On the rate of convergence to bivariate stable laws

Vygantas Paulauskas

Vilnius University, Department of Mathematics and Informatics
and
Institute of Mathematics and Informatics, Vilnius

Abstract

In the paper a distribution function of a sum of independent non-identically distributed bivariate random vectors is approximated by distribution function of a stable vector and the accuracy of such approximation is estimated. The obtained general result is only a little bit worse when compared with known estimates for the case of multivariate independent and identically distributed random vectors or univariate non-identically distributed summands. Also the obtained result is applied for a specific scheme arising when considering the so-called Increment-Ratio Statistics.

MSC 2000 subject classifications. Primary-62F05.
Key words and phrases. Limit theorem, multivariate stable laws, rates of convergence.

1 Introduction and formulation of results

The problem of the rates of convergence in limit theorems for sums of random variables (or random elements with values in more general spaces) can be formulated as follows. Let F_n stand for a distribution function (d.f.) of some sum of the first n random variables and we choose another sequence of distribution functions (d.fs.) H_n (normal, stable or more general infinitely divisible laws), serving as approximating sequence for F_n. In the case where F_n, as n tends to infinity, weakly converges to some limit law, let us say, H, then we usually set $H_n \equiv H$ for all n. Various quantities, such as

$$\sup_x (1+|x|^k)|F_n(x) - H_n(x)|, \ \sup_{f \in \mathscr{F}} \Big| \int_R f(x) d(F_n - H_n)(x) \Big|,$$

where $\mathscr{F}$ is some class of functions, can be used for estimating how good approximation is. Theory of summation of independent real random variables includes limit theorems and their refinements - rates of convergence, asymptotic expansions, large deviations, local limit theorems (by local limit theorems we mean limit theorems for densities of F_n). One can say that this theory is almost completed, for most questions final answers are known, the list of monographs, starting with classical books [13], [14], [27] is rather impressive. Many results are generalized under assumption of some kind of dependence of summands, such as Markov type dependence, martingales, weak dependence, etc.

The situation is different for multi-dimensional or infinite-dimensional random elements. There are only few monographs [9], [31], [24] , [32] devoted to the rate of convergence in the Central Limit Theorem (CLT) for independent random vectors. If the case of Gaussian approximation is investigated comparatively well (rates of convergence and asymptotic expansions are available, see, for example, [24], the survey paper [5] or the recent paper [4]), the case of approximation with stable laws is far from being investigated. But before giving review of known results in this area we shall formulate our result, then we shall be able to compare our result with the results obtained earlier.

Since in the paper a specific method of the proof is used (see Lemma 2.1 and discussion after it), in what follows we consider only two-dimensional case. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ denote

$$(x, y) = x_1 y_1 + x_2 y_2, \ ||x||_r^r = |x_1|^r + |x_2|^r, \ r \ge 1, \ ||x|| = ||x||_2, \ ||x||_\infty = \max(|x_1|, |x_2|),$$

$$V_1 = \{x \in R^2 : ||x|| = 1\}.$$

Let us denote by $\theta = (\theta_1, \theta_2)$ a two-dimensional stable random vector with a characteristic function (ch.f.)

$$g(t) = g(t; \alpha, \lambda, \Gamma) = \exp\Big\{ -\lambda^\alpha \Big(\int_{V_1} |(t, x)|^\alpha \Gamma(dx) + i\beta_{\alpha,\Gamma}(t) \Big) \Big\},$$

where $t \in R^2,\ 0 < \alpha < 2,\ \lambda > 0$ is a scale parameter and Γ is a normalized (that is, $\Gamma(V_1) = 1$) spectral measure, defined on V_1 (the assumption that spectral measure is normalized and concentrated on V_1 is made for convenience, see [30]). Function $\beta_{\alpha,\Gamma}(t)$ is defined as follows:

$$\beta_{\alpha,\Gamma}(t) = \begin{cases} \tan(\pi\alpha/2)\int_{V_1} |(t,x)|^{\alpha-1}(t,x)\Gamma(dx), & \text{if } \ 0 < \alpha < 2, \alpha \neq 1, \\ \int_{V_1} \ln|(t,x)|(t,x)\Gamma(dx), & \text{if } \ \alpha = 1. \end{cases}$$

In [23] this function was called asymmetry function since if $\beta_{\alpha,\Gamma}(s,t) \equiv 0$ then the random vector θ is symmetric. In [30] or [23] one can find more facts about multivariate stable laws. Let $G(x; \alpha, \lambda, \Gamma)$ denote the d.f. of θ.

We consider a sequence of two-dimensional independent random vectors $\xi_j = (\xi_{j,1}, \xi_{j,2}),\ j \geq 1$ with d.fs. F_j and characteristic functions (ch.fs.)

$$f_j(t) = E\exp\{i(t, \xi_j)\}, \quad t = (t_1, t_2).$$

Let $\theta_j = (\theta_{j,1}, \theta_{j,2}),\ j \geq 1$ be a sequence of independent stable random vectors. Let $g_j(t) = g(t; \alpha, \lambda_j, \Gamma_j)$, and $G_j(x) = G(x; \alpha, \lambda_j, \Gamma_j)$ be a ch.f. and a d.f., respectively, of θ_j. Here $0 < \alpha < 2,\ \lambda_j > 0$ and Γ_j is a normalized spectral measure. For non-negative integers i, j, k we set

$$\mu_{i,k;j} = \int_{R^2} x_1^i x_2^k (F_j - G_j)(dx),$$

$$\nu_{j,r} = \int_{R^2} ||x||^r |(F_j - G_j)|(dx), \quad \nu_{j,r}^{(i)} = \int_{R^2} |x_i|^r |(F_j - G_j)|(dx),\ i = 1, 2.$$

Let us denote

$$B_n(\kappa) = \Big(\sum_{k=1}^{n} \lambda_k^{\kappa}\Big)^{1/\kappa}, \quad B_n = B_n(\alpha) \quad S_n = B_n^{-1}\sum_{k=1}^{n} \xi_j, \quad Z_n = B_n^{-1}\sum_{k=1}^{n} \theta_j,$$

and let $\bar{F}_n$ and $\bar{G}_n$ be the d.fs. of S_n and Z_n, respectively. The main goal of the paper is to estimate the quantity

$$\Delta_n := \sup_{x \in R^2} |\bar{F}_n(x) - \bar{G}_n(x)|.$$

Since later on the numbers α and r will be fixed, therefore sometimes we shall skip these numbers, for example, we shall write $\beta_\Gamma(t)$ instead of $\beta_{\alpha,\Gamma}(t)$. Also we assume that pseudomoments $\nu_{j,r}^{(i)}$ are finite for all $j \geq 1$ and $i = 1, 2$. Main quantity by which we want to estimate Δ_n is the following "Lyapunov fraction"

$$L_n = L_n(r) = L_n^{(1)} + L_n^{(2)}, \quad L_n^{(i)} = L_n^{(i)}(r) = B_n^{-r}\sum_{j=1}^{n} \nu_{j,r}^{(i)},\ i = 1, 2.$$

Here it is worth to note that due to equivalence of norms in finite-dimensional space $\nu_{j,r}$ is equivalent to $\nu_{j,r}^{(1)} + \nu_{j,r}^{(2)}$ (here a is equivalent to b means that $c_1 a \leq b \leq c_2 a$ with some constants c_1, c_2). As we are not interested in numerical values of constants appearing in our estimates we shall use the same letter L_n for the quantity $B_n^{-r}\sum_{i=1}^n \nu_{i,r}$. From one-dimensional case we know (see for example [19]) that it is impossible to estimate Δ_n by L_n, therefore we introduce additional quantities

$$\eta_n = B_n^{-1}\max_{1\le k\le n}\lambda_k, \quad \tau_n(\kappa) = \Big(\frac{B_n(\kappa)}{B_n}\Big)^\kappa,$$

where $\kappa > 0$. Note that $\tau_n(\kappa) \le 1$ for $\kappa > \alpha$, $\tau_n(\alpha) = 1$ and $\tau_n(\kappa) \ge 1$ for $\kappa < \alpha$. Also we need some quantity which reflects the dependence between coordinates of the vector Z_n. Let us denote

$$\rho_n = \inf_{||u||=1} B_n^{-\alpha}\sum_{k=1}^n \lambda_k^\alpha\ \gamma_k(u),$$

where

$$\gamma_j(u) = \gamma_j(u;\alpha) := \int_{V_1} |u_1x_1 + u_2x_2|^\alpha \Gamma_j(dx), \quad u = (u_1, u_2).$$

In the proof there will be used the moments

$$\int\int |x_1|^{\beta_1}|x_2|^{\beta_2}F_j(dx_1, dx_2),$$

which are finite for all non-negative $\beta_i,\ i = 1, 2$, such that $\beta_1 + \beta_2 < \alpha$. These moments depend not only on $\beta_i,\ i = 1, 2$, but also on λ_j, therefore we shall use the so-called standardized distribution function $\hat F_j$ corresponding to random vector $\lambda_j^{-1}\xi_j$. Also we assume that there exist constant $\hat C = \hat C(\kappa)$, such that

$$\sup_{j\ge1,\beta_1+\beta_2<\kappa}\int\int |x_1|^{\beta_1}|x_2|^{\beta_2}\hat F_j(dx) \le \hat C. \tag{1.1}$$

Let us denote

$$A_n(\kappa) = L_n^{(1-\kappa)/(r+1)}\eta_n^{(r+\kappa)/(r+1)},\ D_n(\kappa) = \rho_n^{-(r+\kappa)/\alpha}\tau_n(\kappa),\ \kappa > 0.$$

We shall make one remark about notation of constants which will be used in the paper. Generally constants, dependent on fixed parameters, such as $\alpha,\ r$, will be denoted by letter c, so they can be different in different places. If we want to display some constants (this is done mainly to help a reader to follow estimates, some of which are rather complicated, moreover, some calculations are omitted), these constants will be numbered as $c_1, c_2, \ldots$. With a special notation we separate three constants - $\hat C, \bar C$, and C_0. $\hat C$ was just introduced, it is specially denoted for the reason that it is important constant with complicated dependence on parameters of summands. $\bar C$ will be used in all our final estimates of the accuracy of approximation (in the main theorem and its corollaries and Proposition 1.1), thus in different statements it can be different. C_0 will be used two times to define an appropriate value of T, which appears in Lemma 2.1 (see (2.1)).

Now we formulate the main result.

Theorem 1.1 *Let $\varepsilon_1 > 0,\ \varepsilon_2 > 0$ be some small fixed numbers. Suppose that for any $0 < \kappa < \alpha$ (1.1) holds and the following condition is satisfied: for some integer $m,\ [\alpha] \le m \le 1 + [\alpha]$, for some real $r,\ \max(\alpha, m) < r \le \min(1 + m, 1 + \alpha)$, and, for all $j = 1, 2, \ldots, n$,*

$$\mu_{i,k;j} = 0,\ 0 \le i, k \le m \quad i + k \le m, \quad \nu_{j,r} < \infty$$

Then there exists a constant $\bar{C}$ depending on $\varepsilon_1,\ \varepsilon_2,\ \alpha,\ r$, such, that

$$\Delta_n \le \bar{C} \max \Big\{ L_n \min{}^* D_n(\kappa), \min{}^* A_n(\kappa) D_n(\kappa) \Big\}. \tag{1.2}$$

Here and in the sequel

$$\min{}^* := \min_{\varepsilon_1 \le \kappa \le \alpha(1-\varepsilon_2)}$$

Changing a little bit the last step in the proof of the main theorem we can get less precise but more transparent estimates.

Corollary 1.1 *Under conditions of Theorem 1.1 the following estimates hold*

$$\Delta_n \le \bar{C} \min{}^* \Big\{ \max \big(L_n, A_n(\kappa) \big) D_n(\kappa) \Big\} \tag{1.3}$$

$$\Delta_n \le \bar{C} \max \big(L_n, A_n(\kappa_0) \big) D_n(\kappa_0), \tag{1.4}$$

where $\kappa_0 = \arg\min D_n(\kappa)$ and minimum is taken over interval $\varepsilon_1 \le \kappa \le \alpha(1 - \varepsilon_2)$.

We can derive several corollaries from the obtained result. Let us consider sums of independent and identically distributed (i.i.d.) random vectors with weights. Namely, let $\zeta_i = (\zeta_{i,1}, \zeta_{i,2}),\ i \ge 1$ be a sequence of i.i.d. random vectors with a common d.f. $F(x)$ and let $\bar{\zeta}_i = (\bar{\zeta}_{i,1}, \bar{\zeta}_{i,2}),\ i \ge 1$ be another sequence of i.i.d. stable random vectors with a common d.f. $G(x)$ and with ch.f. $g(t; \alpha, 1, \Gamma)$ with some normalized spectral measure Γ. Pseudomoments between d.fs. F and G will be denoted by the same letters only without index j:

$$\mu_{i,k} = \int_{R^2} x_1^i x_2^k (F - G)(dx),$$

$$\nu_r = \int_{R^2} ||x||^r |(F - G)|(dx), \quad \nu_r^{(i)} = \int_{R^2} |x_i|^r |(F - G)|(dx),\ i = 1, 2.$$

Let $\lambda_i,\ i \ge 1$ be a sequence of positive weights and $\xi_j = \lambda_j \zeta_j,\ \theta_j = \lambda_j \bar{\zeta}_j$. Adopting such notation for the weighted random vectors we can use the same notation which was introduced before the formulation of the main result, only, in order to distinguish this case, upper subscript (1) is added: $F_j^{(1)},\ G_j^{(1)},\ \bar{F}_n^{(1)},\ \bar{G}_n^{(1)},\ \Delta_n^{(1)}$. For example,

$$F_j^{(1)}(x) = F(x\lambda_j^{-1}),\ \Delta_n^{(1)} = \sup_{x \in R^2} |\bar{F}_n^{(1)}(x) - \bar{G}_n^{(1)}(x)|,$$

etc. It is necessary to note that now for all $j \ge 1,\ G_j(x) = G(x;\alpha,\lambda_j,\Gamma)$ have the same spectral measure Γ, therefore the parameter $\rho_n^{(1)}$ is independent of n

$$\rho_n^{(1)} \equiv \rho = \inf_{||u||=1} \int_{V_1} |u_1x_1 + u_2x_2|^\alpha \Gamma(dx).$$

Now $L_n^{(1)} = \tau_n(r)\nu_r$, and it is easy to see that the rate of convergence in this case is expressed by means of η_n and $\tau_n(\kappa)$ with a particular value of κ. Namely, denoting

$$A_n^{(1)}(\kappa) = \big(\tau_n(r)\nu_r\big)^{(1-\kappa)/(r+1)}\eta_n^{(r+\kappa)/(r+1)},\quad D_n^{(1)}(\kappa) = \rho^{-(r+\kappa)/\alpha}\tau_n(\kappa),$$

we have the following result.

Corollary 1.2 *Let $\varepsilon_1 > 0,\ \varepsilon_2 > 0$ be some small fixed numbers. Suppose that for introduced i.i.d. random vectors the following condition is satisfied: for some integer $m,\ [\alpha] \le m \le 1+[\alpha]$ and some real $r, \max(\alpha,m) < r \le \min(1+m, 1+\alpha)$,*

$$\mu_{i_1,i_2} = 0,\ 0 \le i_1, i_2 \le m,\ i_1 + i_2 \le m, \quad \nu_r < \infty.$$

Then the following estimates hold

$$\Delta_n^{(1)} \le \bar{C}\min{}^*\Big\{ \max\big(\tau_n(r)\nu_r, A_n^{(1)}(\kappa)\big) D_n^{(1)}(\kappa)\Big\},$$

$$\Delta_n^{(1)} \le \bar{C}\max\big(\tau_n(r)\nu_r, A_n^{(1)}(\kappa_0)\big) D_n^{(1)}(\kappa_0),$$

where $\kappa_0 = \arg\min D_n^{(1)}(\kappa)$ and minimum is taken over interval $\varepsilon_1 \le \kappa \le \alpha(1-\varepsilon_2)$.

In the case of i.i.d. random vectors ξ_i (taking weights $\lambda_i \equiv 1$ in Corollary 1.2) for all notation we shall add the upper subscript (2), for example,

$$F_j^{(2)}(x) \equiv F(x),\ \Delta_n^{(2)} = \sup_{x\in R^2} |\bar{F}_n^{(2)}(x) - \bar{G}_n^{(2)}(x)|,$$

etc. From Corollary 1.2 we have the following result.

Corollary 1.3 *If the conditions of Corollary 1.2 are satisfied, $\varepsilon_1 = \varepsilon_2 = \varepsilon$, and $\lambda_i \equiv 1$, then for any small fixed $\varepsilon > 0$ there exists a constant $\bar{C}$ depending on $\varepsilon,\ \alpha,\ r$, such, that*

$$\Delta_n^{(2)} \le \bar{C}\max\Big\{\nu_r n^{-(r-\alpha)/\alpha}\min{}^*\rho^{-(r+\kappa)/\alpha}n^{1-(\kappa/\alpha)}, \min{}^*\rho^{-(r+\kappa)/\alpha}\nu_r^{(1-\kappa)/(1+r)}n^{\gamma(r,\alpha,\kappa)}\Big\},$$

$$\Delta_n^{(2)} \le \bar{C}n^{\varepsilon-(r-\alpha)/\alpha}\rho^{-(r+\alpha(1-\varepsilon))/\alpha}\max\big\{\nu_r, \nu_r^{1-\alpha(1-\varepsilon)/(1+r)}n^{\gamma_1(r,\alpha,\varepsilon)}\big\}, \tag{1.5}$$

where

$$\gamma(r,\alpha,\kappa) = -\frac{(2+\alpha)(r+\kappa) - 2\alpha(1+r)}{\alpha(1+r)}, \quad \gamma_1(r,\alpha,\varepsilon) = -\frac{(1+\alpha-r)(r+\alpha(1-\varepsilon))}{\alpha(1+r)}.$$

Before presenting one application of the obtained result at first we give a review of some known results, then we shall compare our estimates with them. We shall consider only the rates of convergence in the so-called uniform metrics (that is, supremum of the difference of distributions over some classes of sets, in particular, supremum of the difference of d.fs.), leaving aside the rates of convergence in other metrics (for review of such results we can recommend [34] or [28]), since generally it is difficult to obtain estimates of the right order for uniform metrics using estimates for other types of metrics, moreover, most estimates in terms of probability metrics are formulated in the case of i.i.d. summands (see [28]). We start our review with the univariate case.

It is well-known that pseudomoments (of various types) play an important role in the estimation of the rates of convergence to stable laws. For the first time pseudomoments were used in H. Bergström papers , see [7] and [8], but the systematic use of pseudomoment was initiated by V.M.Zolotarev. Together with his students he laid the foundations of the theory of summation of independent random variables without the so-called classical condition of uniform negligibility of summands, see monograph [34]. We formulate one very general result of Zolotarev (see [33] or Th 6.5.1 in [34]). Suppose we have two sums of independent random variables

$$U_n = \sum_{i=1}^{n} \zeta_i, \quad V_n = \sum_{i=1}^{n} \tau_i,$$

with d.fs. $\tilde{K}_n(x)$ and $\tilde{H}_n(x)$, respectively. Let $K_j(x) = P(\zeta_j \le x)$ and $H_j(x) = P(\tau_j \le x)$. If

$$\int_R x^i (K_j - H_j)(dx) = 0, \quad \nu_j(r) := \int_R |x|^r |(K_j - H_j)|(dx) < \infty,$$

for all $j = 1, \ldots, n, \ i = 1, \ldots, m$, and some $m < r \le m+1$, then

$$\sup_x |\tilde{K}_n(x) - \tilde{H}_n(x)| \le C(m,r)\big(D^r \sum_{j=1}^{n} \nu_j(r)\big)^{1/r+1}, \tag{1.6}$$

where $D = \sup_x \tilde{H}_n'(x)$. In the classical situation of the CLT, when a random variable ξ_i has zero mean, a variance σ_i^2 and finite the third moment $\beta_i := E|\xi_i|^3, \ i \ge 1$. Taking $m = 2, r = 3, H_j = \Phi_j$ - mean zero normal d.f. with variance σ_j^2, $\tilde{H}_n(x) = \Phi$ - standard normal d.f., from (1.6) we get

$$\sup_x |\tilde{F}_n(x) - \Phi(x)| \le C \left(\frac{\sum_{j=1}^{n} \nu_j(3)}{\tilde{B}_n^3} \right)^{1/4}, \tag{1.7}$$

where $\tilde{B}_n^2 = \sum_{j=1}^{n} \sigma_j^2$. Although $\nu_j(3) \le c\beta_j$, due to the presence of exponent 1/4 (therefore in the case of i.i.d. summands (1.7) instead of correct order $n^{-1/2}$ gives only $n^{-1/8}$) from (1.6) we can not derive the classical Berry-Esseen estimate

$$\sup_x |\tilde{F}_n(x) - \Phi(x)| \le C \frac{\sum_{j=1}^{n} \beta_j}{\tilde{B}_n^3}. \tag{1.8}$$

On the other hand, (1.7) can be better than (1.8) if summands ξ_j are close to a normal distribution - take extreme case $K_j = \Phi_j$ for all j, then in (1.7) on both sides of the inequality there are zeros, while (1.8) does not reflect the closeness of the summands to the normal law. In the case of i.i.d. summands ($\sigma_j \equiv 1,\ \nu_j(3) \equiv \nu(3)$) the improvement of both estimates (1.7) and (1.8) was obtained by author in [16]:

$$\sup_x |\tilde{F}_n(x) - \Phi(x)| \leq C \frac{\max(\nu(3), \nu(3)^{1/4})}{n^{1/2}}. \tag{1.9}$$

Several years later generalizations of (1.9)to the case of non-identically distributed summands were obtained, see [15] (the Gaussian case) and [18], [19], [22] (the case of a stable limiting law). Since these estimates have rather complicated form and for the formulation a lot of new notation must be introduced, we do not provide them here; one of these estimates, the most simple, is formulated bellow (see (3.1)). For complete review of estimates of the rate of convergence to univariate stable laws we can recommend [10].

Now we pass to the multivariate case and we restrict ourselves with the case of i.i.d. summands, since it seems that in the case of non-identically distributed summands with infinite moments of the second order the available estimates for uniform distances have not the right order (see, for example, [20], where the estimate (1.6) was generalized to Hilbert space). The first estimates of Δ_n (only now $\bar{F}_n(x)$ is a d.f. of a sum of i.i.d. multivariate summands and supremum is taken over R^k) were obtained in [1], [2], [3], but all they were obtained for very special case where ch.f. of the limit stable law is $\exp(-||t||^\alpha)$, moreover, the order of the estimates with respect to number of summands n was not always the right one. For example, in [1] in the case $r = 1 + [\alpha]$ and $d = 2$ the following rate of convergence was claimed:

$$\Delta_n \leq Cn^{-\frac{r-\alpha}{r\alpha}}. \tag{1.10}$$

If $1 < \alpha < 2$ then $r = 2$ and (1.10) is worse than the right order $-(r - \alpha)/\alpha$ but if $0 < \alpha < 1$ then $r = 1$ and (1.10) is optimal (and better than our estimate (1.5)). It turns out that such result is obtained claiming (essentially without a proof, only using analogy to the Gaussian case, which is clearly misleading) that estimate (2.3) is valid with $\kappa = \alpha$.

In [21] (see also [17]) a general result was presented, stating what quantities and characteristics must be estimated in order to get the right order $n^{-(r-\alpha)/\alpha}$ (if pseudomoment of the order $r,\ \alpha < r \leq 1 + \alpha$ is finite) of the rate of convergence in the multidimensional case. In this paper in some specific cases these quantities were estimated, but completely this problem was solved only in 2000, in a paper [6](there one can find other references of papers related to the problem). In this paper the general d-dimensional case and supremum over all Borel sets are considered and the estimate in the case of $r = 1 + \alpha$ and $d = 2$ with respect to n and ρ is of the order

$$\Delta_n \leq Cn^{-1/\alpha}\rho^{-(1+2\alpha)(1+\alpha)/\alpha}, \tag{1.11}$$

where constant C depends on α and some characteristics of one summand.

Now we can compare indirectly our main result from Theorem 1.1 with known results (the word "indirectly" is used for the reason, that we compare two-dimensional result for non-identically distributed summands with the results in the two-variate i.i.d. case or in the univariate case). At first let us compare (1.11) with (1.5) which with respect to n and ρ has the order

$$n^{\varepsilon-1/\alpha}\rho^{-(1+2\alpha-\varepsilon\alpha)/\alpha}. \tag{1.12}$$

Since ε is a small parameter, it is not difficult to see that for small ρ (comparable with n^{-v} for some $v>0$) the quantity (1.12) may be smaller than (1.11). Of course, comparison of estimate (1.5) with the estimate (1.11) from [6] is not completely correct, since in this paper the remainder term was estimated for all convex sets in R^d, while our estimates are for rectangles only. From Gaussian case we know that dependence of the remainder term on the quantities characterizing degeneracy of distributions is worse for class of convex sets (comparing with rectangles). The same effect should be expected in the case of stable laws, too. Still, this comparison allows to think that in the case of non-identically distributed summands the characteristic ρ_n is the right one and the power $-(r+\kappa)/\alpha$ with some κ from interval $[\varepsilon_1, \alpha(1-\varepsilon_2)]$ is quite good. Also there is one interesting question. In [23] the author had introduced some measure of dependence between coordinates of multivariate symmetric stable random vector, it was called a generalized association parameter; recently this parameter was considered in [12]. It would be interesting to see what is the relation between ρ and g.a.p. and if the latter could be used in the estimates of the rate of convergence.

Comparison of Theorem 1.1 with results in the case of univariate summands is not easy, since, as it was mentioned, estimates obtained in [18], [19], [22] are rather complicated and no one of them dominates others. Since we adapted the method of the proof from [18], for a comparison we shall take the estimate (3.1)). Comparing (3.1) with (1.2) or with (1.4) we see appearance of quantity ρ_n, which is natural (reflects dependence between coordinates), and $\tau_n(\kappa)$, which is caused by the method of the proof - we must use Lemma 2.3 to get product $|t_1t_2|^\kappa$ with positive κ. Also this use of Lemma 2.3 makes change in powers at L_n and η_n: if in (3.1) these powers were $1/(1+r)$ and $r/(1+r)$ (their sum is 1), now these exponents in (1.2) are shifted by $\kappa/(1+r)$ but their sum remains 1. This comparison allows to assert that in the two-dimensional case we loose accuracy only a little bit comparing with univariate case. It is possible to conjecture that under the conditions of Theorem 1.1 for some $b>r/\alpha$ the following estimate is true

$$\Delta_n \le c\max(L_n, L_n^{1/(r+1)}\eta_n^{r/(r+1)})\rho_n^{-b}$$

On the other hand, the appearance of characteristic $\tau_n(\kappa)$ is not so surprising, since in [19] the author obtained the estimate of another type (most probably, more precise), expressed by means of some broken lines, constructed by the values of $\lambda_1,\dots,\lambda_n$. As a corollary from this general result there was obtained an estimate (3.1) only with $\tau_n(r)$ instead of η_n. It is not difficult to see that in some cases $\tau_n(r)$ is smaller, in other cases it is better to use η_n (see [19]). It should be interesting to combine method of the proof from [19]) and lemma 2.1 in the two-dimensional case.

Now we present one application of the obtained result. The problem which originates from the so-called Increment-Ratio Statistics (IRS) was posed by D. Surgailis, and essentially this problem was the main motivation to write the paper. It can be formulated as follows. Let $\varphi_i,\ i \in Z$, be a sequence of i.i.d. random variables with common d.f. F and ch.f. f and let $\psi_i,\ i \in Z$ be another sequence of i.i.d. stable random variables with common d.f. $G = G_\alpha$ and ch.f.

$$g(t) = g(t, \alpha, \beta) = \exp\big\{ -|t|^\alpha (1 - i\beta \text{sign}\, t \tan(\pi\alpha/2))\big\}, \quad |\beta| \le 1, \quad t \in R,$$

for $0 < \alpha < 2,\ \alpha \ne 1$, and for $\alpha = 1$ we assume that $\beta = 0$ (to avoid technical complications with centering). We assume that φ_1 belongs to the normal domain of attraction of ψ_1 and that $E\varphi_1 = 0$ if $\alpha > 1$. Also assume that the following conditions on pseudomoments are satisfied: for $m,\ r$ defined in Theorem 1.1

$$\mu_i := \int_R x^i (F - G)(dx) = 0, \quad 0 \le i \le m, \quad \hat{\nu}_r = \int_R |x|^r |(F - G)|(dx) < \infty. \tag{1.13}$$

Let

$$X_t = \sum_{j=0}^{\infty} a_j \varphi_{t-j}, \quad T_n = \sum_{t=1}^{n} X_t,$$

be a linear process generated by the sequence $\varphi_i,\ i \in Z$ and a sum of the first n values of this process, respectively. Corresponding stable linear process and its sums are defined as follows:

$$Y_t = \sum_{j=0}^{\infty} a_j \psi_{t-j}, \quad U_n = \sum_{t=1}^{n} Y_t$$

Optimal rates of approximation of a distribution of T_n by a distribution of U_n under appropriate conditions on coefficients a_j were obtained in [26] (here it is possible to note that in [26] result from [19], not from [18], as in this paper (see (3.1)) was used). In IRS problem one is interested in quality of approximation of distribution of two-dimensional random vector $(T_n, T_{2n} - T_n)$ by distribution of the corresponding stable vector $(U_n, U_{2n} - U_n)$. Since the sums under consideration are formed by stationary dependent random variables, as the first step we "translate" the problem to the case of independent summands. It is not difficult to see that using our previous notations we need to deal with the following random vectors

$$\bar{S}_{2n} = \bar{B}_n^{-1} \sum_{j \le 2n} \xi_j, \quad \bar{Z}_{2n} = \bar{B}_n^{-1} \sum_{j \le 2n} \theta_j,$$

where $\sum_{j \le 2n} = \sum_{j=-\infty}^{2n}$, and

$$\xi_i = \begin{cases} (b_{n,i}\varphi_i,\ \tilde{b}_{2n,i}\varphi_i), & \text{for} \quad i \le n, \\ (\ \ 0, \quad \tilde{b}_{2n,i}\varphi_i), & \text{for} \quad n < i \le 2n, \end{cases}$$

$$\theta_i = \begin{cases} (b_{n,i}\psi_i,\ \tilde b_{2n,i}\psi_i), & \text{for} \quad i \le n, \\ (\quad 0, \quad \tilde b_{2n,i}\psi_i), & \text{for} \quad n < i \le 2n, \end{cases}$$

$$b_{n,i} = \sum_{t=1\vee i}^{n} a_{t-i}, \quad \tilde b_{2n,i} = \sum_{t=(n+1)\vee i}^{2n} a_{t-i}, \quad \bar B_n(\kappa) = \Big(\sum_{j\le n} |b_{n,i}|^\kappa\Big)^{1/\kappa}, \quad \bar B_n := \bar B_n(\alpha).$$

Unfortunately, any of corollaries formulated above can not be applied directly, although sum S_{2n} looks like a sum of i.i.d. random vectors with weights as in Corollary 1.2 (the difference consisting in infinite summation is not essential, we can truncate and then pass to the limit, see [26]). But if in Corollary 1.2 we multiply random vectors with scalar weights, now we have degenerate vectors (φ_i, φ_i) and we multiply each coordinate by separate weights. Therefore, each vector ξ_i is degenerate on (different) lines, only their sum is not degenerate. It was possible to have coordinate normalizing both in Theorem 1.1 and in Corollary 1.2, but since the estimates even with the scalar normalizing have rather complicated structure, we did not want to make these estimates more complicated. Moreover, due to specific structure of coefficients $b_{n,i}$, $b_{2n,i}$ and easily verified equality

$$\tilde b_{2n,i} = b_{n,i-n}, \tag{1.14}$$

we have

$$\bar B_n^\alpha = \sum_{i\le n} |b_{n,i}|^\alpha = \sum_{i\le 2n} |\tilde b_{2n,i}|^\alpha,$$

therefore both coordinates of $\bar S_{2n}$ are normalized by the same quantity. This is not unexpected, since both coordinates of the vector $(T_n, T_{2n}-T_n)$ contain sum of n stationary random variables. We formulate our result as a proposition. Denote (this time we supply the upper subscript (3))

$$\tau_n^{(3)}(\kappa) = \Big(\frac{\bar B_n(\kappa)}{(\bar B_n)}\Big)^\kappa, \ \eta_n^{(3)} = \frac{\max_{i\le n}|b_{n,i}|}{\bar B_n},$$

$$L_n^{(3)} = \tau_n^{(3)}(r)\nu_r, \quad \rho_n^{(3)} = \inf_{||(u||=1} \bar B_n^{-\alpha} \sum_{i\le n} |u_1 b_{n,i} + u_2 \tilde b_{2n,i}|^\alpha,$$

$$A_n^{(3)}(\kappa) = (L_n^{(3)})^{(1-\kappa)/(r+1)} (\eta_n^{(3)})^{(r+\kappa)/(r+1)}, \quad D_n^{(3)}(\kappa) = (\rho_n^{(3)})^{-(r+\kappa)/\alpha} \tau_n^{(3)}(\kappa).$$

Let us denote

$$\Delta_n^{(3)} := \sup_{x\in R^2} |F_{2n}^{(3)}(x) - G_{2n}^{(3)}(x)|,$$

where $F_{2n}^{(3)}$ and $G_{2n}^{(3)}$ stand for d.fs. of $\bar S_{2n}$ and $\bar Z_{2n}$, respectively.

Proposition 1.1 *Let $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ be some small fixed numbers. Suppose that condition (1.13) is satisfied, then the following estimates hold*

$$\Delta_n^{(3)} \le \bar C \min{}^* \Big\{ \max\big(L_n^{(3)}, A_n^{(3)}(\kappa)\big) D_n^{(3)}(\kappa) \Big\} \tag{1.15}$$

$$\Delta_n^{(3)} \le \bar C \max\big(L_n^{(3)}, A_n^{(3)}(\hat\kappa_0)\big) \hat D_n^{(3)}(\hat\kappa_0), \tag{1.16}$$

where $\hat\kappa_0 = \arg\min D_n^{(3)}(\kappa)$ and minimum is taken over interval $\varepsilon_1 \le \kappa \le \alpha(1-\varepsilon_2)$.

In the section of proofs we will show what steps in the proof of Theorem 1.1 must be checked or changed in order to prove this proposition.

Further analysis of accuracy of estimates (1.15) or (1.16) depends on assumptions on coefficients $(a_i,\ i \ge 0)$, which , in turn, define the so-called notions of short, long and negative memory of a linear process X_t, see, for example, [26]. The behavior of the function $\bar{B}_n(\kappa)$, which is the main characteristic defining the accuracy of approximation, under different assumptions on $(a_i,\ i \ge 0)$ were investigated in [26]. The quantity $\hat{\eta}_n$ is rather simple one and easy to analyze. The quantity $\bar{\rho}_n$ is a specific for the bivariate case. How to get the bound from bellow for $\bar{\rho}_n$ is not evident. On the other hand, from the case of i.i.d. summands with a limit Gaussian law it is known that it is impossible to get optimal with respect to n estimate of the remainder term without presence of quantity, reflecting dependence structure between coordinates of summands. Therefore some quantity, like $\hat{\rho}_n$ or some other, should be present in the estimate of the approximation of $\bar{S}_{2n}$.

2 Auxiliary lemmas

It is well-known the important role that Esseen lemma plays in the problem of the rate of convergence in limit theorems in one-dimensional setting. It allows one to transfer the analysis of d.fs. to analysis of their Fourier transforms. Unfortunately, generalization of Esseen lemma to multivariate case is not so simple. In 1966 there was a paper of S.M. Sadikova [29] where some analog of Esseen lemma in R^2 was proved and by means of this result the rate of convergence in the CLT in the case of i.i.d. bivariate random vectors was proved. In [25] the convergence rate was obtained for non-identically distributed summands.

Let $X = (X_1, X_2),\ Y = (Y_1, Y_2)$ be two random vectors with d.fs. F and G, and ch.fs. f and g, respectively. Denote

$$\hat{f}(t) = f(t) - f((t_1, 0))f((0, t_2)), \quad t = (t_1, t_2)$$

$$A_1 = \sup_{x_1, x_2} \frac{\partial G(x_1, x_2)}{\partial x_1}, \quad A_2 = \sup_{x_1, x_2} \frac{\partial G(x_1, x_2)}{\partial x_2},$$

Lemma 2.1 *([29]) There exists an absolute constant K such that for all $T > 0$*

$$\sup_{x_1, x_2} |F(x_1, x_2) - G(x_1, x_2)| \le K\Big(\int_{-T}^{T} \int_{-T}^{T} \Big|\frac{\hat{f}(t) - \hat{g}(t)}{t_1 t_2}\Big| dt +$$
$$\sup_{x_1} |F(x_1, \infty) - G(x_1, \infty)| + \sup_{x_2} |F(\infty, x_2) - G(\infty, x_2)| + \frac{A_1 + A_2}{T}\Big). \tag{2.1}$$

It is interesting to note that in the author's diploma work in 1967 there was generalization of inequality (2.1) to higher dimensions, but since the expressions in d-dimensional case were too complicated and there was no hope to use such inequality for sums, this result remained unpublished. Although in 1977 N.G. Gamkrelidze published d-dimensional analog of (2.1)(see [11]), as it can be expected, nobody tried to apply this result to obtain the rate of convergence in R^d.

The next lemma is very simple, see , for example, [25].

Lemma 2.2 *Let $u_i, t_i, w_i, z_i,\ i \geq 1$ b be any complex numbers. Then the following inequality holds*

$$\Big|(\prod_{i=1}^{n} z_i - \prod_{i=1}^{n} w_i) - (\prod_{i=1}^{n} u_i - \prod_{i=1}^{n} v_i)\Big| \leq$$
$$\sum_{i=1}^{n} |z_i - w_i| \Big| \prod_{k=1}^{i-1} z_k \prod_{l=i+1}^{n} w_l - \prod_{k=1}^{i-1} u_k \prod_{l=i+1}^{n} v_l \Big| +$$
$$\sum_{i=1}^{n} \Big| \prod_{k=1}^{i-1} u_k \Big| |(z_i - w_i) - (u_i - v_i)| \Big| \prod_{l=i+1}^{n} v_l \Big|. \tag{2.2}$$

To estimate the quantity Δ_n we apply Lemma 2.1 with $\bar{F}_n$ and $\bar{G}_n$ instead of F and G, respectively. Then it is easy to see that in the nominator of the main integral on the right side of (2.1) we have expression present on the left-hand side of (2.2) with

$$z_i = f_i(t'),\ w_i = f_i((t'_1, 0))f_i((0, t'_2)), \quad u_i = g_i(t'),\ v_i = g_i((t'_1, 0))g_i((0, t'_2)),$$

where $t' := (t'_1, t'_2),\ t'_i = B_n^{-1} t_i,\ i = 1, 2$. In the following lemmas we shall keep these notations.

Lemma 2.3 *For any fixed $0 < \kappa < \alpha$ there exists a constant c_1, depending on κ and on the constant $\hat{C}$, defined in (1.1) such that*

$$|z_j - w_j| \leq c_1 |t_1 t_2|^{\kappa/2} \frac{\lambda_j^\kappa}{B_n^\kappa}. \tag{2.3}$$

REMARK 2.1 Although $\hat{C}$ itself depends on κ, we stress the dependence of c_1 on $\hat{C}$, since via this dependence constant c_1 (and final constant $\bar{C}$) tends to infinity when κ tends to α.

Proof. It is easy to see that

$$|z_j - w_j| \leq J_{1,j} + J_{2,j},$$

where

$$J_{1,j} = \Big| \int\int (\exp(it'_1 x_1) - 1)(\exp(it'_2 x_2) - 1) F_j(dx_1, dx_2) \Big|,$$
$$J_{2,j} = |(1 - f_j((t'_1, 0)))(1 - f_j((0, t'_2))|.$$

Now we apply standard estimate

$$|\exp(ix) - 1| \leq c(\beta)|x|^\beta, \quad 0 < \beta \leq 1, \tag{2.4}$$

with an appropriate choice of β (for example, we set $\beta_1 = \beta_2 = \kappa/2$), and we easily estimate both terms $J_{i,j}$, obtaining (2.3). The lemma is proved.

In the sequel $I(A)$ will stand for the indicator of a set A.

Lemma 2.4 *There exists a constant c_2 depending on κ and $\hat{C}$ such that*

$$\begin{aligned} |(z_j - w_j) - (u_j - v_j)| &\le c_2\Big((|t_1|^r|t_2|^\kappa + |t_2|^r|t_1|^\kappa)\eta_n^\kappa\Big(\frac{\nu_{j,r}^1 + \nu_{j,r}^2}{B_n^r}\Big) \\ &+ \Big(|t_1t_2|^{r/2}I(r \le 2) + |t_1t_2|(|t_1|^{r-2} + |t_2|^{r-2})I(r > 2)\Big)\frac{\nu_{j,r}}{B_n^r}\Big). \quad (2.5) \end{aligned}$$

Proof. Simple considerations lead to the estimate

$$|(z_j - w_j) - (u_j - v_j)| \le K_{1,j} + K_{2,j}, \quad (2.6)$$

with

$$K_{1,j} = \Big|\int\int(\exp(it_1'x_1) - 1)(\exp(it_2'x_2) - 1)(F_j(dx) - G_j(dx))\Big|,$$

$$K_{2,j} = |f_j((t_1',0)) - g_j((t_1',0))||1 - f_j((0,t_2'))| + |f_j((0,t_2')) - g_j((0,t_2'))||1 - g_j((t_1',0))|.$$

To estimate $K_{1,j}$ in the case $r \le 2$ we apply (2.4) with $\beta_1 = \beta_2 = r/2$ and another standard inequality

$$|x_1x_2|^{r/2} \le 2^{-r/2}(|x_1|^r + |x_2|^r).$$

Then we get

$$K_{1,j} \le c|t_1t_2|^{r/2}\frac{\nu_{i,r}}{B_n^r}. \quad (2.7)$$

In the case $r > 2$ the estimate (2.4) is not applicable since $r/2 > 1$. Now for $x \in R$ we can write

$$\exp(ix) - 1 = ix + R(x), \quad |R(x)| \le c(\beta)|x|^\beta, \quad 1 < \beta \le 2. \quad (2.8)$$

Applying this inequality with $\beta = r - 1$ and obvious inequality $|a||b|^r$
$< |a|^{1+r} + |b|^{1+r}$, and remembering that

$$\int\int x_1x_2(F_j(dx) - G_j(dx)) = 0,$$

we have

$$\begin{aligned} K_{1,j} &= \Big|\int\int(it_1'x_1 + R(t_1'x_1))(it_2'x_2 + R(t_2'x_2))(F_j(dx) - G_j(dx))\Big| \\ &\le c(|t_1||t_2|^{r-1} + |t_1|^{r-1}|t_2|)\frac{\nu_{i,r}^{(1)} + \nu_{i,r}^{(2)}}{B_n^r}. \quad (2.9) \end{aligned}$$

The quantity $K_{2,j}$ contains only characteristic functions of marginal distributions (one argument always is 0), therefore noting that

$$\int|x_1|^r\Big|\int(F_j(dx_1,dx_2) - G_j(dx_1,dx_2))\Big| \le \int\int|x_1|^r\Big|F_j(dx) - G_j(dx)\Big|$$

and applying estimates from [19] and from previous lemma we easily get

$$K_{2,j} \le c(|t_1|^r|t_2|^\kappa + |t_2|^r|t_1|^\kappa)\eta_n^\kappa\Big(\frac{\nu_{i,r}^1+\nu_{i,r}^2}{B_n^r}\Big). \tag{2.10}$$

From (2.6)-(2.10) we have (2.5). The lemma is proved.

Lemma 2.5 *There exists a constant c_3 such that*

$$V_j := |\prod_{k=1}^{j-1} u_k \prod_{m=j+1}^{n} v_m| \le \exp\{-c_3||t||^\alpha \rho_n + c_3||t||^\alpha B_n^{-\alpha}\lambda_j^\alpha \gamma_j(\bar{t})\}. \tag{2.11}$$

Here and in the sequel $\bar{t} = t||t||^{-1},\ t = (t_1, t_2)$.

Proof. From the definition of u_k and v_k we have

$$\begin{aligned} V_j := \Big|\prod_{k=1}^{j-1} u_k \prod_{m=j+1}^{n} v_m\Big| \quad &\le \quad \exp\Big\{-B_n^{-\alpha}\Big(\sum_{k=1}^{j-1}\lambda_k^\alpha\int_{V_1}|t_1x_1+t_2x_2|^\alpha\Gamma_k(dx) \\ &+ \quad \sum_{k=j+1}^{n}\lambda_k^\alpha\int_{V_1}(|t_1x_1|^\alpha+|t_2x_2|^\alpha)\Gamma_k(dx)\Big)\Big\}. \end{aligned}$$

Since

$$|t_1x_1|^\alpha + |t_2x_2|^\alpha| \ge c(\alpha)|t_1x_1+t_2x_2|^\alpha$$

where $c(\alpha) = 1$ for $0 < \alpha \le 1$ and $c(\alpha) = 2^{1-\alpha}$ for $1 < \alpha \le 2$, thus, taking into account that always $c(\alpha) \le 1$, we get

$$V_j \le \exp\{-c(\alpha)B_n^{-\alpha}||t||^\alpha \sum_{k\ne j}^{n}\lambda_k\gamma_k(\bar{t})\}. \tag{2.12}$$

From (2.12) using definition of ρ_n we easily get (2.11) with $c_3 = c(\alpha)$. The lemma is proved.

3 Proofs

We start with the proof of Theorem 1.1 and Corollary 1.1. Note that in order to estimate Δ_n using Lemma 2.1 we must estimate differences of marginal d.fs. of $\bar{F}_n$ and $\bar{G}_n$. These quantities we estimate by means of Theorem 1 in [18]. We get

$$\sup_{x_1}|\bar{F}_n(x_1,\infty)-\bar{G}_n(x_1,\infty)|+\sup_{x_2}|\bar{F}_n(\infty,x_2)-\bar{G}_n(\infty,x_2)| \le c\max(L_n, L_n^{1/(r+1)}\eta_n^{r/(r+1)}). \tag{3.1}$$

First we assume that

$$\eta_n \le 2L_n. \tag{3.2}$$

We show that under assumption (3.2) for some $0 < \kappa < \alpha$ and for $T_1 := C_0L_n^{-1}\rho_n^{r/\alpha}$ the following estimate holds:

$$\int_{-T_1}^{T_1}\int_{-T_1}^{T_1}\Big|\frac{\bar{f}_n(t)-\bar{g}_n(t)}{t_1t_2}\Big|dt \le cL_n\rho_n^{-(r+\kappa)/\alpha}\tau_n(\kappa). \tag{3.3}$$

This estimate together with (3.1) gives us

$$\Delta_n \le \bar{C} L_n \rho_n^{-(r+\kappa)/\alpha} \tau_n(\kappa), \tag{3.4}$$

with a constant $\bar{C}$ depending on all fixed parameters and also on κ (and unboundedly increasing if κ approaches its boundaries 0 or α).

To estimate the integral in (3.3), for $||t||_\infty \le T_1$ we need to estimate the quantity

$$A_n(t) := |\bar{f}_n(t) - \bar{g}_n(t)| = \Big| \Big(\prod_{i=1}^n z_i - \prod_{i=1}^n w_i \Big) - \Big(\prod_{i=1}^n u_i - \prod_{i=1}^n v_i \Big) \Big|.$$

According to Lemma 2.2

$$A_n(t) \le I_{1,n} + I_{2,n}, \tag{3.5}$$

where

$$I_{1,n} = \sum_{i=1}^n |z_i - w_i| \Big| \prod_{k=1}^{i-1} z_k \prod_{l=i+1}^n w_l - \prod_{k=1}^{i-1} u_k \prod_{l=i+1}^n v_l \Big|,$$

$$I_{2,n} = \sum_{i=1}^n | \prod_{k=1}^{i-1} u_k | |(z_i - w_i) - (u_i - v_i)| | \prod_{l=i+1}^n v_l |.$$

Let us denote

$$\tilde{z}_k = z_k u_k^{-1},\ \tilde{w}_k = w_k v_k^{-1},\ \tilde{u}_k = \tilde{v}_k \equiv 1.$$

Then

$$W_i := | \prod_{k=1}^{i-1} z_k \prod_{l=i+1}^n w_l - \prod_{k=1}^{i-1} u_k \prod_{l=i+1}^n v_l | = \tilde{W}_i V_i,$$

where

$$\tilde{W}_i = | \prod_{k=1}^{i-1} \tilde{z}_k \prod_{l=i+1}^n \tilde{w}_l - \prod_{k=1}^{i-1} \tilde{u}_k \prod_{l=i+1}^n \tilde{v}_l | \le \tilde{W}_{i,1} + \tilde{W}_{i,2},$$

$$\tilde{W}_{i,1} = \sum_{k=1}^{i-1} \Big| \prod_{l=1}^{k-1} \tilde{z}_l \Big| |\tilde{z}_k - \tilde{u}_k|, \quad \tilde{W}_{i,2} = \sum_{k=i+1}^n | \prod_{l=1}^{i-1} \tilde{z}_l \prod_{j=i+1}^{k-1} \tilde{w}_j | |\tilde{w}_k - \tilde{v}_k|.$$

Now, for $||t||_\infty \le T_1 \le C_0 L_n^{-1}$,

$$|u_k|^{-1} \le \exp\big\{ \lambda_k^\alpha B_n^{-\alpha} ||t||^\alpha \gamma_k(\bar{t}) \big\} \le \exp\big\{ (2^{1/2} C_0 \eta_n L_n^{-1})^\alpha \big\} \le c_4,$$

and the standard estimation of characteristic functions gives us the estimate

$$|\tilde{z}_k - \tilde{u}_k| = |u_k|^{-1} |z_k - u_k| \le c B_n^{-r} \big(|t_1|^r \nu_{k,r}^{(1)} + |t_2|^r \nu_{k,r}^{(2)} \big). \tag{3.6}$$

From this estimate we have

$$|\tilde{z}_k| \le 1 + |\tilde{z}_k - \tilde{u}_k| \le \exp\big\{ c B_n^{-r} (|t_1|^r \nu_{k,r}^{(1)} + |t_2|^r \nu_{k,r}^{(2)}) \big\}$$

and

$$\Big|\prod_{l=1}^{k-1}\tilde z_l\Big|\le \exp\big\{cL_n(|t_1|^r+|t_2|^r)\big\} \tag{3.7}$$

Similarly, for $||t||_\infty\le T_1\le C_0L_n^{-1}$, we get

$$|v_k|^{-1}\le c_5$$

and, estimating marginal characteristic functions we have

$$|\tilde w_k-\tilde v_k|=|v_k|^{-1}|w_k-v_k|\le cB_n^{-r}(|t_1|^r\nu_{k,r}^{(1)}+|t_2|^r\nu_{k,r}^{(2)}). \tag{3.8}$$

Using the estimate $|\tilde w_k|\le 1+|\tilde w_k-\tilde v_k|$ and (3.8) together with (3.7) we get for $i<k$

$$\Big|\prod_{l=1}^{i-1}\tilde z_k\prod_{j=i+1}^{k-1}\tilde w_j\Big|\le \exp\big\{c_6L_n(|t_1|^r+|t_2|^r)\big\}. \tag{3.9}$$

Collecting estimates (3.6) -(3.9) we get

$$\tilde W_i\le cL_n(|t_1|^r+|t_2|^r)\exp\big\{c_6L_n(|t_1|^r+|t_2|^r)\big\}.$$

From (2.11), taking into account that, for $||t||_\infty\le T_1\le C_0L_n^{-1}$, the term $||t||^\alpha B_n^{-\alpha}\lambda_j^\alpha\gamma_j(\bar t)$ is bounded by a constant (see the estimation of $|v_k|^{-1}$ or $|u_k|^{-1}$), we have

$$\begin{aligned} W_i &\le cL_n(|t_1|^r+|t_2|^r)\exp\big\{-c_3||t||^\alpha\rho_n+c_6L_n(|t_1|^r+|t_2|^r)\big\}\\ &\le cL_n(|t_1|^r+|t_2|^r)\exp\big\{-c_3||t||^\alpha\rho_n(1-c_7L_nT_1^{r-\alpha}\rho_n^{-1})\big\}. \end{aligned} \tag{3.10}$$

Without loss of generality we can assume that

$$L_n\rho_n^{-(r+\kappa)/\alpha}\le 1,$$

otherwise (3.4) will be true with $\bar C>1$. Then it is easy to see that it is possible to choose C_0 such small that

$$c_7L_nT_1^{r-\alpha}\rho_n^{-1}\le c_7C_0^{r-\alpha}L_n^{1-r+\alpha}\rho_n^{-1+(r+\kappa)(r-\alpha)\alpha^{-1}}\le c_7C_0^{r-\alpha}\rho_n^{-1+(r+\kappa(1-r+\alpha))\alpha^{-1}}\le 1/2,$$

since $\rho_n\le 1$ and the exponent at ρ_n is positive. Then from (3.10) we obtain

$$W_i\le cL_n(|t_1|^r+|t_2|^r)\exp\big\{-(c_3/2)||t||^\alpha\rho_n\big\}$$

and, combining with the estimate (2.3), we get

$$I_{1,n}\le c|t_1t_2|^{\kappa/2}L_n(|t_1|^r+|t_2|^r)\exp\big\{-(c_3/2)||t||^\alpha\rho_n\big\}\tau_n(\kappa). \tag{3.11}$$

Now we estimate $I_{2,n}=\sum_{i=1}^n V_i|(z_i-w_i)-(u_i-v_i)|$. From lemmas 2.4 and 2.5 we have

$$\begin{aligned} I_{2,n} &\le c\Big(|t_1t_2|^{r/2}I(r\le 2)+|t_1t_2|(|t_1|^{r-2}+|t_2|^{r-2})I(r>2)\\ &+(|t_1|^r|t_2|^\kappa+|t_2|^r|t_1|^\kappa)\eta_n^\kappa\Big)L_n\exp\big\{-c_3||t||^\alpha\rho_n\big\}. \end{aligned} \tag{3.12}$$

Collecting estimates (3.5), (3.11) and (3.12) and taking into account that

$$\int_0^\infty s^d \exp(-s^\alpha b) ds = c(d,\alpha) b^{-(d+1)/\alpha},$$

we get (3.3), and therefore (3.4) is proved.

Now we assume that

$$\eta_n > 2L_n. \tag{3.13}$$

We set $T = T_2 := C_0 L_n^{-1/(r+1)} \eta_n^{-r/(r+1)} \rho_n^{(r+\kappa)/\alpha}$ in Lemma 2.1 and we shall prove that

$$\Delta_n \le \bar{C} L_n^{(1-\kappa)/(r+1)} \eta_n^{r'/(r+1)} \rho_n^{-r'/\alpha} \tau_n(\kappa), \tag{3.14}$$

where $r' = r + \kappa$. Note, that C_0 and $\bar{C}$ may be different from corresponding constants obtained in the case (3.2), but in the final estimate we can take bigger $\bar{C}$. This case is a little bit more complicated, although the scheme of estimation is the same. We start with the same estimate (3.5). To estimate $I_{1,n}$ we denote

$$\bar{z}_k = z_k \hat{u}_k^{-1},\ \bar{w}_k = w_k \hat{v}_k^{-1},\ \bar{u}_k = u_k \hat{u}_k^{-1},\ \bar{v}_k = v_k \hat{v}_k^{-1},$$

where

$$\hat{u}_k = g_k\Big(t; \frac{\lambda_k}{B_n} \chi_n^{A\alpha^{-1}}\Big) \quad \hat{v}_k = g_k\Big((t_1, 0); \frac{\lambda_k}{B_n} \chi_n^{A\alpha^{-1}}\Big) g_k\Big((0, t_2); \frac{\lambda_k}{B_n} \chi_n^{A\alpha^{-1}}\Big).$$

Here $\chi_n = L_n/\eta_n,\quad A = \alpha/(r+1)$ and in the notation of ch.f. of stable vector we skip α and Γ_k and leave only the argument and scale parameter. Since, according to (3.13), $\chi_n < 1/2$, it is easy to see that

$$|\bar{u}_k| \le 1, \quad |\bar{v}_k| \le 1.$$

We can write

$$W_i := \Big| \prod_{k=1}^{i-1} z_k \prod_{l=i+1}^{n} w_l - \prod_{k=1}^{i-1} u_k \prod_{l=i+1}^{n} v_l \Big| = \bar{W}_i \bar{V}_i, \tag{3.15}$$

where

$$\bar{W}_i = \Big| \prod_{k=1}^{i-1} \bar{z}_k \prod_{l=i+1}^{n} \bar{w}_l - \prod_{k=1}^{i-1} \bar{u}_k \prod_{l=i+1}^{n} \bar{v}_l \Big| \le \bar{W}_{i,1} + \bar{W}_{i,2}, \tag{3.16}$$

$$\bar{W}_{i,1} = \sum_{k=1}^{i-1} \Big| \prod_{l=1}^{k-1} \bar{z}_l \Big| |\bar{z}_k - \bar{u}_k| \Big| \prod_{l=k+1}^{i-1} \bar{u}_l \prod_{m=i+1}^{n} \bar{v}_m \Big|, \tag{3.17}$$

$$\bar{W}_{i,2} = \sum_{k=i+1}^{n} \Big| \prod_{l=1}^{i-1} \bar{z}_l \prod_{j=i+1}^{k-1} \bar{w}_j \Big| |\bar{w}_k - \bar{v}_k| \Big| \prod_{m=k+1}^{n} \bar{v}_m \Big|, \tag{3.18}$$

and

$$\bar{V}_i := \Big| \prod_{k=1}^{i-1} \hat{u}_k \prod_{m=i+1}^{n} \hat{v}_m \Big|. \tag{3.19}$$

Note that for $||t||_\infty \le T_2$ we have

$$|\hat{u}_k|^{-1} \le \exp\left\{\lambda_k^\alpha B_n^{-\alpha}||t||^\alpha \chi_n^A\right\} \le \exp\left\{2^{\alpha/2}C_0^\alpha\right\} \le c \tag{3.20}$$

and, similarly,

$$|\hat{v}_k|^{-1} \le c.$$

Using these estimates, similarly to (3.6) and (3.8), we have

$$|\bar{z}_k - \bar{u}_k| = |\hat{u}_k|^{-1}|z_k - u_k| \le cB_n^{-r}(|t_1|^r\nu_{k,r}^{(1)} + |t_2|^r\nu_{k,r}^{(2)}), \tag{3.21}$$

$$|\bar{w}_k - \bar{v}_k| = |\hat{v}_k|^{-1}|w_k - v_k| \le cB_n^{-r}(|t_1|^r\nu_{k,r}^{(1)} + |t_2|^r\nu_{k,r}^{(2)}). \tag{3.22}$$

From (3.21) and (3.22), similarly to (3.7), (3.9), we easily get

$$\begin{aligned}\max_{i,k}\max & \left\{ \Big|\prod_{l=1}^{i-1}\bar{z}_l \prod_{j=i+1}^{k-1}\bar{w}_j \prod_{m=k+1}^{n}\bar{v}_m\Big|, \Big|\prod_{l=1}^{k-1}\bar{z}_l \prod_{j=k+1}^{i-1}\bar{w}_j \prod_{m=i+1}^{n}\bar{v}_m\Big|\right\} \\ \le & \exp\left\{c_8L_n(|t_1|^r + |t_2|^r)\right\}. \end{aligned} \tag{3.23}$$

In the same way as in Lemma 2.5, using (3.20), we estimate $\tilde{V}_i$:

$$\bar{V}_i \le c_{10}\exp\{-c_9||t||^\alpha\rho_n\chi_n^A\}. \tag{3.24}$$

Collecting estimates (3.15)- (3.19), (3.21)-(3.24) we get

$$\begin{aligned}\bar{W}_i\bar{V}_i &\le c_{11}\sum_{k\ne i}\frac{\nu_{k,r}}{B_n^r}||t||_r^r\exp\{c_8L_n||t||_r^r - c_9||t||^\alpha\rho_n\chi_n^A\} \\ &\le c_{11}L_n||t||_r^r\exp\{-c_9||t||^\alpha\rho_n\chi_n^A(1 - c_{12}L_nT_2^{r-\alpha}\rho_n^{-1}\chi_n^{-A})\},\end{aligned}$$

If we prove that

$$M := c_{12}L_nT_2^{r-\alpha}\rho_n^{-1}\chi_n^{-A} < 1/2, \tag{3.25}$$

then we will get

$$W_i \le c_{11}||t||_r^rL_n\exp\{-(c_9/2)||(s,t)||^\alpha\rho_n\chi_n^A\}.$$

Combining this estimate with (2.3) we obtain

$$I_{1,n} \le c|t_1t_2|^{\kappa/2}||t||^rL_n\tau_n(\kappa)\exp\{-(c_9/2)||t||^\alpha\rho_n\chi_n^A\}. \tag{3.26}$$

It is easy to see that

$$M \le c_{12}C_0^{r-\alpha}L_n^B\eta_n^C\rho_n^D, \quad B = \frac{1}{r+1}, \quad C = \frac{\alpha - \beta r}{r+1}, \quad D = \frac{(r+\kappa)\beta - \alpha}{\alpha}, \quad \beta = r - \alpha.$$

Since all three quantities L_n, η_n and ρ_n are less than 1, (3.25) holds if $C \ge 0$, $D \ge 0$ and C_0 is chosen sufficiently small. Thus, we need to consider the cases where $C < 0$ or $D < 0$ (it is easy to see that they both can not be negative). Let $C < 0$, that is $\beta > \alpha/r$, then $D > 0$, and since in this case $B > -C$, we have

$$L_n^B \eta_n^C \rho_n^D = \frac{L_n^B}{\eta_n^{-C}} \rho_n^D = \Big(\frac{L_n}{\eta_n}\Big)^{-C} L_n^{B+C} \rho_n^D \le 1,$$

and (3.25) holds. If $D < 0$, that is $\beta < \alpha/(r+\kappa)$, then $C > 0$ and we may assume that

$$L_n^{(1-\kappa)/(r+1)} \eta_n^{r'/(r+1)} \rho_n^{-r'/\alpha} \le 1$$

otherwise (3.14) will be true with $\bar{C} > 1$. From this assumption we have

$$\rho_n^{-1} \le L_n^{B_1} \eta_n^{C_1}, \quad B_1 = -\frac{\alpha(1-\kappa)}{r'(r+1)}, \quad C_1 = -\frac{\alpha}{r+1},$$

therefore

$$L_n^B \eta_n^C \rho_n^D = L_n^B \eta_n^C (\rho_n^{-1})^{-D} \le L_n^{B_2} \eta_n^{C_2}, \quad B_2 = B - B_1 D, \quad C_2 = C - C_1 D.$$

It is not difficult to verify that in this case $B_2 \ge 0,\ C_2 \ge 0$, thus (3.25) holds, too. (3.26) is proved.

For the estimation of $I_{2,n}$ we can not use estimate (2.11) since in the interval $||t||_\infty \le T_2$ we can not estimate the positive term in the exponent. Therefore we write

$$V_i = \Big| \prod_{k=1}^{i-1} u_k \prod_{m=i+1}^{n} v_m \Big| = \Big| \prod_{k=1}^{j-1} \bar{u}_k \prod_{m=j+1}^{n} \bar{v}_m \Big| \bar{V}_i,$$

and remembering that $|\bar{u}_k| \le 1,\ \ |\bar{v}_k| \le 1$, from (3.24) we get

$$V_i \le c_{10} \exp\{-c_9 ||t||^\alpha \rho_n \chi_n^A\}. \tag{3.27}$$

Combining (2.5) and (3.27) we get

$$\begin{aligned} I_{2,n} \ &\le\ c\Big(|t_1 t_2|^{r/2} I(r \le 2) + |t_1 t_2|(|t_1|^{r-2} + |t_2|^{r-2}) I(r > 2) \\ &+\ (|t_1|^r |t_2|^\kappa + |t_2|^r |t_1|^\kappa)\eta_n^\kappa\Big) L_n \exp\Big\{ - c_9 ||t||^\alpha \rho_n \chi_n^A \Big\}. \end{aligned} \tag{3.28}$$

The rest of the proof of (3.14) is similar to the first case: from (3.26) and (3.28) integrating and using (3.13) we get

$$\begin{aligned} \int_{-T_2}^{T_2} \int_{-T_2}^{T_2} \Big| \frac{A_n(t)}{t_1 t_2} \Big| dt \ &\le\ c\big(L_n(\rho_n \chi_n^A)^{-r'/\alpha}(\eta_n^\kappa + \tau_n(\kappa)) + L_n(\rho_n \chi_n^A)^{-r/\alpha}\big) \\ &\le\ c L_n^{(1-\kappa)/(r+1)} \eta_n^{r'/(r+1)} \rho_n^{-r'/\alpha} \Big(\tau_n(\kappa) + \eta_n^\kappa + \chi_n^{\kappa/(r+1)} \rho_n^{\kappa/\alpha} \Big) \\ &\le\ c L_n^{(1-\kappa)/(r+1)} \eta_n^{r'/(r+1)} \rho_n^{-r'/\alpha} \tau_n(\kappa). \end{aligned} \tag{3.29}$$

Since the bound (3.1) for marginal d.fs. and the quantity T_2^{-1} are smaller than the bound obtained in (3.29), from (3.29) we obtain (3.14). To finish the proof of the theorem we must choose the parameter κ. First of all, let us note that constants which depend on κ unboundedly increase if κ approaches its boundaries 0 or α (see, for example (1.1)). Therefore, it is natural to take two fixed small positive quantities ε_1 and ε_2 and to take minimum of the obtained estimates with respect to κ over the interval $\varepsilon_1 \leq \kappa \leq \alpha(1-\varepsilon_2)$. Note that this choice can be made separately in each case ($\eta_n > 2L_n$ or $\eta_n \leq 2L_n$) and in this way we get (1.2). If we choose the parameter κ the same in both cases and take at first maximum between two terms and only then take minimum, we get (1.3) and (1.4) from Corollary 1.1. Theorem 1.1 and Corollary 1.1 are proved.

The estimates of Corollary 1.2 are derived from the estimates of Corollary 1.1 (it is not difficult to get more general estimate from (1.2)), it is sufficient to note that in the case of weighted sums we have

$$L_n^{(1)} = \tau_n(r)\nu_r, \quad \rho_n^{(1)} \equiv \rho$$

and $\tau_n(\kappa)$ and η_n remain unchanged. For a proof of Corollary 1.3 note that if $\lambda_i \equiv 1$ then $\tau_n(\kappa) = n^{(\alpha-\kappa)/\alpha}$ and $\eta_n = n^{-1/\alpha}$. Corollaries 1.2 and 1.3 are proved.

It remains to prove Proposition 1.1. As it was mentioned before the formulation of this proposition, we can not directly apply any of corollaries, therefore we must check all the steps of the proof of the main result. Since most of this checking is routine, we shall point only main points. A reader probably noticed that all characteristics involved in estimates (1.15) and (1.16) contain coefficients of the sum $\bar{S}_{2n}$ only up to n. It is easy to see that one dimensional characteristics $L_{2n}^{(3)}, \eta_{2n}^{(3)}, \tau_{2n}^{(3)}$ for the second coordinate defined by coefficients $b_{2n,i}, \ -\infty < i \leq 2n$ due to the equality (1.14) coincide with the corresponding characteristics defined by $b_{n,i}, \ -\infty < i \leq n$. It is not so evident why $\bar{\rho}_n^{(3)}$ does not depend on $b_{2n,i}, \ n < i \leq 2n$. The explanation is as follows. The characteristic $\bar{\rho}_n^{(3)}$ appears when estimating $A_n(t)$ from (3.5). The quantities $I_{1,n}$ and $I_{2,n}$ are expressed as sums $\sum_{i \leq 2n}$, but due to the structure of summands for $n < i \leq 2n$ we have $z_i = w_i$ and $u_i = v_i$. Therefore all terms with indices in the range $n < i \leq 2n$ in the above mentioned sums vanish and in the estimation of $I_{1,n}$ and $I_{2,n}$ it remains only sums $\sum_{i \leq n}$, therefore the characteristic $\bar{\rho}_n^{(3)}$ depends only on $(b_{n,i}, b_{2n,i}), \ i \leq n$. Since the choice of $T_i, \ i = 1, 2$ is based on the estimation of $A_n(t)$, in the final estimate we get $\bar{\rho}_n$. Calculating pseudomoments one must keep in mind that the distributions of ξ_i and θ_i are concentrated on line $x_2 = kx_1$ with $k = b_{n,i}^{-1} b_{2n,i}$. Also in the proof of Lemmas 2.3-2.5 one must remember that

$$\max_{i \leq n} |b_{n,i}| = \max_{i \leq 2n} |b_{2n,i}|.$$

Thus, taking into account these remarks, one can carry the proof of Proposition 1.1.

References

[1] J. Banys. On the estimates of the remainder term in multivariate integral limit theorem in the case of stable limit law. *Liet. Mat. Rink.*, 9:731–739, 1969.

[2] J. Banys. On the integral limit multivariate theorem in the case of stable limit law. *Liet. Mat. Rink.*, 10:665–672, 1970.

[3] J. Banys. The estimates of the rate of convergence in multivariate limit theorem in the case of symmetric stable limit law. *Liet. Mat. Rink.*, 11:497–507, 1971.

[4] V. Bentkus. A Lyapunov type bound in R^d. *Teor. Verojatn. Primen.*, 49:400–410, 2004.

[5] V. Bentkus, F. Götze, V. Paulauskas, and A. Račkauskas. The accuracy of Gaussian approximation in Banach spaces. *in: Limit Theorems of Probability Theory*, 2000.

[6] V. Bentkus, A. Juozulynas, and V. Paulauskas. Bounds for stable measures of convex shells and stable approximations. *Ann. Probab.*, 28:1280–1300, 2000.

[7] H. Bergström. On the central limit theorem in the case of not equally distributed random variables. *Skand. Aktuarietidskr.*, 33:37–62, 1949.

[8] H. Bergström. On distribution function with a limiting stable distribution function. *Arkiv Mat.*, 2:463–474, 1952.

[9] R.N. Bhattacharya and R. Ranga Rao. *Normal Approximation and Asymptotic Expansions.* John Wiley & Sons, New York, 1976.

[10] G. Christoph and W. Wolf. *Convergence Theorems with a Stable Limit Law*, volume 70 of *Mathematical Research.* Akademie-Verlag, Berlin, 1990.

[11] N.G. Gamkrelidze. Esseen's inequality for multi-dimensional distribution functions. *Teor. Verojat. Primen.*, 22:897–900 (In Russian), 1977. English translation: Theory Probab Appl., 1978, SIAM.

[12] B. Garel and B. Kodia. Signed symmetric covariation coefficient for alpha-stable dependence modeling. *C. R. Acad. Sci. Paris, ser I*, 347:315–320, 2009.

[13] B.V. Gnedenko and A.N. Kolmogorov. *Limit distributions for sums of independent random variables.* Addison-Wesley, Reading, MA, 2nd edition, 1968.

[14] P. Lévy. *Théorie de l'addition des variables aléatoires.* Gautier-Villars, Paris, 1937.

[15] S.V. Nagaev and V.J. Rotar. On stronger estimates of lyapunov type. *Theory Probab. Appl.*, 18:109–121, 1973.

[16] V. Paulauskas. On an improvement of lyapunov theorem. *Lithuanian Math. J. (Litovsk. Matem. Sb)*, 9:323–328, 1969.

[17] V. Paulauskas. an estimate of the rate of convergence in a multidimensional limit theorem in the case of a stable limit law. *Dokl. Akad. Nauk*, 6:1536–1538, 1974.

[18] V. Paulauskas. Estimates of the remainder term in limit theorem in the case of stable limit law. *Lithuanian Math. J.*, 14:127–146, 1974.

[19] V. Paulauskas. Uniform and non-uniform estimates of the remainder term in a limit theorem with a stable limit law. *Lithuanian Math. J.*, 14:661–672, 1974.

[20] V. Paulauskas. On the closeness of the distributions of two sums of independent random variables in hilbert space. *Lithuanian Math. J.*, 15:494–509, 1975.

[21] V. Paulauskas. Rate of convergence in the multidimensional limit theorem with a stable limit law. *Lithuanian Math. J.*, 15:168–184, 1975.

[22] V. Paulauskas. Two non-uniform estimates of the closeness between distributions of two sums of independent random variables. *Lithuanian Math. J.*, 15:254–266, 1975.

[23] V. Paulauskas. Some remarks on multivariate stable distributions. *J. Multivariate Anal.*, 6(3):356–368, 1976.

[24] V. Paulauskas and A. Račkauskas. *Approximation Theory in the Central Limit Theorem: Exact Results in Banach Spaces.* Kluwer Academic Publishers, Dordrecht, 1989.

[25] V. Paulauskas and A. Slushnys. On the estimation of the rate of convergence in the two-dimensional central limit theorem. *Liet. Mat. Rink.*, 8:591–595, 1968.

[26] V. Paulauskas and D. Surgailis. On the rate of approximation in limit theorems for sums of moving averages. *Teor. Verojatn. Primen.*, 52:405–414, 2007.

[27] V.V. Petrov. *Limit Theorems of Probability Theory. Sequences of Independent Random Variables.* Clarendon Press, Oxford, 1995.

[28] S.T. Rachev. *Probability Metrics and the Stability of Stochastic Models.* Jon Wiley & Sons, Chichester, 1991.

[29] S.M. Sadikova. On two-dimensional analogs of an inequality of C.G. Esseen and their application to the central limit theorem. *Teor. Verojatn. Primen.*, 11:369–380, 1966.

[30] G. Samorodnitsky and M. Taqqu. *Stable non-Gaussian Random Processes. Models with Infinite Variance.* Chapman& Hall, New York, 1994.

[31] V.V. Sazonov. *Normal Approximations - Some Recent Advances.* Springer, Berlin, 1981.

[32] V.V. Senatov. *Normal Approximations: New Results, Methods and Problems.* VSP, Utrecht, 1998.

[33] V.M. Zolotarev. On closeness of distributions of two sums of independent random variables. *Theory Probab. Appl.*, 10:519–526, 1965.

[34] V.M. Zolotarev. *Modern Theory of Summation of Random Variables.* VSP, Utrecht, 1997.