,1}^2 = 1$. The same assumption will be assumed for innovations with indices in $\mathbf{Z},\ \mathbf{Z}^3$ and $\mathbf{Z}^4$.

As boundary conditions for model (3), we assume that we have fixed values of $Y_{i,j}$ with $i + j = 0$. In the two-dimensional setting, there are much more possibilities to set boundary conditions (see [16]) but, for the purposes of this note, boundary conditions are not important. For us, it is important that, in model (3), we can express all values $Y_{t,s}$ with $t + s > 0$ recursively by a finite number of these fixed values and values of $\varepsilon_{n,m}$.

It is known (see [15]) that, for model (3), a stationary solution exists in three regions of (a, b)-plane: $\{|a| + |b| < 1\},\ \{|a| + 1 < |b|\},\ \{1 + |b| < |a|\}$ (see Fig. 1).

Four strips (shaded with boundary lines included) between these three regions present the unit roots of model (3). They are obtained in the following way. Take a polynomial $\psi(z_1, \ldots, z_d)$ of complex variables $z_1, \ldots, z_d$. Using the lag operators $L_1, \ldots, L_d$ the general SAR model can be written as

$$\psi(L_1, \ldots, L_d)Y_{\bar{k}} = \varepsilon_{\bar{k}},$$

and the existence of a stationary solution depends on the zeros of the polynomial $\psi(z_1, \dots, z_d)$. If $d = 2$, it is known (see [17,15]) that a necessary and sufficient condition for this general model to have a stationary solution is that $\psi(z_1, z_2)$ is not zero on the set $\{|z_1| = 1, |z_2| = 1\}$ (the boundary of the unit polydisc). In the particular case of model (3) with the polynomial $\varphi(z_1, z_2) = 1 - az_1 - bz_2$, the shaded area presents points where this polynomial has zeros on the boundary of the unit polydisc.

At the end of Introduction, we recall some properties of a stationary solution of (3). In the first region $\{|a| + |b| < 1\}$, the stationary solution can be written in the form

$$Y_{t,s} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j} \varepsilon_{t-j,s-k+j}.$$

Using the notation

$$c(i, j) := c(i, j; a, b) = \binom{j}{i} a^i b^{j-i},$$

we have

$$Var\, Y_{t,s} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} c(j, k)^2.$$

Denote $K = |a| + |b|$. It is easy to show that $Var\, Y_{t,s} \leqslant (1 - K^2)^{-1}$.

In the case of the second region (let us take $a > 0$, $b > 1 + a$), rewrite (3) in the form

$$Y_{t,s-1} = \tilde{a} Y_{t-1,s} + \tilde{b} Y_{t,s} + \tilde{\varepsilon}_{t,s-1} \tag{5}$$

with $\tilde{a} = -ab^{-1}$, $\tilde{b} = b^{-1}$, and $\tilde{\varepsilon}_{t,s-1} = -b^{-1}\varepsilon_{t,s}$. Now $|\tilde{a}| + |\tilde{b}| = (1+a)b^{-1} < 1$; therefore, by a standard argument we have the stationary solution

$$\begin{aligned} Y_{t,s-1} &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} \tilde{a}^j \tilde{b}^{k-j} \tilde{\varepsilon}_{t-j,s+k-1} \\ &= -\sum_{k=0}^{\infty} \frac{1}{b^{k+1}} \sum_{j=0}^{k} \binom{k}{j} (-a)^j \varepsilon_{t-j,s+k}. \end{aligned} \tag{6}$$

It is easy to see that (6) satisfies (3), so formally it seems that there is no difference between (3) and (5), since both have the same stationary solution (6). However, if we consider the boundary conditions and the mechanism how the values of the process $Y_{t,s}$ are generated, models (3) and (5) differ. Let $\mathcal{F}(\cdot)$ denote the σ-algebra generated by random variables in the parenthesis and if X is a random variable, then $X \sim \mathcal{F}$ means that X as a function is measurable with respect to the σ-algebra $\mathcal{F}$. For model (3) (in the case $|a| + |b| < 1$), a stationary solution $Y_{t,s} \sim \mathcal{F}(\varepsilon_{i,j}, i \leqslant t, \ j \leqslant s)$, and a natural boundary set is any line $i + j = c, \ c \in \mathbf{Z}$. For example, having values $Y_{i,j}$ for all i, j satisfying $i + j = 0$, we can generate all values $Y_{t,s}$ with $(t, s) \in \{(t, s) \in \mathbf{Z}^2 : t + s > 0\}$. For model (5) (the case $1 + |a| < |b|$), a stationary solution $Y_{t,s} \sim \mathcal{F}(\varepsilon_{i,j}, i \leqslant t, i + j \geqslant t + s + 1)$, and, as a boundary condition, we can take values $Y_{i,j}$ on any line $j = n$ and then get all values of $Y_{t,s}$ with $s < n$. So the mechanism of autoregression is important, and it is easy to see that if, in (3) with $a > 0, b > 1 + a$, we start to generate $Y_{t,s}$ from $Y_{i,j} = 0, i + j = 0$, we get non stationary process $Y_{t,s}$, since the variance $EY_{t,s}^2$ grows as $t+s$ increases. Similarly, in the case of the third region, we can rewrite (3) in the form

$$Y_{t-1,s} = \tilde{a} Y_{t,s} + \tilde{b} Y_{t,s-1} + \tilde{\varepsilon}_{t-1,s} \tag{7}$$

with $\tilde{a} = a^{-1}$, $\tilde{b} = -ba^{-1}$, and $\tilde{\varepsilon}_{t-1,s} = -a^{-1}\varepsilon_{t,s}$. Again $|\tilde{a}| + |\tilde{b}| < 1$, and we have the stationary solution

$$Y_{t-1,s} = \sum_{k=0}^{\infty} \frac{1}{a^{k+1}} \sum_{j=0}^{k} \binom{k}{j} (-1)^j b^{k-j} \varepsilon_{t+k,s-k+j}.$$

In this case, $Y_{t,s} \sim \mathcal{F}(\varepsilon_{i,j},\ j \leqslant s,\ i+j \geqslant t+s+1)$ and, as a boundary condition, we can take values $Y_{i,j}$ on any line $i=n$ and get all values of $Y_{t,s}$ with $t<n$.

Therefore, it is possible to consider (5) and (7) as modifications of model (3), convenient for different values of parameters a, b, but it is also possible to consider all three models as separate, taking into account the mechanism under which the value of $Y_{t,s}$ at point (t,s) is obtained from neighboring two points. In (3), we take two neighboring points $(t-1,s)$ and $(t,s-1)$, while, in model (5), we take points $(t-1,s+1)$ and $(t,s+1)$ and, in (7), the points $(t+1,s)$ and $(t+1,s-1)$. If we do not require that $\Lambda \subset (\mathbf{Z}^+)^2 \setminus \{\bar{0}\}$, then it is clear that, for each point (t,s), we have eight neighboring points and if, in the generating mechanism, we include only two neighboring points, then there are $\binom{8}{2} = 28$ different models. Clearly, they are not all of the same interest. For example, if we take $(t-1,s)$ and $(t+1,s)$ as neighbors for a point (t,s), then the model

$$Y_{t,s} = aY_{t-1,s} + bY_{t+1,s} + \varepsilon_{t,s}$$

can be regarded as a sequence (with respect to s) of models indexed only by t. In [17], such models in time series were called bilateral, in contrast to unilateral time series models when the value of a process at point t is obtained from values of the process on one side of t. Such a model can also be considered as a panel model: t stands for time and s denotes the panel number.

The paper is organized as follows: in the next two sections we formulate our main results and the last section is devoted to proofs.

2. Main result in case $d = 2$

We return to the case of unit roots in model (3). If we take a point in the interior of the shaded strips (see Fig. 1), it is not difficult to see that the variance of a solution of (3) grows exponentially. Therefore, interesting cases of unit roots for this model are four lines $a+b=1$, $a+b=-1$, $a-b=1$, $b-a=1$, which form the boundary between the "stationary region" and the region of exponential growth. At the points of intersection of these lines we have a unit root for $AR(1)$ model, therefore, the variance of the solution (which is simply a random walk) grows linearly. Let us take model (3) with $|a|+|b|=1$ and, to be specific, let us assume that $a>0$, $b>0$, $a+b=1$. Also assume that $Y_{i,j}=0$ for $i+j=0$ (if we take different boundary conditions, the main picture of the growth of $Var\, Y_{t,s}$ will remain the same). Then

$$Y_{t,s} = \sum_{k=0}^{t+s-1} \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j} \varepsilon_{t-j,s-k+j}$$

and

$$Var\,Y_{t,s} = \sum_{k=0}^{t+s-1} \sum_{j=0}^{k} c(j,k)^2. \tag{8}$$

In order to investigate the growth of $Var\,Y_{t,s}$ as $m = t + s$ tends to infinity, we need to find the asymptotic behavior (as $m \to \infty$) of a function

$$f_m(x) = \sum_{k=0}^{m} g_k(x), \quad 0 \leqslant x \leqslant 1,$$

where $g_k(x) = \sum_{j=0}^{k} a_{k,j}(x), \quad a_{k,j}(x) = \binom{k}{j}^2 x^{2j}(1-x)^{2(k-j)}$, and $x = a$. Note that $f_m(0) = f_m(1) = m$. Using simple combinatorics (identity (32) and the Stirling formula), we get $g_k(1/2) \sim (\pi k)^{-1/2}$ for large k, hence, $f_m(1/2) \simeq \sqrt{m}$ for large m. Here $a_n \sim b_n$ means, as usual, that $\lim a_n b_n^{-1} = 1$ and $a_m \simeq b_m$ means that $C_1 b_m \leqslant a_m \leqslant C_2 b_m$ for sufficiently large m with positive constants $C_i,\ i = 1, 2$, independent of m.

One can expect that the same behavior of the function f_m will be for all fixed x separated from 0 and 1, and only if we let x vary with m (e.g., if we consider a sequence of models), that is, $x = x(m)$ and $x \to 0$ or $x \to 1$ as $m \to \infty$, we must get some intermediate growth between $\sqrt{m}$ and m. This turns out to be the case, and we prove the following statement.

Theorem 1. *Suppose that* $0 < c_1 < 1/2 < c_2 < 1$ *are some fixed constants. Then we have*

$$f_m(x) \simeq \begin{cases} \sqrt{m} & \text{for } c_1 < x < c_2, \\ m^{(1+\gamma)/2} & \text{for } x = m^{-\gamma} \text{ or } x = 1 - m^{-\gamma},\ \ \gamma \in (0,1), \\ m & \text{for } 0 \leqslant x \leqslant m^{-1} \text{ or } 1 - m^{-1} \leqslant x \leqslant 1. \end{cases} \tag{9}$$

The theorem gives us a complete picture of how $Var\,Y_{t,s}$ grows on the set $\{(a,b) : |a|+|b| = 1\}$, that is, on the unit sphere in l_1-norm on (a,b)-plane. We formulate this behavior as a corollary.

Corollary 2. *For model* (3) *with* $|a|+|b| = 1$ *and boundary conditions* $Y_{i,j} = 0$ *for all* $i + j = 0$ *and for large* $m = t + s$, *we have*

$$Var\,Y_{t,s} \simeq \begin{cases} \sqrt{t+s} & \text{for } c_1 < |a| < c_2, \\ (t+s)^{(1+\gamma)/2} & \text{for } |a| = m^{-\gamma} \text{ or } |a| = 1 - m^{-\gamma}, \\ t+s & \text{for } |a| \leqslant m^{-1} \text{ or } 1 - m^{-1} \leqslant |a| \leqslant 1. \end{cases} \tag{10}$$

Here $0 < c_1 < c_2 < 1$ *are fixed constants.*

From Theorem 1 we can derive the following result on the half-line $a \geqslant 0,\ b \geqslant 1,\ 1 + a = b$, extending the line on the unit l_1-sphere to the plane (on the other half-lines, the situation is the same).

Corollary 3. *Let* $a \geqslant 0$, $b \geqslant 1$, *and* $1 + a = b$. *Let us consider* (3) *rewritten in the form* (5) *with boundary conditions* $Y_{in} = 0$ *for all i. Then, for* $s < n$ *and* $n - s$ *large*,

$$Var\, Y_{t,s} \simeq \begin{cases} n - s & \text{for } 0 \leqslant a \leqslant (n-s)^{-1}, \\ (n-s)^{(1+\gamma)/2} & \text{for } a = (n-s)^{-\gamma}, \quad 0 < \gamma < 1, \\ \sqrt{n-s} & \text{for } c_3 < a < c_4, \\ (n-s)^{(1-3\gamma)/2} & \text{for } a = (n-s)^{\gamma}, \quad \gamma > 0. \end{cases} \tag{11}$$

Here c_3 and c_4 are some fixed, respectively, small and large constants. Comparing (11) with (10), we see a new effect: if a (and, at the same time, b) becomes large ($\gamma > 1/3$), then the variance of $Y_{t,s}$ does not grow but tends to zero. This can be explained by the fact that, in (5), $\tilde{b} \to 0$ as $b \to \infty$, so the model becomes close to $AR(1)$ but, at the same time, the innovations $\tilde{\varepsilon}_{t,s}$ tend to zero, therefore, the variability of $Y_{t,s}$ becomes smaller and smaller. This effect can be seen in the $AR(1)$ model $X_t = aX_{t-1} + \varepsilon_t$, $t \in \mathbf{Z}$, with $|a| > 1$ and noncausal solution $X_t = -\sum_{k=0}^{\infty} a^{-(k+1)} \varepsilon_{t+k+1}$ as $a \to \infty$. Another message sent by relations (11) and (10) is the following fact. While in the case of unit root for time series X_t, $t = 0, 1, 2, \ldots$ we have one direction and the unit root case differs from the stationary one by the growth of $Var\, X_t$, in the case of the two-dimensional index together with the growth of $Var\, Y_{t,s}$ the "direction" also is important, that is, how the point (t, s) moves to ∞ on plane. From (10) we see that if, in (t, s) plane, we go from $(0, 0)$ by any direction in the first quadrant, the variance of $Y_{t,s}$ grows, and the most rapid growth is along the line $t = s$, while (11) shows quite different growth of the variance of $Y_{t,s}$, that is, the variance grows as s decreases, starting from n, and does not depend on t. It is easy to see that this direction of maximal growth is perpendicular to the line of boundary conditions or, in other words, depends on the mechanism of generating the value $Y_{t,s}$ from two neighboring points. It is not difficult to provide eight examples of models written in the form

$$Y_{t,s} = aY_{t+i,s+j} + bY_{t+k,s+l} + \varepsilon_{t,s}, \quad a + b = 1, \quad a > 0, \quad b > 0$$

with appropriate combinations of values i, j, k, l from the set $\{0, \pm 1\}$ and appropriate boundary conditions. This will give eight different directions of maximal growth of $Var\, Y_{t,s}$: four directions go along the coordinate axis, and the remaining four can be obtained by rotating the axis by $\pi/4$; the "typical" two-dimensional (that is, a is not close to 0 or 1) growth rates will be the following eight functions:

$$\sqrt{t+s}, \sqrt{t}, \sqrt{s}, \sqrt{n-t}, \sqrt{n-s}, \sqrt{2n-(t+s)}, \sqrt{n+s-t}, \sqrt{n+t-s}.$$

Last five of these functions contain n, since in five models initial conditions are defined on lines, depending on n (chosen in a such way, that it would be possible to generate values of the $X_{t,s}$ on square $[1, n] \times [1, n]$). For example, in Corollary 3 we have initial conditions on line $((i, j) : j = n)$ and the function of the maximal growth is $\sqrt{n-s}$. Taking into account that the rate of growth also changes if the parameters a, b are close to degenerate values ($a = 0$ or $a = 1$, one-dimensional unit root), we see that the unit root case in autoregression on two-dimensional lattice is much more complicated, in comparison with the unit root case for $AR(1)$ model for time series. Despite of this, we still hope that the information obtained in the theorem and corollaries will help to construct some test to identify a unit root, similar to the well-known Dickey–Fuller test for detecting a unit root in $AR(1)$ model. We intend to investigate this problem in the nearest future. [1]

One can suspect that such a big difference in the behavior of the variance of an autoregressive processes is for the reason that we compare the unit root case for model (3) which has two parameters a, b with $AR(1)$ model $X_t = aX_{t-1} + \varepsilon_t$ having only one parameter a. Therefore, it seems more natural to compare (3) with the $AR(2)$ model

$$X_t = aX_{t-1} + bX_{t-2} + \varepsilon_t. \tag{12}$$

It turns out that essential is the dimension of indices but not the number of parameters of a model, since the following result holds.

Proposition 4. *For model* (12) *with* $a + b = 1, a > 0, b > 0$ *and initial conditions* $X_{-1} = X_{-2} = 0$, *we have*

$$1 + C_*^2 t \leqslant Var\, X_t \leqslant 1 + (C^*)^2 t, \tag{13}$$

where $C_* = a$ *and* $C^* = a^2 - a + 1$.

In a recent paper [14] the growth of the variances of more general processes defined by the equation

$$\Delta X_t = u_t$$

is considered. Here $\Delta X_t = X_t - X_{t-1}$ and $\{u_t\}$ is zero-mean weakly stationary process. We included our result, since we give the dependence of the asymptotic of $Var\, X_t$ on a parameter a.

3. The case of higher dimensions

Now we investigate SAR processes in higher dimensions and we show that the growth of the variance of a SAR process in the case of unit root changes rapidly with increasing dimensionality of index. Let us consider the $SAR(3, \Lambda_1)$ model

$$Y_{t,s,v} = aY_{t-1,s,v} + bY_{t,s-1,v} + cY_{t,s,v-1} + \varepsilon_{t,s,v}. \tag{14}$$

It is possible to show that unit roots for this model are in the set of points in the (a, b, c)-space described by the inequalities

$$\begin{aligned}
&||b| - |1+c|| \leqslant |a| \leqslant |b| + |1+c|, \quad && ||c| - |1+b|| \leqslant |a| \leqslant |c| + |1+b|,\\
&||a| - |1+c|| \leqslant |b| \leqslant |a| + |1+c|, \quad && ||c| - |1+a|| \leqslant |b| \leqslant |c| + |1+a|,\\
&||a| - |1+b|| \leqslant |c| \leqslant |a| + |1+b|, \quad && ||b| - |1+a|| \leqslant |c| \leqslant |b| + |1+a|.
\end{aligned}$$

Again, as in the case $d = 2$, we take only the part of the boundary of this set, precisely, we consider model (14) with $a+b+c = 1,\ a \geqslant 0,\ b \geqslant 0,\ c \geqslant 0$, and boundary conditions $Y_{t,s,v} = 0$ for $t + s + v = 0$. If one coefficient is equal to zero, we get the just investigated case of $SAR(2, \Lambda_1)$; if two of them vanish, we have $AR(1)$, so it is necessary to investigate the case where all three parameters a, b, c are separated from 0 and 1. Since now combinatorics becomes more complicated (instead of binomial coefficients now we must deal with multinomial coefficients),

[1] During the second revision the author got a preprint [2], where asymptotic normality is proved for LSE of a and b in the unit root case, but the limit normal law is degenerate.

we provide only a particular result from which we have a general picture about the growth of the variance of the process (14).

Proposition 5. *For model* (14) *with* $a = b = c = 1/3$ *and boundary conditions* $Y_{t,s,v} = 0$ *for* $t + s + v = 0$, *we have*

$$Var\, Y_{t,s,v} \simeq \log(t + s + v). \tag{15}$$

This result shows that, for a simple model on three-dimensional lattice, the growth of the variance of the process on the boundary of the set of unit root points can vary from linear to logarithmic. Therefore, it is rather natural to expect that, in the case $d = 4$, on the boundary of the set of unit root points there will be a region of parameters such that variance of the process will be bounded.

Proposition 6. *Let us consider the model*

$$Y_{t,s,v,u} = aY_{t-1,s,v,u} + bY_{t,s-1,v,u} + cY_{t,s,v-1,u} + eY_{t,s,v,u-1} + \varepsilon_{t,s,v,u} \tag{16}$$

with $a = b = c = e = 1/4$ *and boundary conditions* $Y_{t,s,v,u} = 0$ *for* $t + s + v + u = 0$. *There exists an absolute constant C such that, for all* $t, s, v, u, t + s + v + u > 0$, *we have*

$$Var\, Y_{t,s,v,u} \leqslant C.$$

This proposition leads to an interesting observation. If we consider the set $\{a + b + c + d = 1, 0 < c_1 \leqslant a, b, c, d \leqslant c_2 < 1\}$ in the space of parameters (a, b, c, d), it is not difficult to see that the behavior of $Var\, Y_{t,s,v,u}$ will be the same, i.e., it will be bounded. This means that the series

$$Y_{t,s,u,v} = \sum_{n=0}^{\infty} \sum_{\substack{i,j,k \geqslant 0 \\ 0 \leqslant i+j+k \leqslant n}} c_1(n, i, j, k) a^i b^j c^k d^{n-i-j-k} \varepsilon_{t-i,s-j,u-k,v-n+i+j+k},$$

where

$$c_1(n, i, j, k) = \frac{n!}{i!j!k!(n-i-j-k)!}, \tag{17}$$

converges a.s., since ε's are independent and the corresponding series of variances is convergent. Thus, we have some region in the space of parameters, in which, on one hand, the polynomial $1 - az_1 - bz_2 - cz_3 - ez_4$ has roots on the boundary of unit polydisc and, on the other hand, for the same values of parameters model (16) has a stationary solution. This shows that, starting with dimension 4, the condition that the polynomial $\psi(z_1, \ldots, z_d)$ vanishes on the boundary of the unit polydisc in C^d is not sufficient for the unit root, i.e., nonexistence of a stationary solution.

4. Proofs

Before starting the proof of Theorem 1 we shall make one comment. For analysis of the growth of $Var\, Y_{t,s}$, we use a direct approach, precisely, formula (8). As noted by the anonymous referee, another approach to analyze $Var\, Y_{t,s}$ would be via spectral density. It is easy to see that, for model (3) (with $a > 0,\ b > 0,\ a + b = 1$), the spectral density is

$$f(\lambda_1, \lambda_2) = 4\left(a \sin^2(\lambda_1/2) + b \sin^2(\lambda_2/2) - ab \sin^2((\lambda_1 - \lambda_2)/2)\right)^{-1}. \tag{18}$$

In papers [13,4], there were considered intrinsic autoregressions with spectral density having radial behavior (it grows as negative power of $||\lambda|| := (\lambda_1^2 + \lambda_2^2)^{1/2}$ at the origin). Such an assumption allowed one to investigate the growth of variance. Unfortunately, the behavior of density (18) at the origin is more complicated, and (at least for the author) it is not clear how to get results formulated in Theorem 1 using the spectral density approach.

Proof of Theorem 1. Note that, since $g_k(x) = g_k(1-x)$, it is sufficient to consider the interval $0 \leqslant x \leqslant 1/2$. Consider $a_{nk}(x)$ for sufficiently large n and k. Applying the Stirling formula, we have

$$a_{nk}(x) = \frac{n}{2\pi k(n-k)} \left(\frac{n}{n-k}\right)^{2n} \left(\frac{n-k}{k}\right)^{2k} x^{2k}(1-x)^{2(n-k)}(1+o(1)). \tag{19}$$

Let us take $k = xn + zn^\beta$ (strictly speaking, we must consider $\tilde{k} = [xn + zn^\beta]$, where $[a]$ stands for the integer part of a number a but, since we consider large k, the difference $k - \tilde{k}$ can be neglected). We shall choose parameters $|z| < 1$ and $0 < \beta < 1$ so that $k \in (1, n)$ will be large. Also, instead of writing $1 + o(1)$, sometimes we will use the symbol $\sim$. Then from (19) we have

$$a_{nk}(x) \sim \frac{1}{2\pi n a_n b_n} \left(1 + \frac{z}{n^{1-\beta} a_n}\right)^{2na_n} \left(1 - \frac{z}{n^{1-\beta} b_n}\right)^{2nb_n}, \tag{20}$$

where

$$a_n = 1 - x - \frac{z}{n^{1-\beta}}, \qquad b_n = x + \frac{z}{n^{1-\beta}}.$$

Consider separately

$$U_n = \left(1 + \frac{z}{n^{1-\beta} a_n}\right)^{2na_n} \left(1 - \frac{z}{n^{1-\beta} b_n}\right)^{2nb_n}.$$

Suppose that all parameters x, z, β, n are such that

$$\left|\frac{z}{n^{1-\beta} a_n}\right| < \varepsilon, \qquad \left|\frac{z}{n^{1-\beta} b_n}\right| < \varepsilon \tag{21}$$

for some small $\varepsilon > 0$. Then

$$\begin{aligned}
\ln U_n &= 2na_n \ln\left(1 + \frac{z}{n^{1-\beta} a_n}\right) + 2nb_n \ln\left(1 - \frac{z}{n^{1-\beta} b_n}\right) \\
&= 2na_n \left(\frac{z}{n^{1-\beta} a_n} - \frac{1}{2}\left(\frac{z}{n^{1-\beta} a_n}\right)^2 (1+o(1))\right) \\
&\quad + 2nb_n \left(-\frac{z}{n^{1-\beta} b_n} - \frac{1}{2}\left(\frac{z}{n^{1-\beta} b_n}\right)^2 (1+o(1))\right) \\
&= -\left(\frac{z^2}{n^{1-2\beta} a_n} + \frac{z^2}{n^{1-2\beta} b_n}\right)(1+o(1)).
\end{aligned}$$

Since $a_n + b_n = 1$, we have

$$U_n = \exp\left\{-\frac{z^2}{n^{1-2\beta}}\left(\frac{1}{a_n}+\frac{1}{b_n}\right)(1+o(1))\right\}$$
$$= \exp\left\{-\frac{z^2(1+o(1))}{n^{1-2\beta}a_nb_n}\right\}. \tag{22}$$

We first consider bounds from below. Let us take $0 < c_1 < x < 1/2$. Then $1/3 < a_n < 1$ and $c_1/2 < b_n < 1$ for sufficiently large n, therefore, (21) is satisfied, and we have

$$a_{nk}(x) \geqslant \frac{C}{na_nb_n}U_n = \frac{C}{na_nb_n}\exp\left\{-\frac{z^2(1+o(1))}{n^{1-2\beta}a_nb_n}\right\} \geqslant Cn^{-1} \tag{23}$$

for all $\beta \leqslant 1/2$. Thus, for all $k \in [xn - \sqrt{n}, xn + \sqrt{n}]$, we have $a_{nk}(x) \geqslant Cn^{-1}$, therefore, for $0 < c_1 < x < 1/2$,

$$g_n(x) \geqslant Cn^{-1/2} \quad \text{and} \quad f_m(x) \geqslant C\sqrt{m}.$$

Now let us consider $x = n^{-\gamma},\ 0 < \gamma \leqslant 1$. We take $0 < z < 1$ and choose $\beta < 1 - \gamma$. Again a_n is separated from 0 but $b_n = n^{-\gamma} + zn^{\beta-1}$, therefore, $n^{-\gamma} \leqslant b_n \leqslant 2n^{-\gamma}$. (21) is satisfied and, instead of (23), we have

$$a_{nk}(x) \geqslant \frac{C}{na_nb_n}\exp\left\{-\frac{z^2(1+\varepsilon)n^{2\beta}}{na_nb_n}\right\}.$$

The function

$$h(y) = \frac{1}{y}\exp\left\{-\frac{\tau}{y}\right\}$$

has a maximum at $y = \tau$, $h(\tau) = (e\tau)^{-1}$, $\;h'(y) > 0$ for $0 < y < \tau$, and $h'(y) < 0$ for $y > \tau$. Then $h(y) \sim y^{-1}$ as $y \to \infty$ and $h(y)$ exponentially decreases to zero as $y \to 0$. In our case, $\tau = Cn^{2\beta}$ and $y = na_nb_n$, so $C_1n^{1-\gamma} < y < C_2n^{1-\gamma}$. If we choose $\beta = (1-\gamma)/2$, then, for all $k \in [n^{1-\gamma}, n^{1-\gamma} + n^{(1-\gamma)/2}]$, we get

$$a_{nk}(x) \geqslant Cn^{-(1-\gamma)}.$$

Hence, $g_n(n^{-\gamma}) \geqslant Cn^{\beta}n^{-(1-\gamma)} = Cn^{-(1-\gamma)/2}$. Trying to bound $g_n(x)$, we took $x = n^{-\gamma}$, while we need to estimate $f_m(x)$ at $x = m^{-\gamma}$. This was done taking into account that the value $k = nx$ maximizes the value of $a_{nk}(x)$ over all $1 \leqslant k \leqslant n$. On the other hand, due to the trivial bound

$$f_m(x) \geqslant \sum_{n=m/2}^{m} g_n(x),$$

for us it is sufficient to consider only $m/2 \leqslant n \leqslant m$. Therefore, it is easy to see that all bounds remain valid (up to constants) for $x = m^{-\gamma}$ and, thus, we get

$$f_m(m^{-\gamma}) \geqslant Cm^{(1+\gamma)/2}.$$

Clearly, if $x = m^{-\gamma}$ with $\gamma > 1$, then $g_n(m^{-\gamma}) \geqslant (1 - m^{-\gamma})^{2n} \geqslant C$ for $m/2 \leqslant n \leqslant m$, therefore, $f_m(x) \geqslant Cm$.

So we have proved the bounds from below. Now we shall show that the same rate can be obtained bounding the function f_m from above. Since, for all $0 \leqslant x \leqslant 1$,

$$g_n(x) = \sum_{k=0}^{n} a_{n,k}(x) \leqslant \left(\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{(n-k)} \right)^2 = 1, \tag{24}$$

we have

$$f_m(x) \leqslant Cm.$$

Now consider $0 < c_1 < x < 1/2$. It is easy to verify that, for all $k \in I(x,n) := [xn - \sqrt{n}, xn + \sqrt{n}]$, we have $a_{nk}(x) \leqslant Cn^{-1}$, therefore,

$$\sum_{k \in I(x,n)} a_{nk}(x) \leqslant Cn^{-1/2}.$$

If $k \notin I(x,n)$, then $k = xn + zn^{\beta}$ with $z = \pm 1$ and $\beta > 1/2$. Condition (21) is satisfied, and now we have

$$U_n \leqslant \exp\left\{ -\frac{C}{n^{1-2\beta} a_n b_n} \right\}.$$

Since $1 - 2\beta < 0$, we get that U_n exponentially decreases, therefore, the sum $\sum_{k \notin I(x,n)} a_{nk}(x)$ is negligible comparing with the sum over $I(x,n)$, hence, $g_n(x) \leqslant Cn^{-1/2}$ and

$$f_m(x) \leqslant C\sqrt{m}.$$

It remains the case $x = m^{-\gamma}$, and now we cannot substitute m by n, since we need to bound from above the total sum $\sum_{n=0}^{m} g_n(x)$. Therefore, we divide this sum into three parts

$$f_m(x) = f_{m1}(x) + f_{m2}(x) + f_{m3}(x),$$

where

$$f_{m1}(x) = \sum_{n=0}^{[m^{\zeta}]} g_n(x), \quad f_{m2}(x) = \sum_{n=[m^{\zeta}]+1}^{[m/2]} g_n(x), \quad f_{m3}(x) = \sum_{n=[m/2]+1}^{m} g_n(x),$$

and $\zeta = (1+\gamma)/2$. In the first sum, using trivial bound (24), we have

$$f_{m1}(x) \leqslant m^{\zeta}. \tag{25}$$

In the third sum, as in bounds from below, we can take $x = n^{-\gamma}$. Then it is not difficult to verify that, for all $k \in I_1(x,n) := [n^{1-\gamma}, n^{1-\gamma} + n^{(1-\gamma)/2}]$,

$$a_{nk}(x) \leqslant Cn^{-(1-\gamma)} \quad \text{and} \quad \sum_{k \in I_1(x,n)} a_{nk}(x) \leqslant Cn^{-(1-\gamma)/2}. \tag{26}$$

As above, one can show that, for all k outside the interval $I_1(x,n)$, the term $a_{nk}(x)$ decreases exponentially with respect to n, therefore, the main term in the estimate of $g_n(x)$ is obtained from the bound (26). Namely, if $k \notin I_1(x,n)$, then $k = xn + zn^{\beta}$ with $z = \pm 1$ and $1 - \gamma > \beta > (1-\gamma)/2$, and with $z = 1$ if $1 - \gamma < \beta < 1$. If $1 - \gamma > \beta > (1 - \gamma)/2$, then (21) is satisfied. Then the exponential rate easily follows from the bound

$$a_{nk}(x) \leqslant \frac{C}{na_n b_n} \exp\left\{ -\frac{Cn^{2\beta}}{na_n b_n} \right\},$$

since $2\beta > 1 - \gamma$.

In the case $1 - \gamma < \beta < 1$, condition (21) is not satisfied (at least, ε cannot be taken small), but then returning to formula (20) and directly estimating, one can show the exponential decay again. Thus, we get

$$f_{m3}(m^{-\gamma}) \leqslant Cm^{\zeta}. \tag{27}$$

It remains to estimate $f_{m2}(x)$ for $x = m^{-\gamma},\ 0 < \gamma < 1$. If we take $x = m^{-\gamma}$ and $k = nx$, then it is easy to see that, for such k, we have $a_{nk}(x) \simeq n^{-1}m^{\gamma}$. Again we look for β such that the terms $a_{nk}(x)$ will be of the same order for all $k = nx + zn^{\beta},\ |z| < 1$. We have (20) with

$$a_n = 1 - x - \frac{z}{n^{1-\beta}}, \quad b_n = x + \frac{z}{n^{1-\beta}}.$$

Let us remind that now we consider the values of n that are of order m^{η} with $(1+\gamma)/2 \leqslant \eta \leqslant 1$. For such values of n, the prevailing term in b_n will be $x = m^{-\gamma}$ if $\beta < (1-\gamma)(1+\gamma)^{-1}$. We choose

$$\beta_1 = \frac{1}{2}(1-\gamma) < \frac{1-\gamma}{1+\gamma},$$

then condition (21) is satisfied, and from (20) and (22) we get that, for all $k \in I_2(x, n) := [xn - n^{\beta}, xn + n^{\beta}]$,

$$a_{nk}(m^{-\gamma}) \leqslant \frac{Cm^{\gamma}}{n}.$$

Hence,

$$\sum_{k \in I_2(x,n)} a_{nk}(m^{-\gamma}) \leqslant Cn^{\beta-1}m^{\gamma}.$$

As before, we can show (we omit the calculations) that for $k \notin I_2(x, n),\ a_{nk}(m^{-\gamma})$ exponentially decreases with respect to n, therefore, we get

$$g_n(m^{-\gamma}) \leqslant Cn^{\beta-1}m^{\gamma}.$$

Hence,

$$f_{m2}(x) = \sum_{n=m^{\zeta}}^{m/2} g_n(m^{-\gamma}) \leqslant Cm^{\gamma} \sum_{n=m^{\zeta}}^{m/2} n^{\beta-1} \leqslant Cm^{(1+\gamma)/2}. \tag{28}$$

Collecting estimates (25), (27), and (28), we get the required bound for $f_m(x)$.

Thus, we proved that, for the function f_m, we have the same bounds from above as those from below, therefore, (9) and, thus, the theorem is proved. □

Proof of Corollaries 2 and 3. Corollary 2 follows applying (9) to (8), while, for the proof of Corollary 3, we note that, for model (5) with the boundary conditions $Y_{in} = 0$ for all i, we have

$$Var\, Y_{t,s-1} = b^{-2} \sum_{k=0}^{n-s} \sum_{j=0}^{k} \binom{k}{j}^2 \tilde{a}^{2j} \tilde{b}^{2(k-j)}, \tag{29}$$

and again we apply (9) with $x = \tilde{a}$. A new effect comes from the factor b^{-2} in front of the sum in (29), since, as a and b grow (recall that $b = a + 1$) with n, this factor becomes dominant. □

Proof of Proposition 4. It is not difficult to see that solution of (12) with given initial conditions can be written as follows:

$$X_t = \sum_{k=0}^{t} \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j} \varepsilon_{t-j-2(k-j)}.$$

However, this expression (although having some similarity to the expressions in the two-dimensional lattice case) is not convenient to calculate the variance, since, in the double sum, there are repetitions of ε's. It is known that

$$X_t = \sum_{k=0}^{t} c_k \varepsilon_{t-k},$$

where the coefficients c_k are functions of a and satisfy the relation

$$c_k = a c_{k-1} + b c_{k-2}, \quad k \geqslant 2, \qquad c_0 = 1, \qquad c_1 = a. \tag{30}$$

It is possible to write a general formula for these coefficients:

$$c_{2m} = \sum_{j=0}^{m} \binom{2m-j}{2m-2j} a^{2(m-j)} b^j, \qquad c_{2m+1} = \sum_{j=0}^{m} \binom{2m+1-j}{2m+1-2j} a^{2m+1-2j} b^j.$$

However, these expressions again are too complicated to be useful in estimating the quantity

$$Var\, X_t = \sum_{k=0}^{t} c_k^2. \tag{31}$$

It turns out that recurrent relation (30) is most useful in estimating c_k. Precisely, using this relation and mathematical induction, one can prove that

$$C_* \leqslant c_i \leqslant C^*, \qquad i \geqslant 1,$$

where

$$C_* = a, \qquad C^* = a^2 - a + 1.$$

Therefore, from (31) we get (13), and the proposition is proved. □

Relation (13) means that, for large t, one has $Var\, X_t \simeq t$. Probably, the following is true: there exists a limit of the coefficients c_n as $n \to \infty$ (but the behavior of the coefficients is not so simple, since they are not monotone but oscillating), and there exists a function $C(a)$ such that

$$Var\, X_t = C(a) t (1 + o(1)).$$

Inequalities (13) give the bounds for this function: $C_*^2 \leqslant C(a) \leqslant (C^*)^2$. The function $C(a)$ is not symmetric with respect to the point $a = 1/2$ in the sense that $C(a) \neq C(1-a)$, since $C(1) = 1$, $C(0) = 1/2$ (if $a = 0$, then

$$X_t = X_{t-2} + \varepsilon_t = X_{t-4} + \varepsilon_t + \varepsilon_{t-2} = \varepsilon_t + \varepsilon_{t-2} + \cdots + \varepsilon_i$$

with $i = 1$ or 0, depending on whether t is odd or even). Also looking at the graph of bounds $C_*,\ C^*$ as functions of a, we see that these bounds become less precise for small values of a. This can be easily explained by the fact that, in the limit case $a = 0$, each second coefficient c_i is equal to zero, so, for small a, the better strategy to get a bound from below (close to 1/2) is to consider only coefficients c_{2i}, since each such a coefficient in its expression has term b^i. Then we can get

$$\sum_{i=0}^{t/2} c_{2i}^2 \geqslant \sum_{i=0}^{t/2} b^{2i} = (1-b^2)^{-1}(1-b^{t+2}),$$

and this bound is close to $t/2$ for b close to 1.

One more remark concerning Proposition 4 is appropriate here. We considered only the case $a + b = 1, a > 0, b > 0$ for model (12) and it is known that this is only a part of points on the (a, b)-plane giving unit roots (all points consist of sides of the triangle with vertices at points $(0, 1)$, $(-2, -1)$, $(2, -1)$). Despite of this, we believe that a quantitative picture will be the same also for all points on the boundary of this triangle.

Proof of Proposition 5. It is easy to see that, for model (14) with boundary conditions $Y_{t,s,v} = 0$ for $t + s + v = 0$, we have

$$Var\, Y_{t,s,v} = \sum_{k=0}^{t+s+v-1} \sum_{\substack{i,j \geqslant 0 \\ 0 \leqslant i+j \leqslant k}} c(k, i, j)^2 a^{2i} b^{2j} c^{2(k-i-j)},$$

where

$$c(k, i, j) = \frac{k!}{i!j!(k-i-j)!}.$$

Taking $a = b = c = 1/3$ and denoting $t + s + v = m$, we see that one has to find the asymptotic behavior of the function

$$f(m) = \sum_{n=0}^{m} g(n),$$

where

$$g(n) = \frac{1}{3^{2n}} \sum_{\substack{i,j \geqslant 0 \\ 0 \leqslant i+j \leqslant n}} c(n, i, j)^2.$$

Since some calculations are similar to those in the proof of Theorem 1, we omit some details of calculations. Using the well-known identity

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}, \tag{32}$$

one can get

$$g(n) = \sum_{i=0}^{n} h(i, n), \tag{33}$$

where

$$h(i, n) = \frac{1}{3^{2n}} \binom{n}{i}^2 \binom{2(n-i)}{n-i}.$$

Considering n, i, and $n-i$ large and applying the Stirling's formula, one can show that

$$h(i, n) \sim C \frac{n}{i(n-i)^{3/2}} \frac{2^{2(n-i)}}{3^{2n}} \left(\frac{n}{n-i}\right)^{2n} \left(\frac{n-i}{i}\right)^{2i}. \tag{34}$$

Taking $i = nx$ with $0 < c_1 \leqslant x \leqslant c_2 < 1$, where c_1 and c_2 are fixed constants, it is not difficult to show that the right-hand side of (34) is (as a function of x)

$$h(nx, n) \sim z(x) = \frac{C}{n^{3/2} x(1-x)^{3/2}} v(x)^{2n}, \tag{35}$$

where

$$v(x) = \frac{2^{1-x}}{3x^x (1-x)^{1-x}}.$$

Noting that $v(1/3) = 1$, from (35) we have that $h(n/3, n) \sim z(1/3) = Cn^{-3/2}$. The next step, as in the proof of Theorem 1, is to show that, for all $i \in [n/3 - n^\beta, n/3 + n^\beta]$ with some $\beta > 0$, $h(i, n)$ is of the same order as $h(n/3, n)$. To be specific, let us take $i_0 = n/3 + n^\beta$, then, after some calculations, we get

$$h(i_0, n) = Cn^{-3/2}(U(n, \beta))^{-1},$$

where

$$\ln U(n, \beta) = 2n \ln(1-a) + \frac{2n}{3}(1+2a) \ln(1 + 3a(1-a)^{-1})$$

and $a = 3/(2n^{1-\beta})$. Standard considerations give that

$$\ln U(n, \beta) = 2na^2(1 - O(a)) + o(na^2),$$

and, therefore, the maximal value of β we can take is $1/2$. We get

$$h(i_0, n) \simeq n^{-3/2}$$

and this relation is valid not only for i_0 but also for all $i \in [n/3 - n^\beta, n/3 + n^\beta]$. Hence, from (33) we get

$$g(n) \geqslant Cn^{-1}$$

and this gives a lower bound in (15). To get an upper bound, it is necessary to show that $h(i, n)$ for $i \notin [n/3 - n^\beta, n/3 + n^\beta]$ decreases exponentially with increasing n, and this can be done in a similar way as in the proof of Theorem 1. For example, considering $0 < i < n/3 - n^\beta$ we separately consider three cases: $0 < i \leqslant C$ (in this case, we cannot use Stirling's formula for $i!$), $i = n^\gamma$, $0 < \gamma < 1$, and $i = Cn$ with $C < 1/3$ (in this case we use the fact that $v(C) < 1$). Having this shown, it is not difficult to finish the proof and we leave it to the reader. □

Proof of Proposition 6. It is easy to see that now we have to show that the following function is bounded:

$$f_1(m) = \sum_{n=0}^{m} g_1(n),$$

where

$$g_1(n) = \frac{1}{4^{2n}} \sum_{\substack{i,j,k \geqslant 0 \\ 0 \leqslant i+j+k \leqslant n}} c_1(n,i,j,k)^2,$$

and $c_1(n,i,j,k)$ are defined in (17). Using the same identity as in (32), one can get

$$g_1(n) = \frac{1}{4^{2n}} \sum_{i=0}^{n} \binom{n}{i}^2 \sum_{j=0}^{n-i} \binom{n-i}{j}^2 \binom{2(n-i-j)}{n-i-j}.$$

The inner sum in the last formula has the same form as the quantity g_n from (33), therefore, using the relation $g_n \simeq n^{-1}$ proved in the proof of Proposition 5, we have

$$g_1(n) = \frac{1}{4^{2n}} \sum_{i=0}^{n} \binom{n}{i}^2 3^{2(n-i)} g_{n-i} \leqslant \sum_{i=0}^{n} h_1(i,n),$$

where

$$h_1(i,n) = \frac{Cn3^{2(n-i)}}{i(n-i)^2 4^{2n}} \left(\frac{n}{n-i}\right)^{2n} \left(\frac{n-i}{i}\right)^{2i}.$$

Again, taking $i = nx$, we have

$$h_1(i,n) = \frac{C}{n(1-x)^2}(v_1(x))^{2n},$$

where

$$v_1(x) = \frac{3^{1-x}}{4x^x(1-x)^{1-x}}.$$

It is easy to see that $v_1(1/4) = 1$, $\lim_{x\to 0} v_1(x) = 3/4$, and $\lim_{x\to 1} v_1(x) = 1/4$, therefore, we have that $h_1(n/4, n)$ is of order n^{-2} and that the number of terms $h_1(i,n)$ of such order can be only n^β with $\beta < 1$ (since if $x \neq 1/4$, then $v_1(x) < 1$). This implies that

$$g_1(n) \leqslant Cn^{\beta-2},$$

and the function $f_1(m)$ is bounded. Proposition 6 is proved. □

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